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Forced Perimeter in Elnitksy Polygons

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Abstract: We study tiling-based perimeter and characterize when a given perimeter tile appears in all rhombic tilings of an Elnitsky polygon. Regardless of where on the perimeter this tile appears, its forcing can be described in terms of 321-patterns. We characterize the permutations with maximally many forced right-perimeter tiles, and show that they are enumerated by the Catalan numbers.

Mathematics Subject Classifications: 05B45; 52B60
Key words and phrases: Catalan number; Discrete perimeter; Permutation pattern; Tiling

1. Introduction

In recent work, we developed tiling-based notions of perimeter and area [14]. One focus of that work was on rhombic tilings of Elnitsky polygons, due to their significance for the combinatorics of reduced decompositions of permutations. This paper continues that work through the notion of a "forced" perimeter tile. In addition to the combinatorial significance of such an object, this has an important analogue in the traditional use of isoperimetric quotients to assess geographic compactness for legal and electoral purposes. In those settings, excessive perimeter can be penalizing, despite the fact that some regions—such as those on a coast—necessarily have extensive perimeter, through no manipulative endeavors. (See [4] for more background on the subject.)

The combinatorial significance of perimeter tiles was discussed in [14, Corollary 2.17]. Here we take that relevance as a given, providing motivation for studying these objects, and we devote the rest of this work to an analysis of when perimeter tiles are "forced."

The results that we present here tie together reduced decompositions, pattern avoidance, and Catalan numbers. We begin with a brief overview of Elnitsky polygons in Section 2 and perimeter tiles in Section 3. In Section 4, we introduce the notion of a "forced" perimeter tile and characterize the conditions under which a given right-, left-, top-, or bottom-perimeter tile is forced (Theorem 4.7, Corollary 4.10, Theorem 4.11, and Corollary 4.12, respectively). Each of these results can be phrased in terms of 321-patterns. Having established those conditions, we use Section 5 to look at the special case of permutations with maximally forced right-perimeter tiles. These are characterized in Theorem 5.1, and Theorem 5.7 shows that they are enumerated by Catalan numbers. We conclude the paper with suggestions for some of the many directions in which to extend this research.

2. Elnitsky polygons

We briefly introduce the primary object of this work, assuming a familiarity with permutations, simple reflections, reduced decompositions, commutation and braid relations, inversions, and so on. For further back-

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ground and a hint at the deep and varied mathematical interest in these topics, the reader is referred to [1–3, 5–8, 11, 12, 15].

In [5], Elnitsky related the commutation classes of reduced decompositions of a permutation w to rhombic tilings of a particular polygon $X(w)$, and the bijection that he developed can be exploited to elucidate properties of permutations. It also gives combinatorial meaning to the tiles appearing in such a tiling.

Definition 2.1. Fix $w \in \mathfrak{S}_n$. *Elnitsky's polygon* for w is the equilateral $2n$-gon defined so that:

- sides are labeled $1, \ldots, n, w(n), \ldots, w(1)$ in counterclockwise order,
- the first n of those sides form half of a convex $2n$-gon, and
- sides with the same label are parallel.

We refer to this $2n$-gon as $X(w)$.

The tilings that we consider—and the tilings that Elnitsky permits—follow certain rules.

Definition 2.2. A *rhombic tiling* of $X(w)$ consists of tiles whose edges are all congruent and parallel to the edges of $X(w)$. The set of all rhombic tilings of $X(w)$ is $T(w)$. The *labels* of a tile t appearing in some $T \in T(w)$ are the labels of the sides of $X(w)$ to which the edges of t are parallel.

Corollary 2.3 (see [14, Corollary 2.12])**.** In any rhombic tiling of an Elnitsky polygon, no two tiles have the same labels. Moreover, there exists a tile labeled $\{x < y\}$ if and only if $w^{-1}(x) > w^{-1}(y)$.

Due to this result, it will cause no confusion if we refer to a tile by its labels, writing "the tile $\{x, y\}$" instead of "the tile labeled $\{x, y\}$."

Because we are interested in tilings, it suffices to consider tiling regions that are contiguous. Thus, throughout this paper, each permutation $w \in \mathfrak{S}_n$ will be assumed to satisfy

$$\{w(1), \ldots, w(r)\} \neq \{1, \ldots, r\}$$

for all $r < n$. A permutation $w \in \mathfrak{S}_n$ is *fully supported* if all simple reflections appear in the reduced decompositions for w. Therefore, as discussed in [13], another way of phrasing this assumption is to say that we assume all permutations are fully supported. Fully supported permutations are also called *connected* or *indecomposable*, as in [10].

For the sake of consistency, we orient Elnitsky polygons—in figures and for the sake of discussion—so that the top vertex is the intersection of the sides labeled 1 and $w(1)$, and the counterclockwise path from the top vertex to the bottom vertex is the *leftside boundary.* We mark the top and bottom vertices with dots.

Example 2.4. The Elnitsky polygon $X(34251)$ is illustrated in Figure 1.

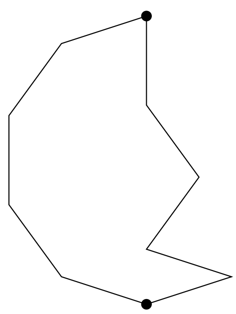


Figure 1: The Elnitsky polygon for 34251.

3. Perimeter tiles

In [14], we introduced the idea of a "perimeter" tile. We reiterate that here, in the context of rhombic tilings of Elnitsky polygons.

Definition 3.1. Fix a permutation w and a rhombic tiling $T \in T(w)$. A tile t that shares at least two consecutive edges with the boundary of $X(w)$ is a *perimeter tile.* We can further specify whether t is a *left-*, *right-*, *top-* or *bottom-perimeter* tile based on whether it shares two consecutive edges with the leftside boundary of $X(w)$, the rightside boundary of $X(w)$, the edges on either side of the top vertex of $X(w)$, or the edges of either side of the bottom vertex of $X(w)$, respectively. This specification is the *type* of the tile.

Note that a perimeter tile may have more than one type. As a trivial example, the sole tile in $T(21)$ is a left-, right-, top-, and bottom-perimeter tile.

The perimeter tiles that appear among all elements of $T(w)$ can vary. Perimeter properties for elements of $T(n \cdots 321)$ were studied in [14], as were the permutations w for which elements of $T(w)$ have minimally many perimeter tiles. In the present work, we consider perimeter tiles from a different perspective: namely, when a given perimeter tile appears among all elements of $T(w)$.

4. Forced tiles

Fix a permutation w and consider the rhombic tilings $T(w)$. The perimeter tiles that appear may vary between tilings, as they do in the two rhombic tilings of $X(321)$ depicted in Figure 2.

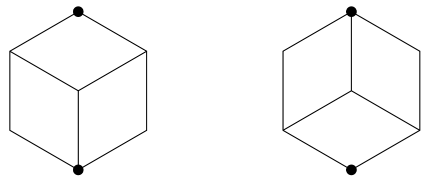


Figure 2: The two rhombic tilings of $X(321)$. Each has three perimeter tiles, and none of the perimeter tiles of a given type are the same in the two tilings.

Definition 4.1. Fix a permutation w. If a specific tile t appears as a right-perimeter tile among all tilings $T \in T(w)$, then t is a *forced right-perimeter* tile for w. Forced left-, top-, and bottom-perimeter tiles are defined analogously.

As we saw in Figure 2, there are some w for which no perimeter tiles are forced, but this is not always the case.

Example 4.2. There are three rhombic tilings of $X(34251)$. As depicted in Figure 3, each of the three has a right- and bottom-perimeter tile $\{1, 5\}$. Therefore $\{1, 5\}$ is a forced right- and bottom-perimeter tile for 34251. There are no other forced perimeter tiles for this permutation.

As we will see, 321-patterns are critical to the determination of forced perimeter tiles, stemming from a previously obtained result.

Definition 4.3. Fix a permutation w and a tiling $T \in T(w)$. This T contains a *subhexagon* if it has a configuration matching either tiling in Figure 2.

Proposition 4.4 (see [11, Theorem 6.4])**.** There is a tiling in $T(w)$ with a subhexagon having sides labeled $x < y < z$ if and only if zyx appears as a 321-pattern in w.

We saw this demonstrated in Figure 3.

Example 4.5. The permutation 34251 has 321-patterns 321 and 421. The first two tilings in Figure 3 have a subhexagon labeled $\{3, 2, 1\}$, and the last two tilings in Figure 3 have a subhexagon labeled $\{4, 2, 1\}$.

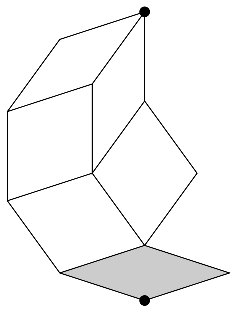


Figure 3: The three rhombic tilings of $X(34251)$. The shaded right- and bottom-perimeter tile appears in all of them.

It follows from Elnitsky's work that if w is 321-avoiding, then $X(w)$ has exactly one rhombic tiling, meaning that all of its perimeter tiles are (trivially) forced. On the other hand, the converse does not hold: a permutation can have all of its perimeter tiles be forced and yet contain the pattern 321.

Example 4.6. The permutation 3614725 has a 321-pattern and $|T(3614725)| = 2$, but all of its perimeter tiles are fixed. This is illustrated in Figure 4.

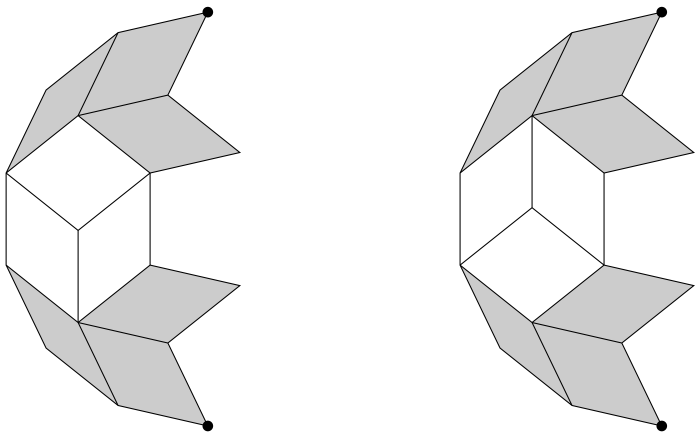


Figure 4: The two rhombic tilings of Elnitsky's polygon for the 321-containing permutation 3614725. The (identical) perimeter tiles in each tiling have been shaded.

The relationship between forced perimeter tiles and 321-patterns can also be phrased in terms of left-to-right and right-to-left maxima and minima. We will refer to these as *LR-* or *RL-max* or *min*.

Theorem 4.7. Fix a permutation $w = \cdots xy \cdots$ with $x > y$. There is a forced right-perimeter tile $\{x, y\}$ if and only if x and y do not appear in a 321-pattern together; equivalently, if and only if x is a LR-max and y is a RL-min.

Proof. First suppose $\{x, y, z\}$ form a 321-pattern in w. Then, by Proposition 4.4, there is a tiling $T \in T(w)$ with a subhexagon whose edges are labeled $\{x, y, z\}$. As demonstrated in Figure 2, this hexagon will always have a tile $\{x, y\}$, but this is not always a right-perimeter tile of the hexagon. Then, by Corollary 2.3, there exists a rhombic tiling of $X(w)$ in which the tile $\{x, y\}$ is not a right-perimeter tile.

Now suppose that $x = w(k)$ is a LR-max and $y = w(k+1)$ is a RL-min, and consider some $T \in T(w)$. The segment labeled x along the rightside boundary of $X(w)$ is the edge of some $\{x, a\} \in T$. In order to fit inside of $X(w)$, we must have either $a > x$ or $a \leq y$. Suppose that $a > x$. Because x is a LR-max, we must have $a = w(h)$ for some $h > k$. But then a tile labeled $\{x, a\}$ would violate Corollary 2.3. A similar argument shows that $a \not< y$. Therefore $a = y$, so the right-perimeter tile $\{x, y\}$ is forced. $\square$

Reduced decompositions of w and of its inverse w^{-1} are left-to-right reflections of each other. Therefore, rhombic tilings of $X(w^{-1})$ can be obtained from rhombic tilings of $X(w)$.

Lemma 4.8 (cf. [5])**.** Let $\tau : T(w) \to T(w^{-1})$ be the map that takes a left-to-right reflection of $T \in T(w)$ and deforms the edges so that the new leftside boundary is convex and all edges with the same label are made to be congruent. This τ is a bijection.

Example 4.9. The polygons $X(34251)$ and $X(53124)$, where $53124 = 34251^{-1}$, are depicted in Figure 5, along with a pair of corresponding tilings.

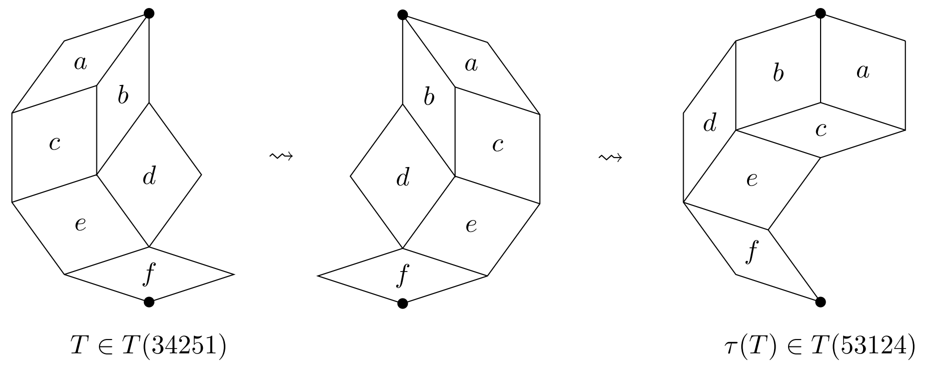


Figure 5: Rhombic tilings of Elnitsky polygons for w and w^{-1} are related by reflection and deformation. To clarify the transformation in this example, the tiles in the first figure have been labeled and their images have been labeled correspondingly in the second and third figure.

We can combine Theorem 4.7 and Lemma 4.8 to characterize forced left-perimeter tiles.

Corollary 4.10. Fix a permutation w. There is a forced left-perimeter tile $\{k, k+1\}$ if and only if $w^{-1}(k) > w^{-1}(k+1)$, and k and $k+1$ do not appear in a 321-pattern together. This is equivalent to $k+1$ being a LR-max in w and k being a RL-min.

Proof. By Lemma 4.8, forcing the left-perimeter tile $\{k, k+1\}$ for w is equivalent to forcing the right-perimeter tile $\{w^{-1}(k), w^{-1}(k+1)\}$ in w^{-1}. By Theorem 4.7, this is equivalent to $w^{-1}(k)$ and $w^{-1}(k+1)$ not appearing in a 321-pattern together in w^{-1}, which is equivalent to k and $k+1$ not appearing in a 321-pattern together in w. □

Forcing top- and bottom-perimeter tiles has a similar flavor to Theorem 4.7 and Corollary 4.10.

Theorem 4.11. Fix a permutation w. There is a forced top-perimeter tile, necessarily $\{1, w(1)\}$, if and only if $w(1)$ and 1 do not appear in a 321-pattern together; equivalently, if and only if the first LR-min after $w(1)$ is 1.

Proof. First suppose that $\{w(1), k, 1\}$ is a 321-pattern in w. Then, as before, Proposition 4.4 means that there is a rhombic tiling of $X(w)$ with a subhexagon whose edges are labeled $\{w(1), k, 1\}$, in which the tile $\{1, w(1)\}$ is not a top-perimeter tile. Thus, no matter where this hexagon is positioned in the polygon $X(w)$, the resulting tiling of $X(w)$ has no top-perimeter tile.

Now suppose that the first LR-min after $w(1)$ is 1, and consider some $T \in T(w)$. The segment labeled $w(1)$ along the rightside boundary of $X(w)$ is the edge of some $\{w(1), a\} \in T$. By Corollary 2.3, we must have $a < w(1)$. If $a > 1$ then, because 1 is the first LR-min after $w(1)$, we must have a tile $\{1, a\}$ in T. However, this would contradict Corollary 2.3, so $a = 1$. Therefore the top-perimeter tile $\{w(1), 1\}$ is forced. □

The conditions for a forced bottom-perimeter tile are analogous.

Corollary 4.12. Fix a permutation $w \in \mathfrak{S}_n$. There is a forced bottom-perimeter tile, necessarily $\{n, w(n)\}$, if and only if $w(n)$ and n do not appear in a 321-pattern together; equivalently, if and only if the first RL-max after $w(n)$ is n.

There is a relationship between right-/left- and top-/bottom-perimeter tiles.

Corollary 4.13. Fix a permutation $w \in \mathfrak{S}_n$. If there is a forced right-perimeter tile $\{w(1), w(2)\}$, then this is a forced top-perimeter tile, too. If there is a forced left-perimeter tile $\{1, 2\}$, then this is a forced top-perimeter tile, too. If there is a forced right-perimeter tile $\{w(n-1), w(n)\}$, then this a forced bottom-perimeter tile, too. If there is a forced left-perimeter tile $\{n-1, n\}$, then this is a forced bottom-perimeter tile, too.

Proof. Suppose that $\{w(1), w(2)\}$ is a forced right-perimeter tile. Then, by Theorem 4.7, $w(2) = 1$. Then, by Theorem 4.11, $\{w(1), w(2) = 1\}$ is a forced top-perimeter tile. Suppose that $\{1, 2\}$ is a forced left-perimeter tile. Then, by Corollary 4.10, $w(1) = 2$, and so by Theorem 4.11, $\{1, 2\}$ is a forced top-perimeter tile. The latter parts of the result follow by symmetry. □

The converses to the statements in Corollary 4.13 do not hold.

Example 4.14. Consider $w = 2341$. This permutation has a forced top-perimeter tile $\{1, 2\}$, but $\{2, 3\}$ is not a forced right-perimeter tile.

5. Optimally forced perimeter

Having characterized how perimeter tiles can be forced in $T(w)$, we conclude by looking at an optimization problem. To be specific, which permutations have maximally many forced right-perimeter tiles, and how many such permutations exist in $\mathfrak{S}_n$? Lemma 4.8 means that analogous questions about left-perimeter tiles can be understood through the answers to these questions.

Consider $w \in \mathfrak{S}_n$. To have maximally many forced right-perimeter tiles would mean $\lfloor n/2 \rfloor$ such tiles. Therefore we will assume $n = 2m$.

Theorem 5.1. A permutation $w \in \mathfrak{S}_{2m}$ has maximally many forced right-perimeter tiles, if and only if $w(2k-1) > w(2k)$ for all k, and

$$w(1) < w(3) < \cdots < w(2m-1) \qquad \text{and} \qquad w(2) < w(4) < \cdots < w(2m). \tag{1}$$

Proof. Suppose that w has maximally many forced right-perimeter tiles. The requirement that $w(2k-1) > w(2k)$ for all k follows from Corollary 2.3, and the inequalities listed in (1) follow from Theorem 4.7.

Now suppose that $w(2k-1) > w(2k)$ for all k, and that the inequalities listed in (1) hold. Corollary 2.3 and Theorem 4.7 mean that there are forced right-perimeter tiles $\{w(2k-1), w(2k)\}$ for all k. □

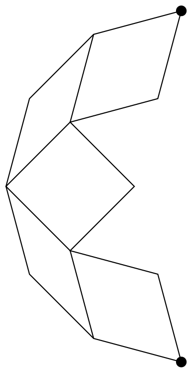
We demonstrate Theorem 5.1 with two examples.

Example 5.2. The permutation 315264 satisfies the requirements of Theorem 5.1, and hence it has three forced right-perimeter tiles. The permutation 325164, on the other hand, does not, because $2 \not< 1 < 4$. Figure 6 depicts the only rhombic tiling of $X(315264)$, in which there are three (forced) right-perimeter tiles, and a rhombic tiling of $X(325164)$ in which there are only two right-perimeter tiles.

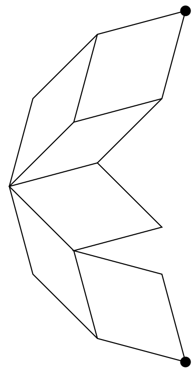
By Theorem 5.1, a permutation w with maximally many forced right-perimeter tiles is 321-avoiding. Therefore, as discussed in Section 4, there is only one rhombic tiling of $X(w)$ for such a permutation, so, in fact, all tiles are forced. Conversely, not all 321-avoiding permutations satsify the requirements of Theorem 5.1. For example, $3412 \in \mathfrak{S}_4$ avoids 321, and hence $X(3412)$ has a single rhombic tiling, but it has only one forced right-perimeter tile.

Recall that we are only considering permutations with full support. Thus the characterization in Theorem 5.1 says that a fully supported permutation $w \in \mathfrak{S}_{2m}$ has maximally many forced right-perimeter tiles if and only if w is a 321-avoiding alternating permutation. Apart from the requirement to be fully supported, this characterization might call the Catalan numbers to mind.

Lemma 5.3 ([9, Item 146])**.** The 321-avoiding alternating permutations in $\mathfrak{S}_{2m}$ are enumerated by the Catalan number C_m.



The only rhombic tiling of $X(315264)$



A rhombic tiling of $X(325164)$

Figure 6: The permutation 315264 has three forced right-perimeter tiles, whereas the permutation 325164 does not.

In fact, we can enumerate fully supported 321-avoiding alternating permutations in $\mathfrak{S}_{2m}$ via a bijection with 321-avoiding alternating permutations in $\mathfrak{S}_{2m-2}$.

Define a map $\phi : \{321\text{-avoiding alternating permutations in } \mathfrak{S}_{2m-2}\} \to \mathfrak{S}_{2m}$ by

$$\big(\phi(v)\big)(i) = \begin{cases} v(i)+1 & \text{if } i < 2m-1 \text{ is odd,} \\ v(i-2)+1 & \text{if } i > 2 \text{ is even,} \\ 1 & \text{if } i = 2, \text{ and} \\ 2n & \text{if } i = 2m-1. \end{cases}$$

We will show, through a sequence of lemmas, that the image of ϕ is a subset of the set of fully supported 321-avoiding alternating permutations in $\mathfrak{S}_{2m}$, and finally that ϕ is a bijection onto this set.

Lemma 5.4. The permutation $\phi(v)$ is fully supported.

Proof. Fix $v \in \{321\text{-avoiding alternating permutations in } \mathfrak{S}_{2m-2}\}$ and set $w := \phi(v)$. Suppose that $\{w(1), \ldots, w(k)\} = \{1, \ldots, k\}$ for some $k < 2m$. Consider first the case that k is even. Then

$$\{v(1)+1, 1, v(3)+1, v(2)+1, \ldots, v(k-1)+1, v(k-2)+1\} = \{1, \ldots, k\},$$

and so $\{v(1), \ldots, v(k-1)\} = \{1, \ldots, k-1\}$. Because k is even and v is alternating, we must have $v(k-1) > v(k)$, so the first $k-1$ positions of v cannot hold the $k-1$ smallest values, which is a contradiction.

Now consider the case that k is odd. Because $w(2m-1) = 2m$, it must be that $k < 2m-1$. Then we have

$$\{v(1)+1, 1, v(3)+1, v(2)+1, \ldots, v(k-2)+1, v(k-3)+1, v(k)+1\} = \{1, \ldots, k\},$$

and so $\{v(1), \ldots, v(k-2), v(k)\} = \{1, \ldots, k-1\}$. Because k is odd and v is alternating, we must have that $v(k-1)$ is less than both $v(k-2)$ and $v(k)$, which is impossible if $\{v(1), \ldots, v(k-2), v(k)\}$ are the smallest values in the permutation.

Therefore w is fully supported. □

Lemma 5.5. The permutation $\phi(v)$ is alternating.

Proof. Fix $v \in \{321\text{-avoiding alternating permutations in } \mathfrak{S}_{2m-2}\}$ and set $w := \phi(v)$. Because v is alternating and avoids 321, we must have $v(i) < v(i+2)$ for all i. Therefore, for all k,

$$\begin{aligned} w(2k+1) &= v(2k+1)+1, \\ w(2k) &= v(2k-2)+1 < v(2k)+1, \text{ and} \\ w(2k+2) &= v(2k)+1 < v(2k+2)+1. \end{aligned}$$

As an alternating permutation, $v(1) > v(w)$ and $v(2k+1) > v(2k), v(2k+2)$ for all $k \geq 1$. Therefore $w(1) > w(2)$ and $w(2k+1) > w(2k), w(2k+2)$ for all $k \geq 1$, as well. □

Lemma 5.6. The permutation $\phi(v)$ is 321-avoiding.

Proof. In an alternating permutation, being 321-avoiding is equivalent to satisfying $w(k) < w(k+2)$ for all k. Fix $v \in \{$321-avoiding alternating permutations in $\mathfrak{S}_{2m-2}\}$ and set $w := \phi(v)$. By Lemma 5.5, w is an alternating permutation.

Because v is a 321-avoiding alternating permutation, we have $v(k) < v(k+2)$ for all k. In particular,

$$v(1)+1 < v(3)+1 < \cdots < v(2m-3)+1 \leq 2m-2+1 < 2m$$

and

$$1 < v(2)+1 < v(4)+1 < \cdots < v(2m-4)+1 < v(2m-2)+1.$$

Therefore, by definition of ϕ, we have $w(k) < w(k+2)$ for all k, and so w is 321-avoiding. □

From these results, we see that the image of ϕ is actually a subset of the fully supported 321-avoiding alternating permutations in $\mathfrak{S}_{2m}$. In fact, we can say much more.

Theorem 5.7. The map ϕ is a bijection from

$$\{\text{321-avoiding alternating permutations in } \mathfrak{S}_{2m-2}\}$$

to

$$\{\text{fully supported 321-avoiding alternating permutations in } \mathfrak{S}_{2m}\},$$

and the number of $w \in \mathfrak{S}_{2m}$ with m forced right-perimeter tiles (equivalently, the number of fully supported 321-avoiding alternating permutations) is the Catalan number C_{m-1}.

Proof. By Lemmas 5.4, 5.5, and 5.6, we have

$$\begin{aligned}\phi : &\{\text{321-avoiding alternating permutations in } \mathfrak{S}_{2m-2}\} \\ &\to \{\text{fully supported 321-avoiding alternating permutations in } \mathfrak{S}_{2m}\}.\end{aligned} \tag{2}$$

In fact, ϕ is a bijection: the preimage of an arbitrary fully supported 321-avoiding alternating permutation $w \in \mathfrak{S}_{2m}$ is the permutation $v \in \mathfrak{S}_{2m-2}$ defined by

$$v(i) = \begin{cases} w(i) - 1 & \text{if } i \text{ is odd, and} \\ w(i+2) - 1 & \text{if } i \text{ is even.} \end{cases}$$

That this v is alternating and 321-avoiding follows from the same types of arguments presented in the proofs of Lemmas 5.5 and 5.6. Thus, the two sets in (2) are equinumerous, and both enumerated by C_{m-1}, thanks to Lemma 5.3. □

We give an example of this enumeration for $m = 4$, including a demonstration on the map ϕ.

Example 5.8. The $C_3 = 5$ fully supported 321-avoiding alternating permutations in $\mathfrak{S}_8$ are listed in Table 1, along with their corresponding (via ϕ) 321-avoiding alternating permutations in $\mathfrak{S}_6$.

321-avoiding alternating permutation in $\mathfrak{S}_6$	ϕ	fully supported 321-avoiding alternating permutation in $\mathfrak{S}_8$
214365	$\mapsto$	31527486
215364	$\mapsto$	31627485
314265	$\mapsto$	41527386
315264	$\mapsto$	41627385
415263	$\mapsto$	51627384

Table 1: There are $C_3 = 5$ permutations in $\mathfrak{S}_6$ that are 321-avoiding and alternating. These are in bijection with the fully supported 321-avoiding alternating permutations in $\mathfrak{S}_8$, by means of the map ϕ.

6. Directions for further research

We have focused on Elnitksy polygons because of the Coxeter-theoretic significance of their tiles. Of course, the notion of forced perimeter tiles exists beyond the context of Elnitsky polygons and rhombic tilings, and this deserves more attention. Certain regions—Elnitsky polygons or otherwise—may have forced non-perimeter tiles, as well. One could also relax the definition of forcing to study when a given tile is "α-forced," where $\alpha \in [0, 1]$ is the proportion of tilings in which the tile appears, and so a forced tile in this paper would be a 1-forced tile in that context.

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