

Generating Functions for Monomial Characters of Wreath Products $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$

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ABSTRACT: Let $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ denote the wreath product of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ with the symmetric group \mathfrak{S}_n . We define generating functions for monomial (induced one-dimensional) characters of $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ and express these in terms of determinants and permanents. This extends work of Littlewood (*The Theory of Group Characters and Representations of Groups*, 1940) and Merris and Watkins (*Linear Algebra Appl.*, **64**, 1985) on generating functions for the monomial characters of \mathfrak{S}_n .

Keywords: Character; Generating function; Wreath product

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1. Introduction

Let $z = (z_{i,j})$ be an $n \times n$ matrix of indeterminates and let \mathfrak{S}_n be the symmetric group. For each linear functional $\theta : \mathbb{C}[\mathfrak{S}_n] \rightarrow \mathbb{C}$, define the generating function

$$\text{Imm}_\theta(z) := \sum_{w \in \mathfrak{S}_n} \theta(w) z_{1,w_1} \cdots z_{n,w_n} \in \mathbb{C}[z] \tag{1}$$

for θ , and call this the θ -*immanant*. Such functions appeared originally in [7, p. 81] for θ equal to irreducible \mathfrak{S}_n -characters χ^λ , and were extended in [14, §3] to general θ . As is the case with many functions, a simple formula for a generating function for θ can be as useful as a simple formula for the numbers $\{\theta(w) \mid w \in \mathfrak{S}_n\}$ themselves.

Particularly simple generating functions for the *monomial* (induced one-dimensional) characters of \mathfrak{S}_n are expressed in terms of integer partitions, ordered set partitions, and submatrices of z . Call a nonnegative integer sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ satisfying $\lambda_1 + \dots + \lambda_r = n$ a *weak composition of n* and write $|\lambda| = n$, $\ell(\lambda) = r$. If the components of λ are weakly decreasing and positive, call it an (*integer*) *partition of n* and write $\lambda \vdash n$. For any weak composition λ of n , call a sequence (I_1, \dots, I_r) of pairwise disjoint subsets of $[n] := \{1, \dots, n\}$ an *ordered set partition of $[n]$ of type λ* if $|I_j| = \lambda_j$ for $j = 1, \dots, r$. (Note that this nonstandard terminology allows empty sets in ordered set partitions, whereas standard terminology [13, pp. 39, 73] does not.) Given subsets I, J of $[n]$, define the (I, J) -*submatrix* of z to be

$$z_{I,J} = (z_{i,j})_{i \in I, j \in J}. \tag{2}$$

The class function space of \mathfrak{S}_n has two standard bases consisting of monomial characters: the *induced trivial character* basis $\{\eta^\lambda = \text{triv} \uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mid \lambda \vdash n\}$ and the *induced sign character* basis $\{\epsilon^\lambda = \text{sgn} \uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mid \lambda \vdash n\}$, where \mathfrak{S}_λ is the Young subgroup of \mathfrak{S}_n indexed by λ . (See, e.g., [9].) Littlewood [7, §6.5] and Merris and Watkins [8] came close to expressing the η^λ - and ϵ^λ -immanants as

$$\begin{aligned}
 \text{Imm}_{\epsilon^\lambda}(z) &= \sum_{(J_1, \dots, J_\ell)} \det(z_{J_1, J_1}) \cdots \det(z_{J_\ell, J_\ell}), \\
 \text{Imm}_{\eta^\lambda}(z) &= \sum_{(J_1, \dots, J_\ell)} \text{per}(z_{J_1, J_1}) \cdots \text{per}(z_{J_\ell, J_\ell}),
 \end{aligned} \tag{3}$$

where the sums are over all ordered set partitions (J_1, \dots, J_ℓ) of $[n]$ of type $\lambda = (\lambda_1, \dots, \lambda_\ell)$. For example, we have

$$\begin{aligned} \text{Imm}_{\epsilon^{21}}(z) &= \det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} z_{3,3} + \det \begin{bmatrix} z_{1,1} & z_{1,3} \\ z_{3,1} & z_{3,3} \end{bmatrix} z_{2,2} + \det \begin{bmatrix} z_{2,2} & z_{2,3} \\ z_{3,2} & z_{3,3} \end{bmatrix} z_{1,1} \\ &= 3z_{1,1}z_{2,2}z_{3,3} - z_{1,2}z_{2,1}z_{3,3} - z_{1,3}z_{2,2}z_{3,1} - z_{1,1}z_{2,3}z_{3,2}, \end{aligned}$$

and $\epsilon^{21}(123) = 3$, $\epsilon^{21}(213) = \epsilon^{21}(321) = \epsilon^{21}(132) = -1$, $\epsilon^{21}(312) = \epsilon^{21}(231) = 0$. While Littlewood, Merris, and Watkins may not have written the equations (3) explicitly, we call them the *Littlewood–Merris–Watkins identities*. These identities have played an important role in the evaluation of (type-*A*) Hecke algebra characters at Kazhdan–Lusztig basis elements [3], [4], [5], the formulation of a generating function for irreducible Hecke algebra characters [6], and the interpretation of coefficients of chromatic symmetric functions [3], [10]. The identity in our main result (Theorem 3.1) plays an important role in the evaluation of hyperoctahedral group characters at elements of the type-*BC* Kazhdan-Lusztig basis [11].

Let $\mathcal{G} = \mathcal{G}(n)$ be the wreath product $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$. (We will suppress d and sometimes n from the notation.) Its class function space has 2^d standard bases consisting of monomial characters, and it is possible to use a matrix of dn^2 indeterminates to construct generating functions analogous to (3) for the elements of these bases. In Section 2 we review \mathcal{G} and its monomial characters; in Section 3 we present our generating functions for these.

2. \mathcal{G} and its monomial characters

The group \mathcal{G} is generated by n elements s_1, \dots, s_{n-1}, t subject to the relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \dots, n-1, \\ t^d &= e, \\ ts_1ts_1 &= s_1ts_1t, \\ s_is_j &= s_js_i && \text{for } |i-j| \geq 2, \\ ts_j &= s_jt && \text{for } j \geq 2, \\ s_is_js_i &= s_js_is_j && \text{for } |i-j| = 1. \end{aligned}$$

A one-line notation for elements of \mathcal{G} , analogous to that for elements of \mathfrak{S}_n , uses sequences of integer multiples of complex d th roots of unity. Let ζ be a primitive d th root of unity, and let S be the set of sequences

$$\{(\zeta^{\gamma_1}w_1, \dots, \zeta^{\gamma_n}w_n) \mid w_1 \cdots w_n \in \mathfrak{S}_n, (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n\}. \tag{4}$$

We define an action of \mathcal{G} on S by letting the generators act on a sequence (a_1, \dots, a_n) as follows.

1. $s_i \circ (a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$,
2. $t \circ (a_1, \dots, a_n) = (\zeta a_1, a_2, \dots, a_n)$.

Letting each element $g \in \mathcal{G}$ act on the sequence $(1, \dots, n)$, we obtain a bijection between \mathcal{G} and S . If $g \circ (1, \dots, n) = (\zeta^{\gamma_1}w_1, \dots, \zeta^{\gamma_n}w_n)$, we define this second sequence to be the *one-line notation* of g , and we write $g = (\gamma, w)$, where $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n$, $w \in \mathfrak{S}_n$. In particular, the identity element e has one-line notation $1 \cdots n$.

Since \mathcal{G} is a finite group, Brauer’s Induced Character Theorem implies that the set of monomial characters of \mathcal{G} spans the *trace space* $\mathcal{T}(\mathcal{G})$ of \mathcal{G} , the set of all linear functionals $\theta : \mathbb{C}[\mathcal{G}] \rightarrow \mathbb{C}$ satisfying $\theta(gh) = \theta(hg)$ for all $g, h \in \mathcal{G}$. (See, e.g., [12].) This includes all \mathcal{G} -characters. $\mathcal{T}(\mathcal{G})$ has dimension equal to the number of conjugacy classes of \mathcal{G} , equivalently, to the number of sequences $\lambda = (\lambda^0, \dots, \lambda^{d-1})$ of d (possibly empty) integer partitions, with

$$|\lambda^0| + \cdots + |\lambda^{d-1}| = n.$$

We call such a sequence a *d-partition* of $[n]$ and write $\lambda \vdash n$.

In order to describe natural bases of $\mathcal{T}(\mathcal{G})$, we introduce certain subgroups of \mathcal{G} which are analogous to Young subgroups of \mathfrak{S}_n . Fix d -partition $\lambda = (\lambda^0, \dots, \lambda^{d-1}) \vdash n$, and define $r_k = \ell(\lambda^k)$ for $k = 0, \dots, d-1$. We will say that an ordered set partition of $[n]$ of type

$$(\lambda_1^0, \dots, \lambda_{r_0}^0, \lambda_1^1, \dots, \lambda_{r_1}^1, \dots, \lambda_1^{d-1}, \dots, \lambda_{r_{d-1}}^{d-1}),$$

has *type* λ . In particular, let $\mathbf{K}(\lambda) = (K_1^0, \dots, K_{r_0}^0, K_1^1, \dots, K_{r_1}^1, \dots, K_1^{d-1}, \dots, K_{r_{d-1}}^{d-1})$ be the ordered set partition of $[n]$ of type λ whose blocks are the $r_0 + \dots + r_{d-1}$ subintervals

$$K_1^0 = [1, \lambda_1^0], \quad K_2^0 = [\lambda_1^0 + 1, \lambda_1^0 + \lambda_2^0], \quad \dots, \quad K_{r_{d-1}}^{d-1} = [n - \lambda_{r_{d-1}}^{d-1} + 1, n] \tag{5}$$

of $[n]$. For $1 \leq i \leq j \leq n$, define the element $t_i = s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} \in \mathcal{G}$, and let

$$\mathcal{G}([i, j]) \cong \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_{j-i+1}$$

be the subgroup of \mathcal{G} generated by $\{t_i, s_i, \dots, s_{j-1}\}$. For $k = 0, \dots, d-1$, use (5) to define the subgroup

$$\mathcal{G}(\lambda, k) := \mathcal{G}(K_1^k) \cdots \mathcal{G}(K_{r_k}^k) \cong \mathcal{G}(\lambda_1^k) \times \cdots \times \mathcal{G}(\lambda_{r_k}^k),$$

of \mathcal{G} , and finally define the *Young subgroup*

$$\mathcal{G}(\lambda) := \mathcal{G}(\lambda, 0) \cdots \mathcal{G}(\lambda, d-1) \cong \prod_{k=0}^{d-1} (\mathcal{G}(\lambda_1^k) \times \cdots \times \mathcal{G}(\lambda_{r_k}^k))$$

of \mathcal{G} . Each element $y \in \mathcal{G}(\lambda)$ factors uniquely as $y_0 \cdots y_{d-1}$ with $y_k \in \mathcal{G}(\lambda, k)$.

Several natural representations of \mathcal{G} are defined by using symmetric group representations and induction from $\mathcal{G}(\lambda)$. First, observe that the subgroup of \mathcal{G} generated by s_1, \dots, s_{n-1} is isomorphic to \mathfrak{S}_n , and that each r -dimensional \mathfrak{S}_n -representation ρ can trivially be extended to an r -dimensional \mathcal{G} -representation in at least d ways: by defining $\rho(t) = \zeta^k I$ for $k = 0, \dots, d-1$. If the character of the \mathfrak{S}_n -representation is χ , call its extension $\delta_k \chi$. Thus the two one-dimensional \mathfrak{S}_n -representations

$$\begin{aligned} 1 : s_i &\mapsto 1 & (w &\mapsto 1 \text{ for all } w \in \mathfrak{S}_n), \\ \epsilon : s_i &\mapsto -1 & (w &\mapsto (-1)^{\text{inv}(w)} \text{ for all } w \in \mathfrak{S}_n) \end{aligned}$$

yield $2d$ one-dimensional \mathcal{G} -representations:

$$\begin{aligned} \delta_k : (s_i, t) &\mapsto (1, \zeta^k), & (g = (\gamma, w) &\mapsto \zeta^{k(\gamma_1 + \dots + \gamma_n)} \text{ for all } g \in \mathcal{G}), \\ \delta_k \epsilon : (s_i, t) &\mapsto (-1, \zeta^k), & (g = (\gamma, w) &\mapsto (-1)^{\text{inv}(w)} \zeta^{k(\gamma_1 + \dots + \gamma_n)} \text{ for all } g \in \mathcal{G}), \end{aligned}$$

for $k = 0, \dots, d-1$. Here, $\text{inv}(w)$ denotes the Coxeter length of w . (See, e.g., [2, p.15].) Next, observe that for any d -tuple (H_0, \dots, H_{d-1}) of subgroups of a group G which satisfy

$$H := H_0 \cdots H_{d-1} \cong H_0 \times \cdots \times H_{d-1}, \tag{6}$$

and any d -tuple $(\theta_0, \dots, \theta_{d-1})$ of characters of these, we have that the function $\theta = \theta_0 \otimes \cdots \otimes \theta_{d-1}$ defined by $\theta(h_0 \cdots h_{d-1}) = \theta_0(h_0) \cdots \theta_{d-1}(h_{d-1})$ is a character of H , and $\theta \uparrow_H^G$ is a character of G . In particular, the Young subgroup $\mathcal{G}(\lambda)$ has the form (6) with $H_k = \mathcal{G}(\lambda, k)$. For every d -tuple $\beta = (\beta_0, \dots, \beta_{d-1}) \in \{1, \epsilon\}^d$ of one-dimensional symmetric group characters we have the one-dimensional $\mathcal{G}(\lambda)$ -character

$$\delta_0 \beta_0 \otimes \cdots \otimes \delta_{d-1} \beta_{d-1}, \tag{7}$$

the corresponding monomial \mathcal{G} -character

$$\beta^\lambda := (\delta_0 \beta_0 \otimes \cdots \otimes \delta_{d-1} \beta_{d-1}) \uparrow_{\mathcal{G}(\lambda)}^{\mathcal{G}},$$

and the basis $\{\beta^\lambda \mid \lambda \vdash n\}$ of $\mathcal{T}(\mathcal{G})$. The irreducible character basis $\{\chi^\lambda \mid \lambda \vdash n\}$ of $\mathcal{T}(\mathcal{G})$ can be defined somewhat similarly. Given $\lambda = (\lambda^0, \dots, \lambda^{d-1}) \vdash n$, define the d -partition $\lambda^\bullet = (|\lambda^0|, \dots, |\lambda^{d-1}|)$, and the $\mathcal{G}(\lambda^\bullet)$ -character

$$\delta_0 \chi^{\lambda^0} \otimes \cdots \otimes \delta_{d-1} \chi^{\lambda^{d-1}},$$

where χ^{λ^k} is the irreducible $\mathfrak{S}_{|\lambda^k|}$ -character indexed by the partition λ^k . The corresponding induced characters

$$\chi^\lambda = (\delta_0 \chi^{\lambda^0} \otimes \cdots \otimes \delta_{d-1} \chi^{\lambda^{d-1}}) \uparrow_{\mathcal{G}(\lambda^\bullet)}^{\mathcal{G}}$$

are the irreducible characters of \mathcal{G} . (See, e.g., [1, p.219].)

For the purpose of creating generating functions for characters β^λ , it will be convenient to realize each as the character of a submodule of the group algebra $\mathbb{C}[\mathcal{G}]$, with \mathcal{G} acting by left multiplication. To do this, we consider an arbitrary finite group G , a subgroup H , an H -character θ , and the element

$$T_H^\theta := \sum_{h \in H} \theta(h^{-1})h \in \mathbb{C}[G].$$

Proposition 2.1. *Let H be a subgroup of a finite group G and let ρ be a one-dimensional complex representation of H with character θ ($= \rho$). Let $U = (u_1, \dots, u_r)$ be a transversal of representatives of cosets of H in G . Let G act by left multiplication on the submodule*

$$V := \text{span}_{\mathbb{C}}\{u_i T_H^\theta \mid 1 \leq i \leq r\} \tag{8}$$

of $\mathbb{C}[G]$. Then V is a G -module with character $\theta \uparrow_H^G$.

Proof. To see that V is a G -module, consider the action of $g \in G$ on the j th element of the defining basis of V . Let $u_i H$ be the unique coset satisfying $gu_j H = u_i H$, i.e., $u_i^{-1}gu_j \in H$. Then we have

$$\begin{aligned} gu_j T_H^\theta &= gu_j \sum_{h \in H} \theta(h^{-1})h = u_i \sum_{h \in H} \theta(h^{-1})u_i^{-1}gu_j h = u_i \sum_{h' \in H} \theta((h')^{-1})u_i^{-1}gu_j h' \\ &= \theta(u_i^{-1}gu_j)u_i T_H^\theta, \end{aligned}$$

since $\theta = \rho$ is a homomorphism. It follows that in the j th column of the matrix representing g , all components are 0 except for the i th, which is $\theta(u_i^{-1}gu_j)$. But this is precisely the formula for entries of the matrix $\rho \uparrow_H^G(g)$. (See, e.g., [9, Defn. 1.12.2].) \square

For $\chi = \theta \uparrow_H^G$, Proposition 2.1 allows us to express T_G^χ as a sum of conjugates of T_H^θ .

Lemma 2.2. Let groups G, H , transversal $U = (u_1, \dots, u_r)$, H -character θ , and G -module V be as in Proposition 2.1, and let $A = (a_{i,j})$ be the matrix of $g \in G$ with respect to the defining basis (8) of V . Then $a_{i,j}$ equals the coefficient of g^{-1} in $u_j T_H^\theta u_i^{-1}$. In particular if χ is the character of V , then we have the identity

$$\sum_{i=1}^r u_i T_H^\theta u_i^{-1} = \sum_{g \in G} \chi(g)g^{-1}.$$

in $\mathbb{C}[G]$.

Proof. By the proof of Proposition 2.1, we have $a_{i,j} = \theta(u_i^{-1}gu_j)$ if some $h \in H$ satisfies $g = u_i h u_j^{-1}$, and is 0 otherwise. On the other hand, we have

$$u_j T_H^\theta u_i^{-1} = \sum_{h \in H} \theta(h^{-1})u_j h u_i^{-1}. \tag{9}$$

If there is no $h \in H$ satisfying $g^{-1} = u_j h u_i^{-1}$, then the coefficient of g^{-1} in (9) is 0. Otherwise, the coefficient of g^{-1} is

$$\theta(h^{-1}) = \theta(u_i^{-1}gu_j).$$

It follows that $a_{i,j}$ is equal to the coefficient of g^{-1} in $u_j T_H^\theta u_i^{-1}$. Thus $\chi(g) = \sum_i a_{i,i}$ is equal to the coefficient of g^{-1} in $\sum_i u_i T_H^\theta u_i^{-1}$. \square

For $G = \mathcal{G}$, $H = \mathcal{G}(\lambda)$, and θ as in (7), the module V (8) has a particularly nice form. The element T_H^θ factors as $T_{\mathcal{G}(\lambda,0)}^{\delta_0 \beta_0} \cdots T_{\mathcal{G}(\lambda,d-1)}^{\delta_{d-1} \beta_{d-1}}$, and each coset $u\mathcal{G}(\lambda)$ of $\mathcal{G}(\lambda)$ has a unique representative $g = (\gamma, w)$ satisfying $\gamma_1 = \cdots = \gamma_n = 0$ and $w_i < w_{i+1}$ for $i, i+1$ belonging to the same block of $\mathbf{K}(\lambda)$, i.e.,

$$w_1 < \cdots < w_{\lambda_1^0}, \quad w_{\lambda_1^0+1} < \cdots < w_{\lambda_1^0+\lambda_2^0}, \dots, \quad w_{n-\lambda_{r,d-1}^{d-1}+1} < \cdots < w_n. \tag{10}$$

Letting $\mathcal{G}(\lambda)^-$ be the set of such coset representatives, we have

$$V = V(\lambda, \beta) = \text{span}_{\mathbb{C}}\{u T_{\mathcal{G}(\lambda,0)}^{\delta_0 \beta_0} \cdots T_{\mathcal{G}(\lambda,d-1)}^{\delta_{d-1} \beta_{d-1}} \mid u \in \mathcal{G}(\lambda)^-\},$$

and the following special case of Lemma 2.2.

Corollary 2.3. Fix a d -partition $\lambda \vdash n$. For each one-dimensional $\mathcal{G}(\lambda)$ -character θ of the form (7), the monomial \mathcal{G} -character $\beta^\lambda = \theta \uparrow_{\mathcal{G}(\lambda)}^{\mathcal{G}}$ satisfies

$$\sum_{u \in \mathcal{G}(\lambda)^-} u T_{\mathcal{G}(\lambda)}^\theta u^{-1} = \sum_{g \in \mathcal{G}} \beta^\lambda(g^{-1})g. \tag{11}$$

For $d = 1, 2$, the group \mathcal{G} (equal to the symmetric group or the hyperoctahedral group) has real-valued irreducible characters. Therefore each group element is conjugate to its inverse, and the final sum of (11) may be expressed as $\sum_{g \in \mathcal{G}} \beta^\lambda(g)g$.

3. Main result

To construct generating functions analogous to (3) for monomial characters of \mathcal{G} , we first extend the ring $\mathbb{C}[z] = \mathbb{C}[z_{1,1}, z_{1,2}, \dots, z_{n,n}]$ and its $n!$ -dimensional subspace $\text{span}\{z_{1,w_1} \cdots z_{n,w_n} \mid w \in \mathfrak{S}_n\}$ which make the definition (1) possible. The one-line notation (4) for elements of \mathcal{G} suggests that we define a ring $\mathbb{C}[x]$ using the dn^2 indeterminates

$$x = \{x_{i,\zeta^k p} \mid i, p \in [n], k \in \mathbb{Z}/d\mathbb{Z}\}$$

and the $d^n n!$ -dimensional subspace

$$\text{span}\{x_{1,g_1} \cdots x_{n,g_n} \mid g \in \mathcal{G}\} \tag{12}$$

of $\mathbb{C}[x]$, where $g_1 \cdots g_n$ has the form (4). We call (12) the \mathcal{G} -immanant space.

One can think of x as an $n \times dn$ matrix of indeterminates, and of each monomial in the \mathcal{G} -immanant space as a collection of n entries of x , with one entry per row and one entry per column (mod d). For example let $(n, d) = (4, 3)$, so that $\zeta = e^{2\pi i/3}$. Let us economize notation by writing $\dot{m} := \zeta m$, $\ddot{m} := \zeta^2 m$, e.g., $x_{2,\ddot{3}} = x_{2,\zeta^2 3}$. Then the 48 indeterminates are

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,\dot{1}} & x_{1,\dot{2}} & x_{1,\dot{3}} & x_{1,\dot{4}} & x_{1,\ddot{1}} & x_{1,\ddot{2}} & x_{1,\ddot{3}} & x_{1,\ddot{4}} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,\dot{1}} & x_{2,\dot{2}} & x_{2,\dot{3}} & x_{2,\dot{4}} & x_{2,\ddot{1}} & x_{2,\ddot{2}} & x_{2,\ddot{3}} & x_{2,\ddot{4}} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,\dot{1}} & x_{3,\dot{2}} & x_{3,\dot{3}} & x_{3,\dot{4}} & x_{3,\ddot{1}} & x_{3,\ddot{2}} & x_{3,\ddot{3}} & x_{3,\ddot{4}} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,\dot{1}} & x_{4,\dot{2}} & x_{4,\dot{3}} & x_{4,\dot{4}} & x_{4,\ddot{1}} & x_{4,\ddot{2}} & x_{4,\ddot{3}} & x_{4,\ddot{4}} \end{bmatrix}. \tag{13}$$

Some monomials belonging to the \mathcal{G} -immanant space are

$$x_{1,2}x_{2,4}x_{3,3}x_{4,1}, \quad x_{1,\dot{4}}x_{2,\dot{3}}x_{3,\dot{2}}x_{4,\dot{1}}, \quad x_{1,3}x_{2,\dot{1}}x_{3,2}x_{4,\ddot{4}}, \quad x_{1,\dot{1}}x_{2,\ddot{2}}x_{3,4}x_{4,\dot{3}}.$$

For $u \in \mathfrak{S}_n, g \in \mathcal{G}$, we will find it convenient to write monomials in the \mathcal{G} -immanant space as

$$x^{u,g} := x_{u_1,g_1} \cdots x_{u_n,g_n}.$$

By the action defined after (4) and commutativity, these monomials satisfy

$$x^{s_i u, s_i g} = x_{u_1,g_1} \cdots x_{u_{i-1},g_{i-1}} x_{u_{i+1},g_{i+1}} x_{u_i,g_i} x_{u_{i+2},g_{i+2}} \cdots x_{u_n,g_n} = x^{u,g}$$

and thus

$$x^{u,g} = x^{u^{-1}u, u^{-1}g} = x^{e, u^{-1}g} \tag{14}$$

for all $u \in \mathfrak{S}_n, g \in \mathcal{G}$. It follows that for any fixed $u \in \mathfrak{S}_n$, the \mathcal{G} -immanant subspace of $\mathbb{C}[x]$ may also be expressed as $\text{span}_{\mathbb{C}}\{x^{u,g} \mid g \in \mathcal{G}\}$. The left- and right-regular representations of \mathcal{G} ,

$$h_1 \circ g \circ h_2 = h_1 g h_2$$

for $g, h_1, h_2 \in \mathcal{G}$, define left- and right-actions of \mathcal{G} on the \mathcal{G} -immanant space,

$$h_1 \circ x^{e,g} \circ h_2 = x^{e, h_1 g h_2}. \tag{15}$$

Now we state a natural \mathcal{G} -analog of the definition (1). For any function $\theta : \mathcal{G} \rightarrow \mathbb{C}$, define the *type- \mathcal{G} θ -immanant* to be the generating function

$$\text{Imm}_\theta^{\mathcal{G}}(x) = \sum_{g \in \mathcal{G}} \theta(g^{-1})x^{e,g} \tag{16}$$

for evaluations of θ . Our counterintuitive use of g^{-1} in place of g is necessitated by Proposition 2.1 – Corollary 2.3. (See also [15, Eq. (1)].) By the comment following Corollary 2.3, symmetric group and hyperoctahedral group ($\mathfrak{B}_n \cong \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$) immanants can be written without inverses:

$$\text{Imm}_{\theta}^{\mathfrak{S}_n}(x) = \sum_{w \in \mathfrak{S}_n} \theta(w)x^{e,w}, \quad \text{Imm}_{\theta}^{\mathfrak{B}_n}(x) = \sum_{g \in \mathfrak{B}_n} \theta(g)x^{e,g}.$$

For economy, we will suppress \mathfrak{S}_n from the notation of symmetric group immanants.

Immanants for the $2d$ one-dimensional characters $\delta_0, \dots, \delta_{d-1}, \delta_0\epsilon, \dots, \delta_{d-1}\epsilon$ of \mathcal{G} may be expressed very simply in terms of $n \times n$ matrices whose entries are linear combinations of the indeterminates x . In particular, define the d $n \times n$ matrices $Q_0(x), \dots, Q_{d-1}(x)$ by $Q_k(x) = (q_{i,j,k}(x))_{i,j \in [n]}$, where

$$q_{i,j,k}(x) = x_{i,j} + \zeta^{-k}x_{i,\zeta j} + \zeta^{-2k}x_{i,\zeta^2 j} + \dots + \zeta^{-(d-1)k}x_{i,\zeta^{(d-1)j}}.$$

Then we have

$$\begin{aligned} \text{per}(Q_k(x)) &= \sum_{g=(\gamma,w) \in \mathcal{G}} \zeta^{-k(\gamma_1+\dots+\gamma_n)}x^{e,g} = \text{Imm}_{\delta_k}^{\mathcal{G}}(x), \\ \det(Q_k(x)) &= \sum_{g=(\gamma,w) \in \mathcal{G}} (-1)^{\text{inv}(w)}\zeta^{-k(\gamma_1+\dots+\gamma_n)}x^{e,g} = \text{Imm}_{\delta_k\epsilon}^{\mathcal{G}}(x). \end{aligned} \tag{17}$$

Returning to our $(n, d) = (4, 3)$ example, we have the matrices

$$\begin{aligned} Q_0(x) &= \begin{bmatrix} x_{1,1} + x_{1,i} + x_{1,\ddot{i}} & x_{1,2} + x_{1,\dot{2}} + x_{1,\ddot{2}} & x_{1,3} + x_{1,\dot{3}} + x_{1,\ddot{3}} & x_{1,4} + x_{1,\dot{4}} + x_{1,\ddot{4}} \\ x_{2,1} + x_{2,i} + x_{2,\ddot{i}} & x_{2,2} + x_{2,\dot{2}} + x_{2,\ddot{2}} & x_{2,3} + x_{2,\dot{3}} + x_{2,\ddot{3}} & x_{2,4} + x_{2,\dot{4}} + x_{2,\ddot{4}} \\ x_{3,1} + x_{3,i} + x_{3,\ddot{i}} & x_{3,2} + x_{3,\dot{2}} + x_{3,\ddot{2}} & x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} & x_{3,4} + x_{3,\dot{4}} + x_{3,\ddot{4}} \\ x_{4,1} + x_{4,i} + x_{4,\ddot{i}} & x_{4,2} + x_{4,\dot{2}} + x_{4,\ddot{2}} & x_{4,3} + x_{4,\dot{3}} + x_{4,\ddot{3}} & x_{4,4} + x_{4,\dot{4}} + x_{4,\ddot{4}} \end{bmatrix}, \tag{18} \\ Q_1(x) &= \begin{bmatrix} x_{1,1} + \zeta^2x_{1,i} + \zeta x_{1,\ddot{i}} & x_{1,2} + \zeta^2x_{1,\dot{2}} + \zeta x_{1,\ddot{2}} & x_{1,3} + \zeta^2x_{1,\dot{3}} + \zeta x_{1,\ddot{3}} & x_{1,4} + \zeta^2x_{1,\dot{4}} + \zeta x_{1,\ddot{4}} \\ x_{2,1} + \zeta^2x_{2,i} + \zeta x_{2,\ddot{i}} & x_{2,2} + \zeta^2x_{2,\dot{2}} + \zeta x_{2,\ddot{2}} & x_{2,3} + \zeta^2x_{2,\dot{3}} + \zeta x_{2,\ddot{3}} & x_{2,4} + \zeta^2x_{2,\dot{4}} + \zeta x_{2,\ddot{4}} \\ x_{3,1} + \zeta^2x_{3,i} + \zeta x_{3,\ddot{i}} & x_{3,2} + \zeta^2x_{3,\dot{2}} + \zeta x_{3,\ddot{2}} & x_{3,3} + \zeta^2x_{3,\dot{3}} + \zeta x_{3,\ddot{3}} & x_{3,4} + \zeta^2x_{3,\dot{4}} + \zeta x_{3,\ddot{4}} \\ x_{4,1} + \zeta^2x_{4,i} + \zeta x_{4,\ddot{i}} & x_{4,2} + \zeta^2x_{4,\dot{2}} + \zeta x_{4,\ddot{2}} & x_{4,3} + \zeta^2x_{4,\dot{3}} + \zeta x_{4,\ddot{3}} & x_{4,4} + \zeta^2x_{4,\dot{4}} + \zeta x_{4,\ddot{4}} \end{bmatrix}, \\ Q_2(x) &= \begin{bmatrix} x_{1,1} + \zeta x_{1,i} + \zeta^2x_{1,\ddot{i}} & x_{1,2} + \zeta x_{1,\dot{2}} + \zeta^2x_{1,\ddot{2}} & x_{1,3} + \zeta x_{1,\dot{3}} + \zeta^2x_{1,\ddot{3}} & x_{1,4} + \zeta x_{1,\dot{4}} + \zeta^2x_{1,\ddot{4}} \\ x_{2,1} + \zeta x_{2,i} + \zeta^2x_{2,\ddot{i}} & x_{2,2} + \zeta x_{2,\dot{2}} + \zeta^2x_{2,\ddot{2}} & x_{2,3} + \zeta x_{2,\dot{3}} + \zeta^2x_{2,\ddot{3}} & x_{2,4} + \zeta x_{2,\dot{4}} + \zeta^2x_{2,\ddot{4}} \\ x_{3,1} + \zeta x_{3,i} + \zeta^2x_{3,\ddot{i}} & x_{3,2} + \zeta x_{3,\dot{2}} + \zeta^2x_{3,\ddot{2}} & x_{3,3} + \zeta x_{3,\dot{3}} + \zeta^2x_{3,\ddot{3}} & x_{3,4} + \zeta x_{3,\dot{4}} + \zeta^2x_{3,\ddot{4}} \\ x_{4,1} + \zeta x_{4,i} + \zeta^2x_{4,\ddot{i}} & x_{4,2} + \zeta x_{4,\dot{2}} + \zeta^2x_{4,\ddot{2}} & x_{4,3} + \zeta x_{4,\dot{3}} + \zeta^2x_{4,\ddot{3}} & x_{4,4} + \zeta x_{4,\dot{4}} + \zeta^2x_{4,\ddot{4}} \end{bmatrix}. \end{aligned}$$

Immanants for the other monomial characters of \mathcal{G} may be expressed as sums of products of \mathfrak{S}_p -immanants of $p \times p$ submatrices of $Q_0(x), \dots, Q_{d-1}(x)$, or of $\mathcal{G}(p)$ -immanants of $p \times dp$ submatrices of x . In terms of the submatrix notation (2), these submatrices will have the forms $Q_k(x)_{M,M}$ and x_{M,C_dM} for some subset $M \subset [n]$, where

$$C_d = \{\zeta^k \mid k \in \mathbb{Z}/d\mathbb{Z}\}, \quad C_dM = \{\zeta^k m \mid k \in \mathbb{Z}/d\mathbb{Z}, m \in M\}.$$

For example, with $(n, d) = (4, 3)$ and $M = \{2, 3\}$, we have

$$C_3 = \{1, \zeta, \zeta^2\}, \quad C_3M = \{2, 3, \dot{2}, \dot{3}, \ddot{2}, \ddot{3}\},$$

and the matrices (18) and (13) have submatrices

$$\begin{aligned} Q_0(x)_{M,M} &= \begin{bmatrix} x_{2,2} + x_{2,\dot{2}} + x_{2,\ddot{2}} & x_{2,3} + x_{2,\dot{3}} + x_{2,\ddot{3}} \\ x_{3,2} + x_{3,\dot{2}} + x_{3,\ddot{2}} & x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} \end{bmatrix}, \\ x_{M,C_3M} &= \begin{bmatrix} x_{2,2} & x_{2,3} & x_{2,\dot{2}} & x_{2,\dot{3}} & x_{2,\ddot{2}} & x_{2,\ddot{3}} \\ x_{3,2} & x_{3,3} & x_{3,\dot{2}} & x_{3,\dot{3}} & x_{3,\ddot{2}} & x_{3,\ddot{3}} \end{bmatrix}. \end{aligned}$$

Now we may state \mathcal{G} -analogs of the Littlewood-Merris-Watkins generating functions (3) in terms of the monomial characters $\eta^\mu = 1 \uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_p}$ and $\epsilon^\mu = \epsilon \uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_p}$ of \mathfrak{S}_p , for $p \leq n$.

Theorem 3.1. Fix d -partition $\lambda = (\lambda^0, \dots, \lambda^{d-1}) \vdash n$, and let $a_k = |\lambda^k|$. Fix symmetric group character sequence $\beta = (\beta_0, \dots, \beta_{d-1}) \in \{1, \epsilon\}^d$ and define

$$\beta_k^{\lambda^k} = \beta_k \uparrow_{\mathfrak{S}_{\lambda^k}}^{\mathfrak{S}_{a_k}} \in \{\epsilon^{\lambda^k}, \eta^{\lambda^k}\}, \quad k = 0, \dots, d-1.$$

Then we have

$$\text{Imm}_{\beta^\lambda}^{\mathcal{G}}(x) = \sum_{(I_0, \dots, I_{d-1})} \text{Imm}_{\beta_0^{\lambda^0}}(Q_0(x)_{I_0, I_0}) \cdots \text{Imm}_{\beta_{d-1}^{\lambda^{d-1}}}(Q_{d-1}(x)_{I_{d-1}, I_{d-1}}), \quad (19)$$

where the sum is over all ordered set partitions of $[n]$ of type $\lambda^\bullet = (a_0, \dots, a_{d-1})$.

Proof. Define the $\mathcal{G}(\lambda)$ -character $\theta = \delta_0 \beta_0 \otimes \cdots \otimes \delta_{d-1} \beta_{d-1}$ and let $\beta^\lambda = \theta \uparrow_{\mathcal{G}(\lambda)}^{\mathcal{G}}$. By (16), (15), and Corollary 2.3, we can express the left-hand side of (19) as

$$\sum_{g \in \mathcal{G}} \beta^\lambda(g^{-1}) x^{e, g} = \sum_{g \in \mathcal{G}} \beta^\lambda(g^{-1})(g \circ x^{e, e}) = \sum_{u \in \mathcal{G}(\lambda)^-} (u T_{\mathcal{G}(\lambda)}^\theta u^{-1} \circ x^{e, e}). \quad (20)$$

Now consider the right-hand side of (19), and define $r_k = \ell(\lambda^k)$. By (3), we may rewrite this as a sum of products of permanents and determinants,

$$\sum_{\mathbf{J}} \left(\prod_{i=0}^{r_0} \text{Imm}_{\beta_0}(Q_0(x)_{J_i^0, J_i^0}) \right) \cdots \left(\prod_{i=0}^{r_{d-1}} \text{Imm}_{\beta_{d-1}}(Q_{d-1}(x)_{J_i^{d-1}, J_i^{d-1}}) \right), \quad (21)$$

where the sum is over all ordered set partitions $\mathbf{J} = (J_1^0, \dots, J_{r_0}^0, \dots, J_1^{d-1}, \dots, J_{r_{d-1}}^{d-1})$ of $[n]$ of type λ , and where $\text{Imm}_\epsilon = \det$, $\text{Imm}_1 = \text{per}$. For all i, k , the indeterminates that appear in $Q_k(x)_{J_i^k, J_i^k}$ are $x_{J_i^k, C_d J_i^k}$. By (17), we may again rewrite (21) as a sum

$$\sum_{\mathbf{J}} \left(\prod_{i=1}^{r_0} \text{Imm}_{\delta_0 \beta_0}^{\mathcal{G}(\lambda_i^0)}(x_{J_i^0, C_d J_i^0}) \right) \cdots \left(\prod_{i=1}^{r_{d-1}} \text{Imm}_{\delta_{d-1} \beta_{d-1}}^{\mathcal{G}(\lambda_i^{d-1})}(x_{J_i^{d-1}, C_d J_i^{d-1}}) \right) \quad (22)$$

in which each factor of each term has the form

$$\text{Imm}_{\delta_k \beta_k}^{\mathcal{G}(\lambda_i^k)}(x_{J_i^k, C_d J_i^k}) = \begin{cases} \sum_{g=(\gamma, w) \in \mathcal{G}(J_i^k)} \zeta^{-k(\gamma_1 + \cdots + \gamma_n)} (x_{J_i^k, C_d J_i^k})^{e, g} & \text{if } \beta_k = 1, \\ \sum_{g=(\gamma, w) \in \mathcal{G}(J_i^k)} \zeta^{-k(\gamma_1 + \cdots + \gamma_n)} (-1)^{\ell(w)} (x_{J_i^k, C_d J_i^k})^{e, g} & \text{if } \beta_k = \epsilon. \end{cases}$$

Define the set partition $\mathbf{K} = (K_1^0, \dots, K_{r_0}^0, \dots, K_1^{d-1}, \dots, K_{r_{d-1}}^{d-1})$ of type λ as in (5), and for each ordered set partition \mathbf{J} of type λ define $u = u(\mathbf{J}) \in \mathcal{G}(\lambda)^-$ to be the element whose one-line notation has the λ_i^k consecutive letters K_i^k in positions J_i^k , for $k = 0, \dots, d-1$ and $i = 1, \dots, r_k$. In particular, u^{-1} is the element in $\mathfrak{S}_n \subset \mathcal{G}$ whose one-line notation contains the increasing rearrangement of J_i^k in the consecutive positions K_i^k for $k = 0, \dots, d-1$ and $i = 1, \dots, r_k$. By (10), the map $\mathbf{J} \mapsto u(\mathbf{J})$ defines a bijective correspondence between ordered set partitions of type λ and $\mathcal{G}(\lambda)^-$. Thus in the expansion of the product (22), the monomials which appear are precisely the set $\{x^{u^{-1}, y u^{-1}} \mid y \in \mathcal{G}(\lambda)\}$. Factoring $y = y_0 \cdots y_{d-1}$ with $y_k \in \mathcal{G}(\lambda, k)$, we may express the coefficient of each such monomial as

$$\delta_0 \beta_0 (y_0^{-1}) \cdots \delta_{d-1} \beta_{d-1} (y_{d-1}^{-1}) = \theta(y^{-1}).$$

Using these facts and (14), (15), we may rewrite (21) as

$$\sum_{u \in \mathcal{G}(\lambda)^-} \sum_{y \in \mathcal{G}(\lambda)} \theta(y^{-1}) x^{u^{-1}, y u^{-1}} = \sum_{u \in \mathcal{G}(\lambda)^-} \sum_{y \in \mathcal{G}(\lambda)} \theta(y^{-1}) (u y u^{-1} \circ x^{e, e}) = \sum_{u \in \mathcal{G}(\lambda)^-} (u T_{\mathcal{G}(\lambda)}^\theta u^{-1} \circ x^{e, e})$$

to see that it is equal to (20). \square

We illustrate with an example. Consider the group $\mathcal{G} = \mathbb{Z}/3\mathbb{Z} \wr \mathfrak{S}_6$. Its trace space $\mathcal{T}(\mathcal{G})$ has dimension equal to the number of 3-partitions of 6, and its immanant space

$$\text{span}_{\mathbb{C}}\{x_{1, g_1} \cdots x_{6, g_6} \mid (g_1, \dots, g_6) \in \mathcal{G}\}$$

requires the $6^2 \cdot 3 = 108$ indeterminates $\{x_{i, m}, x_{i, \dot{m}}, x_{i, \ddot{m}} \mid i, m \in [6]\}$ where we define $\dot{m} := \zeta m$, $\ddot{m} := \zeta^2 m$, as in (13). The $2^3 = 8$ monomial character bases correspond to the triples of one-dimensional symmetric group characters $(1, 1, 1), (1, 1, \epsilon), (1, \epsilon, 1), \dots, (\epsilon, \epsilon, \epsilon)$, so that the basis corresponding to $(\epsilon, \epsilon, 1)$ is

$$\{(\epsilon, \epsilon, 1)^\lambda = (\epsilon \otimes \delta_1 \epsilon \otimes \delta_2) \uparrow_{\mathcal{G}(\lambda)}^{\mathcal{G}} \mid \lambda \vdash 6\}.$$

Consider the basis element $(\epsilon, \epsilon, 1)^{(21,1,2)}$. To evaluate $(\epsilon, \epsilon, 1)^{(21,1,2)}(g)$ for all $g \in \mathcal{G}$, we write its immanant $\text{Imm}_{(\epsilon, \epsilon, 1)^{(21,1,2)} }^{\mathcal{G}}(x)$ as a sum of 60 terms

$$\begin{aligned} & \text{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\text{Imm}_{\epsilon^1}(Q_1(x)_{4,4})\text{Imm}_{\eta^2}(Q_2(x)_{56,56}) \\ & + \text{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\text{Imm}_{\epsilon^1}(Q_1(x)_{5,5})\text{Imm}_{\eta^2}(Q_2(x)_{46,46}) \\ & + \text{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\text{Imm}_{\epsilon^1}(Q_1(x)_{6,6})\text{Imm}_{\eta^2}(Q_2(x)_{45,45}) \\ & + \text{Imm}_{\epsilon^{21}}(Q_0(x)_{124,124})\text{Imm}_{\epsilon^1}(Q_1(x)_{3,3})\text{Imm}_{\eta^2}(Q_2(x)_{56,56}) \\ & \qquad \qquad \qquad \vdots \\ & + \text{Imm}_{\epsilon^{21}}(Q_0(x)_{456,456})\text{Imm}_{\epsilon^1}(Q_1(x)_{3,3})\text{Imm}_{\eta^2}(Q_2(x)_{12,12}), \end{aligned} \tag{23}$$

each corresponding to an ordered set partition of [6] of type (3, 1, 2). Consider the term corresponding to the ordered set partition (136, 4, 25). It is a product of the three factors

$$\begin{aligned} \text{Imm}_{\epsilon^{21}}(Q_0(x)_{136,136}) &= \det \begin{bmatrix} x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}} & x_{1,3} + x_{1,\dot{3}} + x_{1,\ddot{3}} \\ x_{3,1} + x_{3,\dot{1}} + x_{3,\ddot{1}} & x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} \end{bmatrix} (x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}}) \\ &+ \det \begin{bmatrix} x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}} & x_{1,6} + x_{1,\dot{6}} + x_{1,\ddot{6}} \\ x_{6,1} + x_{6,\dot{1}} + x_{6,\ddot{1}} & x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}} \end{bmatrix} (x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}}) \\ &+ \det \begin{bmatrix} x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} & x_{3,6} + x_{3,\dot{6}} + x_{3,\ddot{6}} \\ x_{6,3} + x_{6,\dot{3}} + x_{6,\ddot{3}} & x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}} \end{bmatrix} (x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}}), \\ \text{Imm}_{\epsilon^1}(Q_1(x)_{4,4}) &= x_{4,4} + \zeta^2 x_{4,\dot{4}} + \zeta x_{4,\ddot{4}}, \\ \text{Imm}_{\eta^2}(Q_2(x)_{25,25}) &= \text{per} \begin{bmatrix} x_{2,2} + \zeta x_{2,\dot{2}} + \zeta^2 x_{2,\ddot{2}} & x_{2,5} + \zeta x_{2,\dot{5}} + \zeta^2 x_{2,\ddot{5}} \\ x_{5,2} + \zeta x_{5,\dot{2}} + \zeta^2 x_{5,\ddot{2}} & x_{5,5} + \zeta x_{5,\dot{5}} + \zeta^2 x_{5,\ddot{5}} \end{bmatrix}. \end{aligned} \tag{24}$$

It is easy to see that this term, like all others in (23), contributes 3 to the coefficient of $x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6}$. Thus we have

$$(\epsilon, \epsilon, 1)^{(21,1,2)}(123456) = 180.$$

Now consider the computation of $(\epsilon, \epsilon, 1)^{(21,1,2)}(62345\dot{1})$. Terms in (23) with nonzero contributions to the coefficient of

$$x_{1,6}x_{2,2}x_{3,3}x_{4,\dot{4}}x_{5,5}x_{6,\dot{1}} \tag{25}$$

are those corresponding to ordered set partitions in which 1 and 6 belong to the same block. (Otherwise the variables $x_{1,6}$, $x_{6,\dot{1}}$ will not appear in the term.) Each such ordered set partition has one of the forms

$$(1a6, 4, bc), \quad (1a6, b, 4c), \quad (ab4, c, 16), \quad (235, 4, 16).$$

There are three terms corresponding to ordered set partitions of the first form, including (24). Multiplying the three factors in (24), we find the desired monomial (25) as

$$(-x_{1,6}x_{6,\dot{1}})(x_{3,3})(\zeta^2 x_{4,\dot{4}})(x_{2,2}x_{5,5}),$$

i.e., the term contributes $-\zeta^2$ to the coefficient. The remaining two terms having ordered set partitions of the form $(1a6, 4, bc)$ contribute $-\zeta^2$ as well. Terms corresponding to the six ordered set partitions $(1a6, b, 4c)$ contribute $-\zeta$ each,

$$(-x_{1,6}x_{6,\dot{1}})(x_{a,a})(x_{b,b})(\zeta x_{4,\dot{4}}x_{c,c}),$$

terms corresponding to the three ordered set partitions $(ab4, c, 16)$ contribute 3ζ each,

$$(3x_{a,a}x_{b,b}x_{4,\dot{4}})(x_{c,c})(x_{1,6}\zeta x_{6,\dot{1}}),$$

and the term corresponding to the ordered set partition $(235, 4, 16)$ contributes $3\zeta^2\zeta = 3$,

$$(3x_{2,2}x_{3,3}x_{5,5})(\zeta^2 x_{4,\dot{4}})(x_{1,6}\zeta x_{6,\dot{1}}).$$

Thus we have

$$(\epsilon, \epsilon, 1)^{(21,1,2)}(62345\dot{1}) = -3\zeta^2 - 6\zeta + 9\zeta + 3 = 6 + 6\zeta.$$

It would be interesting to extend Theorem 3.1 to obtain a generating function for the monomial characters of Hecke algebras of wreath products [1], as was done for monomial characters of the Hecke algebra of \mathfrak{S}_n in [6, Thm. 2.1].

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References

- [1] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations*, Adv. Math. 106:2 (1994), 216–243.
- [2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, vol. 231 of Graduate Texts in Mathematics, Springer, New York, 2005.
- [3] S. Clearman, M. Hyatt, B. Shelton and M. Skandera, *Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements*, Electron. J. Combin. 23:2 (2016), Paper 2.7.
- [4] A. Clearwater and M. Skandera, *Total nonnegativity and Hecke algebra trace evaluations*, Sémin. Lothar. Combin. 82B (2019), Article #42.
- [5] R. Kaliszewski, J. Lambright and M. Skandera, *Bases of the quantum matrix bialgebra and induced sign characters of the Hecke algebra*, J. Algebraic Combin. 49:4 (2019), 475–505.
- [6] M. Konvalinka and M. Skandera, *Generating functions for Hecke algebra characters*, Canad. J. Math. 63:2 (2011), 413–435.
- [7] D.E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, New York, 1940.
- [8] R. Merris and W. Watkins, *Inequalities and identities for generalized matrix functions*, Linear Algebra Appl. 64 (1985), 223–242.
- [9] B. Sagan, *The Symmetric Group*, Springer, New York, 2001.
- [10] M. Skandera, *Characters and chromatic symmetric functions*, arXiv:2010.00458.
- [11] M. Skandera, *Graph coloring and hyperoctahedral group character evaluations*, preprint.
- [12] V.P. Snaith, *Explicit Brauer Induction, with applications to algebra and number theory*, vol. 40 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1994.
- [13] R. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.
- [14] R. Stanley, *Positivity problems and conjectures*, In Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, 295–319.
- [15] J. Stembridge, *Immanants of totally positive matrices are nonnegative*, Bull. London Math. Soc. 23 (1991), 422–428.