Enumerative Combinatorics and Applications
ecajournal.haifa.ac.il

ECA **1:3** (2021) Article #S2R19
https://doi.org/10.54550/ECA2021V1S3R19

Transformation à la Foata for Special Kinds of Descents and Excedances

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Received: January 6, 2021, **Accepted**: March 17, 2021, **Published**: March 26, 2021

ABSTRACT: A pure excedance in a permutation $\pi = \pi_1\pi_2\dots\pi_n$ is a position $i < \pi_i$ such that there is no $j < i$ with $i \leq \pi_j < \pi_i$. We present a one-to-one correspondence on the symmetric group that transports pure excedances to descents of a special kind. As a byproduct, we prove that the popularity of pure excedances equals those of pure descents on permutations, while their distributions are different.

Keywords: Cycle; Descent; Distribution; Excedance; Permutation; Popularity; Statistic
2020 Mathematics Subject Classification: 05A05; 05A15; 05A19

1. Introduction and notations

The distribution of the number of descents has been widely studied on several classes of combinatorial objects such as permutations [14], cycles [7,8], and words [3,10]. Many interpretations of this statistic appear in several fields as Coxeter groups [4,11] or lattice path theory [12]. One of the most famous result involves the *Foata fundamental transformation* [9] to establish a one-to-one correspondence between descents and excedances on permutations. This bijection provides a more straightforward proof than those of MacMahon [14] for the equidistribution of these two Eulerian statistics.

In this paper, we present a bijection *à la Foata* on the symmetric group that exchanges pure excedances with special kind of descents defined as a mesh pattern p_2 [6] (see below for the definition of this pattern). Then, we deduce that the popularities (but not the distributions) of pure descents [2] and pure excedances are the same. This common popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number. Finally, we conjecture the existence of a bijection on the symmetric group that exchanges pure excedances and p_2 while preserving the number of cycles.

Let S_n be the set of permutations of length n, *i.e.*, all bijections from $[n] = \{1, 2, \dots, n\}$ into itself. The one-line representation of a permutation $\pi \in S_n$ is $\pi = \pi_1\pi_2\dots\pi_n$ where $\pi_i = \pi(i)$, $1 \leq i \leq n$. For $\sigma \in S_n$, the *product* $\sigma \cdot \pi$ is the permutation $\sigma(\pi_1)\sigma(\pi_2)\dots\sigma(\pi_n)$. A *$\ell$-cycle* $\pi = \langle i_1, i_2, \dots, i_\ell\rangle$ in S_n is a n-length permutation satisfying $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_{\ell-1}) = i_\ell, \pi(i_\ell) = i_1$ and $\pi(j) = j$ for $j \in [n]\backslash\{i_1, i_2, \dots, i_\ell\}$. For $1 \leq k \leq n$, we denote by $C_{n,k}$ the set of all n-length permutations admitting a decomposition in a product of k disjoint cycles. The set $C_{n,k}$ is counted by the signless Stirling numbers of the first kind $c(n, k)$ defined by

$$c(n, k) = (n - 1)\, c(n - 1, k) + c(n - 1, k - 1)$$

where $c(n, k) = 0$ if $n = 0$ or $k = 0$, except $c(0, 0) = 1$ (see [16,17] and Sequence A132393 in [15]). These numbers also enumerate n-length permutations π having k *left-to-right maxima*, *i.e.*, positions $i \in [n]$ such that $\pi_j < \pi_i$ for $j < i$ (see [16]), and permutations $\pi \in S_n$ with $k - 1$ *pure descents*, *i.e.*, descents $\pi_i > \pi_{i+1}$ where there is no $j < i$ such that $\pi_j \in [\pi_{i+1}, \pi_i]$ (see [2]). Note that a pure descent can be viewed as an occurrence of the mesh pattern $(21, L_1)$ where $L_1 = \{1\} \times [0, 2] \cup \{(0, 1)\}$. Indeed, for a k-length permutation σ and a subset $R \subseteq [0, k] \times [0, k]$, an occurrence of the mesh pattern (σ, R) in a permutation π is an occurrence of σ in π with the additional restriction that no element of π lies inside the shaded regions defined by R, where $(i, j) \in R$ means the square having bottom left corner (i, j) in the graphical representation $\{(i, \sigma_i), i \in [k]\}$ of σ. For instance, an occurrence of the mesh pattern p_1 in Figure 1 corresponds to an occurrence of a pure descent. See [6] for a more detailed definition of mesh patterns.

Regarding this interpretation of pure descents in terms of mesh patterns, we define other kinds of descents by the mesh patterns $p_i = (21, L_i)$, $p'_i = (21, R_i)$ with $L_i = \{1\} \times [0, 2] \cup \{(0, i)\}$ and $R_i = \{1\} \times [0, 2] \cup \{(2, i)\}$ for $0 \leq i \leq 2$. Modulo the trivial symmetries on permutations (reverse and complement), it is straightforward to see that p_0, p_1 and p_2 are respectively in the same distribution class as p'_2, p'_1 and p'_0. Then, we deal with only mesh patterns p_i, $i \in [0,2]$. We refer to Figure 1 for a graphical illustration. On the other hand, we define a *pure excedance* as an occurrence of an excedance, *i.e.* $\pi_i > i$, with the additional restriction that there is no point (j, π_j) such that $1 \leq j \leq i-1$ with $i \leq \pi_j < \pi_i$. Although such a pattern (called *pex*) is not a mesh pattern, we can represent it graphically as shown in Figure 1.

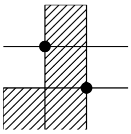
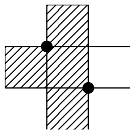
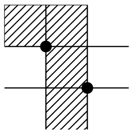
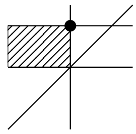
$p_0 =$  $p_1 =$  $p_2 =$  $pex =$ 

Figure 1: Illustration of the mesh patterns p_0, p_1, p_2 and pex; p_1 and pex correspond respectively to a pure descent and a pure excedance.

A *statistic* is an integer-valued function from a set $\mathcal{A}$ of n-length permutations (we use the boldface to denote statistics). For a pattern p, we define the pattern statistic $\mathbf{p} : \mathcal{A} \to \mathbb{N}$ where the image $\mathbf{p}\ \pi$ of $\pi \in \mathcal{A}$ by $\mathbf{p}$ is the number of occurrences of p in π. The *popularity* of p in $\mathcal{A}$ is the total number of occurrences of p over all objects of $\mathcal{A}$, that is $\sum_{a \in \mathcal{A}} \mathbf{p}\ a$ (see [5] for instance). Below, we present statistics that we use throughout the paper:

$$\begin{array}{ll}
\mathbf{exc}\ \pi & = \text{number of excedances in } \pi, \\
\mathbf{pex}\ \pi & = \text{number of pure excedances in } \pi, \\
\mathbf{des}\ \pi & = \text{number of descents in } \pi, \\
\mathbf{des}_i\ \pi & = \text{number of patterns } p_i \text{ in } \pi,\ 0 \leq i \leq 2, \\
\mathbf{fix}\ \pi & = \text{number of fixed points in } \pi, \\
\mathbf{cyc}\ \pi & = \text{number of cycles in the decomposition of } \pi, \\
\mathbf{pcyc}\ \pi & = \text{number of pure cycles (i.e. cycles of length at least two) in } \pi, \\
 & = \mathbf{cyc}\ \pi - \mathbf{fix}\ \pi
\end{array}$$

We organize the paper as follows. In Section 2, we focus on patterns p_i, $0 \leq i \leq 2$. We prove that the statistics $\mathbf{des}_0$ and $\mathbf{des}_1$ are equidistributed by giving algebraic and bijective proofs. Next, we provide the bivariate exponential generating function for the distribution of p_2, and we deduce that p_2 has the same popularity as p_0 and p_1, without having the same distribution. In Section 3, we present a bijection on S_n that transports pure excedances into patterns p_2. Notice that the Foata's first transformation [9] is not a candidate for such a bijection. As a consequence, pure descents and pure excedances are equipopular on S_n, but they do not have the same distribution. Combining all these results, we deduce that patterns p_i, $0 \leq i \leq 2$, and *pex* are equipopular on the symmetric group S_n. Finally we present two conjectures about the equidistribution of $(\mathbf{cyc}, \mathbf{des}_2)$ and $(\mathbf{cyc}, \mathbf{pex})$, and that of $(\mathbf{des}, \mathbf{des}_2)$ and $(\mathbf{exc}, \mathbf{pex})$.

2. The statistics $\mathbf{des}_i$, $0 \leq i \leq 2$

For $0 \leq i \leq 2$, let $A^i_{n,k}$ be the set of n-length permutations having k occurrences of p_i, and denote by $a^i_{n,k}$ its cardinality. Let $A^i(x,y)$ be the bivariate exponential generating function $\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} a^i_{n,k} \frac{x^n}{n!} y^k$. In [2, 13], it is proved that $a^1_{n,k}$ equals the signless Stirling numbers of the first kind $c(n, k+1)$ (see Sequence A132393 in [15]). Indeed, a permutation $\sigma \in A^1_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^1_{n-1,k-1}$, then we increase by one all values of π greater than or equal to π_{n-1}, and we add π_{n-1} at the end;

(ii) if $\pi \in A^1_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value $x \leq n$, $x \neq \pi_{n-1}$ and we add x at the end.

Then, we deduce the recurrence relation $a^1_{n,k} = a^1_{n-1,k-1} + (n-1)a^1_{n-1,k}$ with $a^1_{n,0} = (n-1)!$ for $n \geq 1$, $a^1_{0,0} = 1$ and the bivariate exponential generating function is

$$A^1(x,y) = \frac{1}{y(1-x)^y} - \frac{1}{y} + 1$$

which proves that $a^1_{n,k} = c(n, k+1)$.

Below, we prove that $a^1_{n,k}$ also counts n-length permutations having k occurrences of the pattern p_0.

Theorem 2.1. *The number $a^0_{n,k}$ of n-length permutations having k occurrences of pattern p_0 equals $a^1_{n,k} = c(n, k+1)$.*

Proof. An n-length permutation $\sigma \in A^0_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^0_{n-1,k-1}$, then we increase by one all values of π and we add 1 at the end;

(ii) if $\pi \in A^0_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value x, $1 < x \leq n$, and we add x at the end.

We deduce the recurrence relation $a^0_{n,k} = a^0_{n-1,k-1} + (n-1)a^0_{n-1,k}$ with the initial condition $a^0_{n,0} = (n-1)!$, and then $a^0_{n,k} = a^1_{n,k} = c(n, k+1)$. ☐

Now, we focus on the distribution of the pattern p_2. Table 1 provides exact values for small sizes.

Theorem 2.2. *We have*

$$A^2(x, y) = \frac{e^{x(1-y)}}{(1-x)^y},$$

and the general term $a^2_{n,k}$ satisfies for $n \geq 2$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k}$$

with the initial conditions $a^2_{n,0} = 1$ and $a^2_{n,k} = 0$ for $n \geq 0$ and $k > \lfloor \frac{n}{2} \rfloor$ (see Table 1 and Sequence A136394 in [15]).

Proof. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ denote a permutation of length n having k occurrences of pattern p_2. Let $u_{n,k}$ (resp. $v_{n,k}$) be the number of such permutations satisfying $\sigma_n = n$ (resp. $\sigma_n < n$). Obviously, we have

$$a^2_{n,k} = u_{n,k} + v_{n,k}.$$

A permutation σ with $\sigma_n = n$ can be uniquely constructed from an $(n-1)$-length permutation π as $\sigma = \pi_1\pi_2 \ldots \pi_{n-1}n$. No new occurrences of p_2 are created, and we obtain

$$u_{n,k} = a^2_{n-1,k}.$$

A permutation σ satisfying $\sigma_n < n$ can be uniquely obtained from an $(n-1)$-length permutation π by adding a value $x < n$ on the right side of its one-line notation, after increasing by one all the values greater than or equal to x. This construction creates a new pattern p_2 if and only if π ends with $n-1$. Thus, we deduce

$$v_{n,k} = (n-1)u_{n-1,k-1} + (n-1)v_{n-1,k}.$$

Combining the equations, we obtain for $n \geq 2$ and $k \geq 1$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k},$$

which implies the following differential equation

$$\frac{\partial A^2(x,y)}{\partial x} = (y-1)xA^2(x,y) + \frac{\partial \left(xA^2(x,y)\right)}{\partial x}, \text{ where } A^2(x,0) = 1.$$

A simple calculation provides the claimed closed form for the generating function $A^2(x, y)$. ☐

Corollary 2.1. *For $0 \leq i \leq 2$, the patterns p_i are equipopular on S_n. Their popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.*

Proof. The generating function of the popularity is directly deduced from the bivariate generating function of pattern distribution by calculating

$$\left.\frac{\partial A^1(x,y)}{\partial y}\right|_{y=1} = \left.\frac{\partial A^2(x,y)}{\partial y}\right|_{y=1}.$$

☐

The statistic $\mathbf{des}_2$ has a different distribution from $\mathbf{des}_0$ and $\mathbf{des}_1$, but the three patterns p_0, p_1, p_2 have the same popularity. Below we present a bijection on S_n that transports the statistic $\mathbf{des}_2$ to the statistics $\mathbf{pcyc} = \mathbf{cyc} - \mathbf{fix}$.

$k\backslash n$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1		1	5	20	84	409	2365	16064
2				3	35	295	2359	19670
3						15	315	4480
4								105
...								...
$\sum$	1	2	6	24	120	720	5040	40320

Table 1: Number of n-length permutations having k occurrences of p_2 for $0 \leq k \leq 4$ and $1 \leq n \leq 8$.

Theorem 2.3. *There is a one-to-one correspondence ϕ on S_n such that for any $\pi \in S_n$, we have*

$$\mathbf{des}_2\ \pi = \mathbf{pcyc}\ \phi(\pi).$$

Proof. Let π be a permutation of length n having k occurrences of p_2. We decompose

$$\pi = B_0\pi_{i_1}A_1B_1\pi_{i_2}A_2B_2\pi_{i_3}\dots\pi_{i_k}A_kB_k,$$

where

- $\pi_{i_1} < \pi_{i_2} < \dots < \pi_{i_k}$ are the tops of the occurrences of p_2, *i.e.* values $\pi_{i_j} > \pi_{i_j+1}$ such that there does not exist $\ell < i_j$ such that $\pi_\ell > \pi_{i_j}$,
- A_j is a maximal sequence such that all its values are lower than π_{i_j},
- for $0 \leq j \leq k$, B_j is an increasing sequence such that $\pi_{i_j} < \min B_j$ and $\max B_j < \pi_{i_{j+1}}$.

Now we construct an n-length permutation $\phi(\pi)$ with k pure cycles as follows:

$$\phi(\pi) = \langle \pi_{i_1}A_1\rangle \cdot \langle \pi_{i_2}A_2\rangle \cdots \langle \pi_{i_k}A_k\rangle.$$

For instance, if $\pi = 125346879$ then $\phi(\pi) = \langle 5, 3, 4\rangle \cdot \langle 8, 7\rangle$. The map ϕ is clearly a bijection on S_n such that $\mathbf{des}_2\ \pi$ equals the number of pure cycles in $\phi(\pi)$. □

Note that ϕ^{-1} is closely related to the Foata fundamental transformation [9].

3. The statistic pex of pure excedances

In order to prove the equidistribution of **pex** and $\mathbf{des}_2$, regarding Theorem 2.3, it suffices to construct a bijection on S_n that transports pure excedances to pure cycles. Here, we first exhibit a bijection on the set D_n of n-length derangements (permutations without fixed points), then we extend it to the set of all permutations S_n.

Any permutation $\pi \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\pi = \langle t_1, 1\rangle \cdot \langle t_2, 2\rangle \cdots \langle t_n, n\rangle$$

where t_i are integers such that $1 \leq t_i \leq i$. The transposition array of π is defined by $T(\pi) = t_1t_2\dots t_n$, which induces a bijection $\pi \longmapsto T(\pi)$ from S_n to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. By Lemma 1 from [1], the number of cycles of a permutation π is given by the number of fixed points in $T(\pi)$. Moreover, it is straightforward to check the two following properties:

- if $t_i = i$, then $\pi_i = i$ if and only if there is no number $j > i$ such that $t_j = t_i = i$;
- if $t_i = i$ and $\pi_i \neq i$, then i is the minimal element of a cycle of length at least two in π.

So, we deduce the following lemma.

Lemma 3.1. *The transposition array $t_1t_2\dots t_n \in T_n$ corresponds to a derangement if and only if: $t_i = i \Rightarrow$ there is a $j > i$ such that $t_j = i$.*

Given a derangement $\pi = \pi_1\pi_2\dots\pi_n \in D_n$ and its graphical representation $\{(i, \pi_i), i \in [n]\}$. We say that the square $(i, j) \in [n] \times [n]$ is *free* if all following conditions hold:

(i) Neither π_i nor i is a position of a pure excedance;

(ii) (i, j) is not on the first diagonal, *i.e.* $j \neq i$;

(iii) there does not exist $k > i$ such that $\pi_k = j$;

(iv) j is not a pure excedance such that $j < i$ and $\pi^{-1}(j) < i$;

(v) there does not exist $k < i$, with $\pi_k = j > i$ such that all values of the interval $[i, j-1]$ appear on the right of π_i in π.

Whenever at least one of the statements above is not satisfied, we say that the square (i, j) is *unfree*. Notice that if i and π_i are not the positions of a pure excedance, then the square (i, π_i) is always free. So, for a column i of the graphical representation of π such that i and π_i are not the positions of a pure excedance, we label by j the jth free square from the bottom to the top. We refer to Figure 2 for an example of this labeling.

Now we define the map λ from D_n to the set $T_n^\bullet$ of transposition arrays of length n satisfying the property of Lemma 3.1.

For a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in D_n$, we label its graphical representation as defined above, and $\lambda(\pi) = \lambda_1\lambda_2\ldots\lambda_n$ is obtained as follows:

- if i is a pure excedance in π, then we set $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i$;
- otherwise, λ_i is the sum of the label of the free square (i, π_i) with the number of pure excedances $k < i$ such that $\pi^{-1}(k) < i$.

For instance, if $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ then we obtain $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$ (see Figure 2).

Let us consider i, $1 \le i \le n$. If i is a pure excedance of π, then we fix $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i < \pi^{-1}(i)$. Otherwise, the square (i, i) is unfree, and all squares (i, π_k), $i+1 \le k \le n$, are unfree, which implies that the number of free squares in the ith column is less than or equal to i. This means that $\lambda(\pi)$ lies in T_n. Note that, by construction, all labeled squares do not correspond to any pure excedance. Now let us prove that the square (i, π_i) cannot be labeled i. Indeed, if $\pi_i < i$ then the label of (i, π_i) is necessarily at most $\pi_i \le i-1$; otherwise, if $\pi_i > i$ then the fact that i is not a pure excedance implies that there is $\pi_j \in [i, \pi_i - 1]$ with $j < i$. Let us choose the lowest j with this property. Using (v), the square (i, j) is unfree, which implies that the label of (i, π_i) is less than or equal to n minus the minimal number of unfree squares (i, j) in column i, that is $n - (n-i+1) = i-1$. Moreover, the transposition array $\lambda(\pi)$ has exactly $\mathbf{pex}\ \pi$ fixed points, and for any fixed point i there necessarily exists $j = \pi^{-1}(i) > i$ such that $\lambda_j = \lambda_i = i$. This implies that $\lambda(\pi) \in T_n^\bullet$.

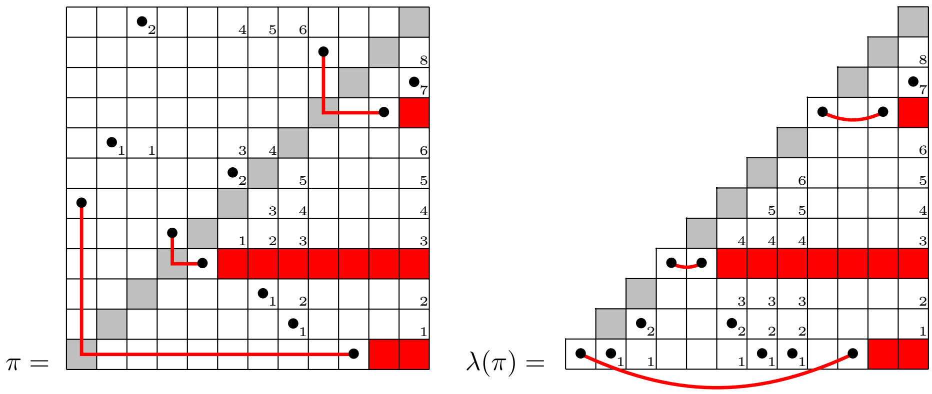


Figure 2: Illustration of the bijection λ for $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ and $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$.

Theorem 3.1. *The map λ from D_n to $T_n^\bullet$ is a bijection such that*

$$\mathbf{pex}\ \pi = \mathbf{fix}\ \lambda(\pi).$$

Proof. Since the cardinality of $T_n^\bullet$ equals that of D_n, and the image of D_n by λ is contained in $T_n^\bullet$, it suffices to prove the injectivity.

Let π and σ, $\pi \ne \sigma$, be two derangements in D_n. If π and σ do not have the same pure excedances, then, by construction, $\lambda(\pi)$ and $\lambda(\sigma)$ do not have the same fixed points, and thus $\lambda(\pi) \ne \lambda(\sigma)$.

Now, let us assume that π and σ have the same pure excedances. If there is a pure excedance i such that $\pi^{-1}(i) \ne \sigma^{-1}(i)$ then the definition implies $\lambda(\pi) \ne \lambda(\sigma)$. Otherwise the two permutations have the same pure excedances i, and for each of them we have $\pi^{-1}(i) = \sigma^{-1}(i)$. Let j be the greatest integer such that $\pi_j \ne \sigma_j$ (without loss of generality, we assume $\pi_j < \sigma_j$). In this case, j is not a pure excedance for the two permutations. Thus, $\lambda(\pi)_j$ (resp. $\lambda(\sigma)_j$) is the sum of the label of (j, π_j) (resp. (j, σ_j)) with the number of pure excedances

$k < j$ such that $\pi^{-1}(k) < j$ (resp. $\sigma^{-1}(k) < j$). Since we have $\pi_j < \sigma_j$, the label of (j, π_j) is less than the label of (j, σ_j), and the number of pure excedances $k < j$ such that $\pi^{-1}(k) < j$ is less than or equal to the number of pure excedances $k < j$ such that $\sigma^{-1}(k) < j$. Then we have $\lambda(\pi)_j < \lambda(\sigma)_j$. Then λ is an injective map, and thus a bijection. $\square$

Theorem 3.2. *There is a one-to-one correspondence ψ on the set D_n of n-length derangements such that for any $\pi \in D_n$,*

$$\mathbf{pex}\ \pi = \mathbf{cyc}\ \psi(\pi).$$

Proof. Considering Theorem 2.3 and Theorem 3.1, we define for any $\pi \in D_n$, $\psi(\pi) = \phi(\sigma)$ where σ is the permutation having $\lambda(\pi)$ as transposition array. $\square$

Theorem 3.3. *The two bistatistics* $(\mathbf{pex}, \mathbf{fix})$ *and* $(\mathbf{pcyc}, \mathbf{fix})$ *are equidistribiuted on S_n.*

Proof. Considering Theorem 3.2, we define the map $\bar{\psi}$ on S_n. Let π' be the permutation obtained from π by deleting all fixed points and after rescaling as a permutation. Let $I = \{i_1, i_2, \ldots, i_k\}$ be the set of fixed points of π. Then, we set $\pi'' = \psi(\pi')$. So, $\sigma = \bar{\psi}(\pi)$ is obtained from π'' by inserting fixed points $i \in I$ after a shift of all other entries in order to produce a permutation in S_n. By construction, we have $\mathbf{pex}\ \pi = \mathbf{pcyc}\ \sigma$ and $\mathbf{fix}\ \pi = \mathbf{fix}\ \sigma$ which completes the proof. $\square$

A byproduct of this theorem is

Corollary 3.1. *The statistics* $\mathbf{cyc}$ *and* $\mathbf{pex} + \mathbf{fix}$ *are equidistributed on S_n.*

Also, a direct consequence of Theorems 2.3 and 3.3 is

Theorem 3.4. *The two statistics* $\mathbf{pex}$ *and* $\mathbf{des}_2$ *are equidistributed on S_n.*

Notice that Foata's first transformation is not a candidate for proving the equidistribution of $\mathbf{pex}$ and $\mathbf{des}_2$, while it transports $\mathbf{exc}$ to $\mathbf{des}$. Combining Theorem 3.4 and Corollary 2.1 we have the following.

Corollary 3.2. *For $0 \le i \le 2$, the patterns p_i and pex are equipopular on S_n (see Sequence A001705 in [15]).*

Finally, we present two conjectures for future works.

Conjecture 3.1. *The two bistatistics* $(\mathbf{des}_2, \mathbf{cyc})$ *and* $(\mathbf{pex}, \mathbf{cyc})$ *are equidistributed on S_n.*

Conjecture 3.2. *The two bistatistics* $(\mathbf{des}_2, \mathbf{des})$ *and* $(\mathbf{pex}, \mathbf{exc})$ *are equidistributed on S_n.*

It is interesting to remark that $(\mathbf{des}, \mathbf{cyc})$ and $(\mathbf{exc}, \mathbf{cyc})$ are not equidistributed. Indeed, there are 3 permutations in S_3 having $\mathbf{exc} = 1$ and $\mathbf{cyc} = 2$, namely 132, 213, 321, but only 2 permutations with $\mathbf{des} = 1$ and $\mathbf{cyc} = 2$, videlicet 132 and 213. So, if the Conjectures 3.1 and 3.2 are true then their proofs are probably independent.

Acknowledgements

We would like to greatly thank Vincent Vajnovszki for having offered us Conjecture 3.2 and the anonymous referees for their helpful comments and suggestions.

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