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(Augmented) Chow rings of (q-)uniform matroids

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Abstract: Chow rings and augmented Chow rings of matroids play important roles in the celebrated proofs of two long-standing conjectures: (1) the Adiprasito-Huh-Katz proof of the Heron-Rota-Welsh Conjecture and (2) the Braden-Huh-Matherne-Proudfoot-Wang proof of the Dowling-Wilson Top-Heavy Conjecture. These two matroid invariants have since been extensively studied. In 2021, Hameister, Rao, and Simpson gave a nice combinatorial interpretation of the Hilbert series of the Chow ring of the (q-)uniform matroid in terms of permutations and the q-Eulerian polynomials studied by Shareshian and Wachs. We present an analogous interpretation for the augmented Chow ring in terms of decorated permutations and q-binomial Eulerian polynomials.

Our proof relies on a Feichtner-Yuzvinsky type basis for the augmented Chow ring of a matroid (introduced in our previous work and in independent work of Eur, Huh and Larson). This basis is also used to obtain closed form formulas for the Hilbert series of the augmented Chow ring of the uniform matroid evaluated at -1. These are analogous to our simplification of formulas of Hameister, Rao, and Simpson for the Chow ring. We also obtain symmetric function analogs of the above results.

1. Introduction

The notion of the *Chow ring of an atomic lattice* was introduced by Feichtner and Yuzvinsky in [4]. Adiprasito, Huh, and Katz applied Feitchner and Yuzvnisky's construction to the lattice of flats of a matroid and call it the *Chow ring of a matroid.* They studied the Chow ring of a matroid in their seminal work [1] in order to resolve the long-standing Heron-Rota-Welsh conjecture. Later in the subsequent work of solving the Dowling-Wilson Top-Heavy conjecture, Braden, Huh, Matherne, Proudfoot, and Wang [2] introduced the *augmented Chow rings of a matroid.* In both works, they showed that the Chow ring and the augmented Chow ring of a matroid satisfy Poincaré duality, Hard Lefschetz theorem, and the Hodge-Riemann relations respectively.

Since the settlement of the Heron-Rota-Welsh Conjecture and the Dowling-Wilson Top Heavy Conjecture, there have been many new developments in different aspects of the study of the Chow ring of a matroid. Among these works, Hameister, Rao, and Simpson [5] compute the Hilbert series of the Chow rings of uniform matroids and their q-analogues and obtain some interesting combinatorial expression (see Section 2.3). In this extended abstract, we study the parallel story of the augmented Chow ring of a q-uniform matroid. We also find some of their results can be simplified. Furthermore, we study the representations of $\mathfrak{S}_n$ the (augmented) Chow rings of the uniform matroids carry and obtain the symmetric function analogs of the expressions.

2. Background

2.1. **Permutations and the Eulerian story.** Denote by $\mathfrak{S}_n$ the set of permutations on $[n] := \{1, 2, \ldots, n\}$. The *Eulerian polynomial* can be defined as the generating function of the distribution of the excedence number over $\mathfrak{S}_n$, that is $A_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{exc}(\sigma)}$.

Motivated by their study on Rees product of posets, Shareshian and Wachs [**?**] introduced a set statistic called DEX and define the *Eulerian quasisymmetric function*

$$Q_n(\mathbf{x}, t) := \sum_{\sigma\in\mathfrak{S}_n} F_{n,\mathsf{DEX}(\sigma)}(\mathbf{x})t^{\mathsf{exc}(\sigma)},$$

where $F_{n,S}(\mathbf{x})$ for $S \subset [n-1]$ is Gessel's *fundamental quasisymmetric function*. It turns out that $Q_n(\mathbf{x}, t)$ is a polynomial in t with symmetric function coefficients and has a connection with the permutohedron, see [6] for detail. For all $n \geq 0$ the stable principal specialization of $Q_n(\mathbf{x}, t)$ gives the following q-Eulerian polynomial

$$\prod_{i=1}^{n} \left(1-q^i\right) \mathrm{ps}_q(Q_n(\mathbf{x}, t)) = A_n(q, t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)}t^{\mathsf{exc}(\sigma)}.$$

2.2. **Decorated Permutations and the binomial Eulerian story.** The *binomial Eulerian polynomial* $\widetilde{A}_n(t) := 1 + t\sum_{k=1}^{n}\binom{n}{k}A_k(t)$ shares many similar combinatorial and geometric properties with the Eulerian polynomial. Shareshian and Wachs [9] introduced the symmetric function analogue of $\widetilde{A}_n(t)$,

$$\widetilde{Q}_n(\mathbf{x}, t) := h_n(\mathbf{x}) + t\sum_{k=1}^{n} h_{n-k}(\mathbf{x})Q_k(\mathbf{x}, t) \tag{1}$$

and show that its stable principal specialization gives the following q-binomial Eulerian polynomial

$$\prod_{i=1}^{n}(1-q^i)\mathrm{ps}_q(\widetilde{Q}_n(\mathbf{x}, t)) = \widetilde{A}_n(q, t) :- 1 + t\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k(t).$$

As an analog of the set of permutations $\mathfrak{S}_n$ in the Eulerian story, we consider the set $\widetilde{\mathfrak{S}}_n$ of *decorated permutations* of $[n]$. A *decorated permutation* of $[n]$ is a permutation on a subset of $[n]$.

Example 2.1. For example, $215 \in \widetilde{\mathfrak{S}_5}$ is a permutation on $\{1, 2, 5\} \subset [5]$ that maps 1 to 2, 2 to 1, 5 to 5. We can express it in *two-line notation* and in *one-line notation* respectively as the following

$$215 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 0 & 0 & 5 \end{pmatrix} = 21005.$$

In particular, denote θ the permutation on $\emptyset$ in $\widetilde{\mathfrak{S}_5}$. It has the following two-line notation and one-line notation

$$\theta := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 00000.$$

Remark 2.2. Our definition of decorated permutations is different from Postinkov's in [7] where he introduced them as permutations with two kinds of fixed points. An easy bijection is given by replacing the 0s in our one-line notation with the second kind of fixed points.

Liao [6] showed that DEX can be extended to $\widetilde{\mathfrak{S}}_n$ and

$$\widetilde{Q}_n(\mathbf{x},t) = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} F_{n,\mathsf{DEX}(\sigma)}(\mathbf{x})t^{\mathsf{exc}(\sigma)+1}.$$

Similarly, the q-binomial Eulerian polynomial has the combinatorial expression

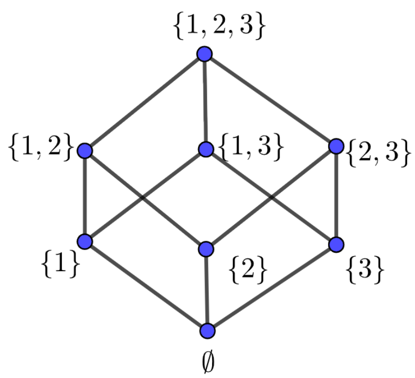
$$\widetilde{A}_n(q,t) = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)}t^{\mathsf{exc}(\sigma)+1}.$$

2.3. **Chow ring of a matroid.** Let M be a loopless matroid on the ground set $[n]$ with the collection of flats $\mathcal{F}(M)\subset 2^{[n]}$. The collection $\mathcal{F}(M)$ with the inclusion relation $\subset$ forms a finite geometric lattice $\mathcal{L}(M)=(\mathcal{F}(M),\subset)$ which we call the *lattice of flats*.

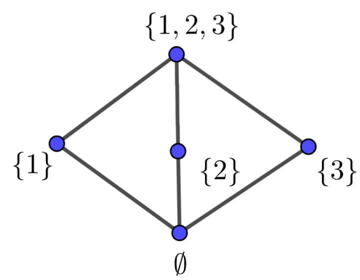
In this extended abstract, our main character is the *uniform matroid*. For $1\le k\le n$, the uniform matroid $U_{n,k}$ is a matroid on $[n]$ with the collection of flats

$$\mathcal{F}(U_{n,k}) = \{S\subset[n] : |S|\le k-1\}\cup\{[n]\}$$

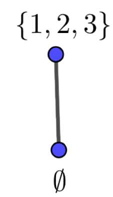
such that $\mathrm{rk}(S)=|S|$ if $|S|\le k-1$ and $\mathrm{rk}([n])=k$. The lattice of flats of $U_{n,k}$ is obtained from the Boolean lattice by removing all subsets of cardinality at least k and then joining $[n]$ as the maximal flat. In particular, the matroid $U_{n,n}$ has the Boolean lattice as its lattice of flats, hence $U_{n,n}$ is also called the *Boolean matroid* B_n. See the following diagrams of $n=3$ for examples.



(A) Boolean lattice $\mathcal{L}(U_{3,3})$



(B) $\mathcal{L}(U_{3,2})$



(C) $\mathcal{L}(U_{3,1})$

On the other hand, let $\mathbb{F}_q$ be the finite field of order q where q is some prime power. The *q-uniform matroid* $U_{n,k}(q)$ is a matroid on the vector space $\mathbb{F}_q^n$ whose lattice of flats is given by the collection of subspaces of $\mathbb{F}_q^n$ of dimension at most $k-1$ together with the maximal subspace $\mathbb{F}_q^n$. In particular, the lattice of flats of $U_{n,n}(q)$ is the lattice of all subspaces of $\mathbb{F}_q^n$, also known as the q-Boolean lattice $B_n(q)$.

The Chow ring $A(M)$ of M is a quotient of a polynomial ring that encoded the information of $\mathcal{L}(M)$ and is defined as follows

$$A(M) := \frac{\mathbb{Q}[x_F]_{F\in\mathcal{L}(M)-\{\emptyset\}}/\left\langle x_Fx_G : F,G \text{ are incomparable in } \mathcal{L}(M)\right\rangle}{\left\langle \sum_{F:i\in F} x_F : 1\le i\le n\right\rangle}.$$

In the setting of the Chow ring of an atomic lattice, Feichtner and Yuzvnisky [4] found a Gröbner basis for the ideal being mod out in the definition of such a Chow ring. Applying their discovery to $A(M)$ gives the following $\mathbb{Q}$-basis $FY(M)$ for $A(M)$

$$FY(M) := \left\{ x_{F_1}^{a_1}x_{F_2}^{a_2}\dots x_{F_k}^{a_k} : \substack{\emptyset=F_0\subsetneq F_1\subsetneq F_2\subsetneq\dots\subsetneq F_k\subseteq[n],\\ 1\le a_i\le \mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1} \right\}. \tag{2}$$

Using the FY-basis, Hameister, Rao, Simpson [5] showed that the Hilbert series of the Chow ring of the q-uniform matroid $U_{n,r}(q)$ for $1 \le r \le n$ can be expressed combinatorially as

$$\mathsf{Hilb}(A(U_{n,r}(q)),t) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)} - \sum_{j=r}^{n-1} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{fix}(\sigma)\ge n-j}} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{j-\mathsf{exc}(\sigma)}. \tag{3}$$

In particular, when $r = n$ the matroid $U_{n,n}(q)$ are exactly q-Boolean matroid $B_n(q)$. They also found that for the special case $r = n$ and $r = n-1$, the Hilbert series are

$$\mathsf{Hilb}(A(U_{n,n}(q),t) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)} = A_n(q,t),$$
$$\mathsf{Hilb}(A(U_{n,n-1}(q)),t) = \sum_{\sigma\in D_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)-1}$$

where D_n is the set of derangements in $\mathfrak{S}_n$. On the other hand, they also show

$$\mathsf{Hilb}(A(U_{n,r}(q)),-1) = \begin{cases} 0 & \text{, if } r \text{ is even;} \\ \displaystyle\sum_{k=0}^{\frac{r-1}{2}} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix}_q E_{2k}(q) & \text{, if } r \text{ is odd.} \end{cases} \tag{4}$$

where $E_{2k}(q) := \sum_{\sigma\in\mathsf{RAlt}_n} q^{\mathsf{inv}(\sigma)}$ and $\mathsf{RAlt}_n \subset \mathfrak{S}_n$ is the subset of reverse alternating permutations. This is related to the *Charney-Davis quantity* of $A(U_{n,r}(q))$. A different expression of the sum in the case of odd r is also given:

$$1 + [n]_q! \sum_{k=0}^{\frac{r-1}{2}} \frac{(-1)^k}{[n-2k]_q!} \begin{vmatrix} \frac{1}{[2]_q!} & 1 & 0 & \cdots & 0 \\ \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[2k-2]_q!} & \frac{1}{[2k-4]_q!} & \frac{1}{[2k-6]_q!} & \cdots & 1 \\ \frac{1}{[2k]_q!} & \frac{1}{[2k-2]_q!} & \frac{1}{[2k-4]_q!} & \cdots & \frac{1}{[2]_q!} \end{vmatrix}. \tag{5}$$

2.4. **Augmented Chow ring of a matroid.** The *augmented Chow ring of a matroid* can be regarded as an extension of the Chow ring but contains information both from the lattice of flats and the collection of independent subsets. Let M be a matroid on $[n]$ with the lattice of flats $\mathcal{L}(M)$ and the collection of independent subsets $\mathcal{I}(M)$. The *augmented Chow ring of* M is defined to be

$$\widetilde{A}(M) := \frac{\mathbb{Q}\left[\{x_F\}_{F\in\mathcal{L}(M)\setminus[n]} \cup \{y_1,y_2,\ldots,y_n\}\right]/(I_1+I_2)}{\langle y_i - \sum_{F:i\notin F} x_F\rangle_{i=1,2,\ldots,n}}$$

where $I_1 = \langle x_F x_G : F, G \text{ are incomparable in } \mathcal{L}(M)\rangle$, $I_2 = \langle y_i x_F : i \notin F\rangle$. It is shown in the previous work of the author [6] and independently by Eur, Huh, and Larson [3] that the augmented Chow ring $\widetilde{A}(M)$ can be regarded as the Chow ring of the lattice of flats of the free coextension of M with specific building set. Hence they found the following Feichtner-Yuzvinisky-like basis for $\widetilde{A}(M)$.

Lemma 2.3 ([6],[3])**.** *Given a matroid M, the augmented Chow ring $\widetilde{A}(M)$ has the following basis*

$$\widetilde{FY}(M) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_k}^{a_k} : {}^{\emptyset=F_0\subsetneq F_1\subsetneq F_2\subsetneq\ldots\subsetneq F_k}_{1\le a_1\le \mathrm{rk}_M(F_1),\ a_i\le \mathrm{rk}_M(F_i)-\mathrm{rk}_M(F_{i-1})-1 \text{ for } i\ge 2} \right\}.$$

Then it is easy to get a formula for the Hilbert series of $\widetilde{A}(M)$.

Corollary 2.4. *The Hilbert series of* $\widetilde{A}(M)$ *is*

$$\mathsf{Hilb}(\widetilde{A}(M),t) = 1 + \sum_{\emptyset \neq F_1 \subset \ldots \subset F_\ell} \frac{t(1-t^{\mathrm{rk}(F_1)})}{1-t} \prod_{i=2}^{\ell} \frac{t(1-t^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})})}{1-t}.$$

3. Hilbert series: q-uniform matroids

In this section, we introduce our main results for q-uniform matroids. We use the FY-basis in Lemma 2.3 and a similar recurrence argument from Hameister, Rao, and Simpson [5] to obtain the Hilbert series expression of $\widetilde{A}(U_{n,r}(q),t)$, but we have two expressions in different flavors.

Theorem 3.1. *The Hilbert series* $\mathsf{Hilb}(\widetilde{A}(U_{n,r}(q)),t)$ *has the following two expressions*

(i) $1 + t\sum_{j=0}^{r-1} \begin{bmatrix} n \\ j \end{bmatrix}_q A_j(q,t)(1+t+\ldots+t^{r-j-1})$,

(ii) $\sum_{\sigma\in\widetilde{\mathfrak{S}}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)+1} - \sum_{j=r}^{n-1} \sum_{\substack{\sigma\in\widetilde{\mathfrak{S}}_n \\ \mathsf{fix}_2(\sigma)\geq n-j}} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{j-\mathsf{exc}(\sigma)}$,

where $\mathsf{fix}_2(\sigma)$ *is the number of* 0*s in the one-line notation of* σ*.*

Corollary 3.2. *For the special case that* $r=n$ *and* $r=n-1$ *we have the following*

(i) $\mathsf{Hilb}(\widetilde{A}(U_{n,n}(q)),t) = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)+1}$;

(ii) $\mathsf{Hilb}(\widetilde{A}(U_{n,n-1}(q)),t) = \sum_{\sigma\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)}$.

We also found an expression different from (3) for the Chow ring of the q-uniform matroid.

Theorem 3.3. *Let* $d_j(q,t) = \sum_{\sigma\in D_n} q^{\mathsf{maj}(\sigma)-\mathsf{exc}(\sigma)} t^{\mathsf{exc}(\sigma)}$*, then*

$$\mathsf{Hilb}(A(U_{n,r}(q)),t) = \sum_{j=0}^{r-1} \begin{bmatrix} n \\ j \end{bmatrix}_q d_j(q,t)(1+t+\ldots+t^{r-j-1}).$$

Using Corollary 2.4, we evaluating $\mathsf{Hilb}(\widetilde{A}(U_{n,r}(q)),t)$ at $t=-1$ which gives us the following.

Theorem 3.4. *For any positive integer* n *and* $1 \leq r \leq n$*,*

$$\mathsf{Hilb}(\widetilde{A}(U_{n,r}(q)),-1) = \begin{cases} (-1)^{\frac{r}{2}} \displaystyle\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{DES}(\sigma)=\{1,3,5,\ldots,r-1\}}} q^{\mathsf{inv}(\sigma)}, & \text{if } r \text{ is even;} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

When r *is even the evaluation can also be expressed as a determinant*

$$(-1)^{\frac{r}{2}}[n]_q! \begin{vmatrix} \frac{1}{[1]_q!} & 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{[3]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 & 0 \\ \frac{1}{[5]_q!} & \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{[r-1]_q!} & \frac{1}{[r-2]_q!} & \frac{1}{[r-4]_q!} & \cdots & \frac{1}{[2]_q!} & 1 \\ \frac{1}{[n]_q!} & \frac{1}{[n-1]_q!} & \frac{1}{[n-3]_q!} & \cdots & \frac{1}{[n-r+3]_q!} & \frac{1}{[n-r+1]_q!} \end{vmatrix}.$$

Furthermore, we found (4) and (5) can be simplified in a similar way.

Theorem 3.5. *For any positive integer n and $1 \le r \le n$,*

$$\mathsf{Hilb}(A(U_{n,r}(q)),-1) = \begin{cases} 0 & \text{, if } r \text{ is even;} \\ (-1)^{\frac{r-1}{2}} \displaystyle\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{DES}(\sigma)=\{2,4,6,\ldots,r-1\}}} q^{\mathsf{inv}(\sigma)} & \text{, if } r \text{ is odd} \end{cases}.$$

When r is odd the evaluation can also be expressed as a determinant

$$(-1)^{\frac{r-1}{2}}[n]_q! \begin{vmatrix} \frac{1}{[2]_q!} & 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 & 0 \\ \frac{1}{[6]_q!} & \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{[r-1]_q!} & \frac{1}{[r-3]_q!} & \frac{1}{[r-5]_q!} & \cdots & \frac{1}{[2]_q!} & 1 \\ \frac{1}{[n]_q!} & \frac{1}{[n-2]_q!} & \frac{1}{[n-4]_q!} & \cdots & \frac{1}{[n-r+3]_q!} & \frac{1}{[n-r+1]_q!} \end{vmatrix}.$$

4. Frobenius series: uniform matroids

In this section, we consider the uniform matroids $U_{n,r}$ on $[n]$. Note that both FY-bases $FY(U_{n,r})$ in (2) and $\widetilde{FY}(U_{n,r})$ in Lemma 2.3 are $\mathfrak{S}_n$-invariant, hence the $\mathfrak{S}_n$-action induce a representation of $\mathfrak{S}_n$ on both $A(U_{n,r})$ and $\widetilde{A}(U_{n,r})$. We compute the graded Frobenius series of these representations of $\mathfrak{S}_n$ and find that the results give $\mathfrak{S}_n$-equivariant analogs of the expression in Section 3.

Let $Q_n^0(\mathbf{x},t) := \sum_{\sigma\in D_n} F_{n,\mathsf{DEX}(\sigma)}(\mathbf{x})t^{\mathsf{exc}(\sigma)}$. Then the following table illustrates the relationship between the polynomials, their extensions, and the corresponding combinatorial models they represent.

	polynomial		$q-$analogue		symmetric function
D_n	$d_n(t)$		$d_n(q,t)$		$Q_n^0(\mathbf{x},t)$
$\cap$					
$\mathfrak{S}_n$	$A_n(t)$	$\xleftarrow{q\to 1}$	$A_n(q,t)$	$\xleftarrow{\mathrm{ps}_q}$	$Q_n(\mathbf{x},t)$
$\cap$					
$\widetilde{\mathfrak{S}}_n$	$\widetilde{A}_n(t)$		$\widetilde{A}_n(q,t)$		$\widetilde{Q}_n(\mathbf{x},t)$

Our main results in this section are the following.

Theorem 4.1. *The graded Frobenius series* $\mathsf{grFrob}(A(U_{n,r}),t)$ *has the following two expressions*

(i) $\displaystyle\sum_{j=0}^{r-1} h_{n-j}Q_j^0(\mathbf{x},t)(1+t+\ldots+t^{r-j-1})$,

(ii) $\displaystyle Q_n(\mathbf{x},t) - \sum_{j=r}^{n-1} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{fix}(\sigma)\ge n-j}} F_{n,\mathsf{DEX}(\sigma)}t^{j-\mathsf{exc}(\sigma)}$.

Theorem 4.2. *The graded Frobenius series* $\mathsf{grFrob}(\widetilde{A}(U_{n,r}),t)$ *has the following two expressions*

(i) $\displaystyle h_n + t\sum_{j=0}^{r-1} h_{n-j}Q_j(\mathbf{x},t)(1+t+\ldots+t^{r-j-1})$,

(ii) $\widetilde{Q}_n(\mathbf{x},t) - \sum_{j=r}^{n-1} \sum_{\substack{\sigma\in\widetilde{\mathfrak{S}}_n \\ \mathsf{fix}_2(\sigma)\geq n-j}} F_{n,\mathsf{DEX}(\sigma)} t^{j-\mathsf{exc}(\sigma)}$.

An interesting observation is that the stable principal specialization of the symmetric functions in this section will give the (q,t)-polynomial in Section 3. Hence we actually have two different ways of getting q-analogue information from uniform matroids on $[n]$. One is considering the q-uniform matroids on $\mathbb{F}_q^n$ directly, and the other one is considering the representation of $\mathfrak{S}_n$ on the uniform matroids and taking the specialization of them. The two different paths somehow give the same q-analogues.

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