ICECA

International Conference
Enumerative Combinatorics and Applications
University of Haifa – Virtual – September 4-6, 2023

Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

Bishal Deb

Department of Mathematics, University College London, London WC1E 6BT, UK

bishal.deb.19@ucl.ac.uk

A continued fraction of Jacobi-type (J-fraction) is of the form

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\cdots}}}\;, \tag{1}$$

where a_n are its coefficients when expanded as a formal power series. Euler [4, section 21] discovered a Stieltjes-type continued fraction for $a_n = n!$ which can be contracted (see [13, p. V-31] for the contraction formula) to obtain a J-fraction for $a_n = n!$ with coefficients $\gamma_n = 2n+1$ and $\beta_n = n^2$. One can introduce new variables in this J-fraction by replacing

- $\gamma_n = 2n+1$ with $\gamma_0 = z$, $\quad \gamma_n = ([x_2 + (n-1)u_2] + [y_2 + (n-1)v_2] + w$ for $n \geq 1$;
- and $\beta_n = n^2$ with $\beta_n = [x_1 + (n-1)u_1][x_2 + (n-1)v_1]$;

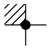
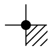
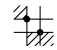
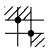
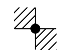
and then ask what permutation statistics are enumerated by the 10 variables x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z. Sokal and Zeng systematically answered this question in [11]. In fact, they provide two interpretations for this J-fraction. However, their second interpretation was left as a conjecture [11, Conjecture 2.3] and they could only prove it with a specialisation. We have proved this conjecture in [2].

Statement of result

Given a permutation $\sigma \in \mathfrak{S}_n$, an index i can be classified as per the ***cycle classification*** into the following five disjoint categories: cycle peak if $\sigma^{-1}(i) < i > \sigma(i)$; $\quad$ cycle valley if $\sigma^{-1}(i) > i < \sigma(i)$;

cycle double rise if $\sigma^{-1}(i) < i < \sigma(i)$; cycle double fall if $\sigma^{-1}(i) > i > \sigma(i)$; and fixed point if $\sigma^{-1}(i) = i = \sigma(i)$.

Additionally, an index i can also be classified using the ***record classification***. Following [8, p. 4] we also reformulate these statistics in terms of mesh patterns.

- record (or left-to-right maximum) if $\sigma(j) < \sigma(i)$ for all $j < i$; i.e., an occurrence of pattern  ;
- antirecord (or right-to-left minimum) if $\sigma(j) > \sigma(i)$ for all $j > i$; i.e., an occurrence of pattern  ;
- exclusive record if it is a record and not also an antirecord; i.e., an occurrence of pattern  ;
- exclusive antirecord if it is an antirecord and not also a record; i.e., an occurrence of pattern  ;
- record-antirecord if it is both a record and an antirecord; i.e., an occurrence of pattern  ;
- neither-record-antirecord if it is neither a record nor an antirecord ; i.e., an occurrence of pattern  , which is the pattern 321.

Every index i thus belongs to exactly one of the latter four types.

Furthermore, one can apply the record and cycle classifications simultaneously, to obtain 10 disjoint categories of the ***record-and-cycle classification***: exclusive records that are either cycle valleys (ereccval) or cycle double rises (ereccdrise); exclusive antirecords that are either cycle peaks (eareccpeak) or cycle double falls (eareccdfall); record-antirecords (these are always fixed points) (rar); neither-record-antirecords that are either cycle peaks (nrcpeak) or are cycle valleys (nrcval) or cycle double rises (nrcdrise) or cycle double falls (nrcdfall) or fixed points (nrfix).

Using the record-and-cycle classification and the count of cycles the following 11-variable polynomial $\widehat{Q}_n$ [11, Equation (2.29)] can be defined

$$\begin{aligned}\widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,z,w,\lambda) \;=\; & \textstyle\sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times \\ & u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}\end{aligned} \tag{2}$$

The polynomials $\widehat{Q}_n$ have a nice J-fraction:

Theorem 0.1 ([11, Conjecture 2.3], [2, Theorem 3.1])**.** *The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,\mathbf{w},\lambda)\, t^n \;=\;$$

$$\cfrac{1}{1-\lambda w_0 t-\cfrac{\lambda x_1 y_1 t^2}{1-(x_2+y_2+\lambda w_1)t-\cfrac{(\lambda+1)(x_1+u_1)y_1 t^2}{1-(x_2+y_2+u_2+v_2+\lambda w_2)t-\cfrac{(\lambda+2)(x_1+2u_1)y_1 t^2}{1-\cdots}}}} \tag{3}$$

with coefficients

$$\gamma_0 = \lambda w_0 \tag{4a}$$

$$\gamma_n = [x_2+(n-1)u_2] + [y_2+(n-1)v_2] + \lambda w_n \qquad \textit{for } n \geq 1 \tag{4b}$$

$$\beta_n = (\lambda + n - 1)[x_1+(n-1)u_1]y_1 \tag{4c}$$

Overview of proof

We first provide an overview of the Foata–Zeilberger bijection [7], and then briefly mention how we reinterpet it to obtain the count of cycles in a permutation.

Let $\sigma \in \mathfrak{S}_n$ be a permutation on n letters. This permutation σ partitions the set $[n]$ into excedance indices ($F = \{i \in [n] : \sigma(i) > i\}$), anti-excedance indices ($G = \{i \in [n] : \sigma(i) < i\}$), and fixed points ($H$). Similarly, σ also partitions $[n]$ into excedance values ($F' = \{i \in [n] : i > \sigma^{-1}(i)\}$), anti-excedance values ($G' = \{i \in [n] : i < \sigma^{-1}(i)\}$), and fixed points. Clearly, $\sigma \upharpoonright F\colon\ F \to F'$, $\sigma \upharpoonright G\colon\ G \to G'$, and $\sigma \upharpoonright H\colon\ H \to H$ are bijections, and the permutation σ can be obtained from the following data:

- Two partitions of the set $[n] = F \cup G \cup H = F' \cup G' \cup H$.
- The two subwords of σ: $\sigma(x_1)\ldots\sigma(x_m)$ and $\sigma(y_1)\ldots\sigma(y_l)$, where $G = \{x_1 < x_2 < \ldots < x_m\}$ and $F = \{y_1 < y_2 < \ldots < y_l\}$.

In their construction, Foata and Zeilberger [7] use this data to describe a bijection between $\mathfrak{S}_n$ to a set of labelled Motzkin paths of length n. One then uses Flajolet's theorem [5] to obtain continued fractions from this bijection while keeping track of a multitude of simultaneous permutation statistics.

The Foata–Zeilberger bijection consists of the following steps (following [11, Section 6.1]):

- Step 1: A Motzkin path ω is described from σ. The description of ω completely depends on the sets F, F', G, G', H.
- Step 2: The labels ξ associated to ω are obtained from σ. It turns out that the description of the labels depend on $\sigma \upharpoonright F\colon\ F \to F'$, $\sigma \upharpoonright G\colon\ G \to G'$, and the set H, separately.
- Step 3: This step describes the construction of the inverse map $(\omega, \xi) \mapsto \sigma$ and can be further broken down as follows:
  - Step 3(a): The sets F, F', G, G', H are read off from the path ω.
  - Step 3(b): This description is the crucial part of the construction (at least for our purposes). We use the notion of *inversion tables* to construct the words σ: $\sigma(x_1)\ldots\sigma(x_m)$ and $\sigma(y_1)\ldots\sigma(y_l)$, the former is constructed using "right-to-left" inversion table and the latter is constructed using "left-to-right" inversion table.

It is, a priori, unclear how one might be able to track the number of cycles of σ in this construction. We resolve this issue by reinterpreting Step 3(b). We describe a "history" of this construction using *Laguerre digraphs* [6, 10].

A Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1. Clearly, any subgraph of a Laguerre digraph is also a Laguerre digraph. A permutation σ in cycle notation is equivalent to a Laguerre digraph L ([12, pp. 22-23]). The directed edges of L are precisely $u \to \sigma(u)$.

For a subset $S \subseteq [n]$, we let $L|_S$ denote the subgraph of L containing the same set of vertices $[n]$, but only the edges $u \to \sigma(u)$, with $u \in S$ (we are allowed to have $\sigma(u) \notin S$). Let $u_1, \ldots, u_n$ be a rewriting of $[n]$. We consider the "history" $L|_\emptyset \subset L|_{\{u_1\}} \subset L|_{\{u_1,u_2\}} \subset \ldots \subset L|_{\{u_1,\ldots,u_n\}} = L$ as a process of building up the permutation σ by successively considering the status of vertices $u_1, u_2, \ldots, u_n$. Thus, at each step we insert a new edge into the digraph, and at the end of this process, the resulting digraph obtained is the digraph of σ.

The crucial part of our construction is that the rewriting $u_1, \ldots, u_n$ is obtained as follows: we first go through H in increasing order (we call this stage (a)), we then go through G in increasing order (stage (b)), finally we go through F but in decreasing order (stage (c)). This total order is suggested by the inversion tables. On building up the permutation σ using this history, we will see that the cycles can only be formed during stage (c) and we can now count the number of cycles. Our total order on $[n]$ only depends on the sets F, G, H, and hence, only on the path ω and not on the labels ξ which is important for our proof to work.

Twist in the story and final remarks.

The continued fractions for permutations in [11] were classified as "second" or "first" depending on whether or not they involved the count of cycles. The proofs of the first and second continued fractions involved two different bijections: the first continued fractions used a variant of the Foata–Zeilberger bijections, whereas the second continued fractions used the Biane bijection [1]. However, our proof for the conjectured "second" continued fraction proceeds by employing the "first" bijection but then reinterpreting it differently. This was a surprise to us.

We can adapt our proof technique to also resolve [9, Conjecture 12] from 1996, and [3, Conjecture 4.1]; both of these are continued fractions generalising the Genocchi and median Genocchi numbers, respectively. More details can be found in [2].

References

[1] P. Biane, Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine, European J. Combin. 14 (1993), 277–284.

[2] B. Deb, Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants, preprint (April 2023), arxiv: 2304.14487.

[3] B. Deb and A.D. Sokal, Classical continued fractions for some multivariate polynomials generalizing the Genocchi and median Genocchi numbers, preprint (December 2022), arXiv:2212.07232.

[4] L. Euler, De seriebus divergentibus, Novi Commentarii Academiae Scientiarum Petropolitanae 5 (1760), 205–237; reprinted in *Opera Omnia*, ser. 1, vol. 14, pp. 585–617. [Latin original and

English and German translations available at http://eulerarchive.maa.org/pages/E247.html]

[5] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980), 125–161.

[6] D. Foata and V. Strehl, Combinatorics of Laguerre polynomials, in *Enumeration and Design*, edited by D.M. Jackson and S.A. Vanstone (Academic Press, Toronto, 1984), pp. 123–140.

[7] D. Foata and D. Zeilberger, Denert's permutation statistic is indeed Euler-Mahonian, Stud. Appl. Math. 83 (1990), 31–59.

[8] B. Han and J. Zeng, Equidistributions of mesh patterns of length two and Kitaev and Zhang's conjectures, Adv. Appl. Math. 127 (2021), 102149.

[9] A. Randrianarivony and J. Zeng, Some equidistributed statistics on Genocchi permutations, Electron. J. Combin. 3:2 (1996), Research Paper #22.

[10] A.D. Sokal, Multiple Laguerre polynomials: Combinatorial model and Stieltjes moment representation, Amer. Math. Soc. 150 (2022), 1997–2005.

[11] A.D. Sokal and J. Zeng, Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, Adv. Appl. Math. 138 (2022), 102341.

[12] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Second Edition, Cambridge University Press, 2012.

[13] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Notes de conférences données à l'Université du Québec à Montréal, septembre-octobre 1983. Available on-line at http://www.xavierviennot.org/xavier/polynomes_orthogonaux.html