ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **1:3** (2021) Article #S2R20
https://doi.org/10.54550/ECA2021V1S3R20

The Degree of Stiefel Manifolds

Taylor Brysiewicz[†] and Fulvio Gesmundo[‡]

[†] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany*
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
Email: taylor.brysiewicz@mis.mpg.de

[‡] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany*
QMATH, Dept. Mathematical Sciences, U. Copenhagen, Universitetsparken 5, 2100 Copenhagen O., Denmark
Email: fulvio.gesmundo@mis.mpg.de

Received: November 9, 2020, **Accepted**: April 1, 2021, **Published**: April 9, 2021

Abstract: We compute the degree of Stiefel manifolds, that is, the variety of orthonormal frames in a finite dimensional vector space. Our approach employs techniques from classical algebraic geometry, algebraic combinatorics, and classical invariant theory.

Keywords: Degree; Fano scheme; Gelfand–Tsetlin polytope; Hilbert polynomial; Non-intersecting lattice paths; Parseval frame; Stiefel manifold
2020 Mathematics Subject Classification: 14M17; 05A10; 52B20; 15B10

1. Introduction

Frames are a generalization of bases of (real or complex) vector spaces, where one considers spanning sets that satisfy certain conditions. Formally, a collection of vectors $\{v_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ with inner product $\langle -,-\rangle$ is a *frame* if there exist *frame constants* $A, B \in \mathbb{R}_{>0}$ such that

$$A\|v\|^2 \leq \sum_{i\in I} |\langle v, v_i\rangle|^2 \leq B\|v\|^2$$

where $\|\cdot\|$ is the norm induced by the inner product. This set of inequalities is called the *frame condition* and guarantees that $\{v_i\}_{i\in I}$ spans $\mathcal{H}$. If the set $I = \{1,\dots,n\}$ is finite, then $\mathcal{H}$ is finite dimensional and the frame $\{v_i\}_{i=1,\dots,n}$ is called a finite frame.

A frame is called *tight* if $A = B$ and *Parseval* if $A = B = 1$. Frames are extensively studied in linear algebra, functional analysis and operator theory. They find numerous applications in signal processing where they are used to represent signals in compact form while guaranteeing certain desired robustness properties [22].

From a computational point of view, a finite frame in $\mathbb{R}^k$ is encoded by a $k \times n$ matrix Φ whose columns are the coordinates of the frame vectors $\{v_i\}_{i=1,\dots,n}$. The corresponding frame is tight with frame constant A if $\Phi\Phi^T = A\cdot \mathrm{id}_k$ and Parseval if $\Phi\Phi^T = \mathrm{id}_k$, where id_k denotes the $k\times k$ identity matrix. This characterizes all finite Parseval frames as the solutions of $\binom{k+1}{2}$ quadratic equations in the entries of a $k\times n$ matrix. In particular, it realizes the set of Parseval frames as an algebraic subvariety of the space of $k \times n$ matrices known as the *Stiefel manifold*. We consider its Zariski closure $\mathrm{St}(k,n)$ in the space of complex $k\times n$ matrices, $\mathrm{Mat}_{k\times n}(\mathbb{C})$.

Equivalently, Stiefel manifolds can be realized as collections of k orthonormal vectors in an n-dimensional vector space, recorded by the rows of the matrix Φ. In this setting, if $k < n$, the variety $\mathrm{St}(k,n)$ is naturally identified with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-k)$ where $\mathrm{SO}(n-k)$ is regarded as the stabilizer of k fixed (complex) orthonormal vectors (see Section 2). This perspective allows for the use of powerful tools from representation theory and classical invariant theory in the study of Stiefel manifolds.

Long-standing open problems in finite frame theory have been recently solved by understanding spaces of frames as embedded algebraic varieties [8,25,29]. Nonetheless, one of the fundamental invariants of an embedded variety, its *degree*, remains unknown for almost all spaces of frames. When $n = k$, the Stiefel manifold $\mathrm{St}(k,n)$ coincides with the orthogonal group $\mathrm{O}(n)$, and its degree as a subvariety of the space of $n\times n$ matrices was computed in [2]. The main purpose of this paper is to compute the degree of Stiefel manifolds in general.

Theorem 1.1. *Let $n \geq k$.*

- *Suppose $n \leq 2k-1$ and write $n = 2r$ or $n = 2r+1$ depending on the parity. Then*
$$\deg \mathrm{St}(k,n) = 2^k \cdot L_{k,n}$$
where $L_{k,n}$ denotes the number of collections of non-intersecting lattice paths from the points
$$A = \{(-a_i, 0) : i = 1, \ldots, r\} \quad \textit{to} \quad B = \{(0, b_j) : k = 1, \ldots, r\}$$
with
$$(a_1, \ldots, a_r) = (\underbrace{k-1, k-2, , \ldots, k-(n-k)}_{n-k}, \underbrace{2k-n-2, 2k-n-4, \ldots, n-2r}_{r-(n-k)}),$$
$$(b_1, \ldots, b_r) = (n-2, n-4, , \ldots, n-2r).$$

- *Suppose $n \geq 2k-1$. Then*
$$\deg \mathrm{St}(k,n) = 2^{\binom{k+1}{2}}.$$

While this theorem gives a combinatorial interpretation to the degrees of Stiefel manifolds, a bijective proof remains elusive. Such proof amounts to establishing a bijection between intersection points of suitable linear sections of the Stiefel manifolds and collections of non-intersecting lattice paths. This may give a simpler and more direct proof of Theorem 1.1. Moreover, it would bolster the use of homotopy methods for studying Stiefel manifolds and their subvarieties. Indeed, an explicit bijection immediately gives a representation of any Stiefel manifold via a *witness set*, the fundamental data type in numerical algebraic geometry [3].

2. Preliminaries

2.1 Degree, Hilbert function and Hilbert polynomial

We introduce some basic notions about the degree of algebraic varieties. The material of this section is classical and we refer to [18, Lecture 18] and [12, Section I.1.9] for formal definitions and an exposition of the theory. We include some basics here for the reader's convenience and to introduce some notation and convention.

We use homogeneous coordinates $x_0, \ldots, x_N$ on the projective space $\mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$. The affine space $\mathbb{A}^N$ is identified with the affine chart $\{x_0 \neq 0\}$ of $\mathbb{P}^N$ and its complement $H_\infty = \{x_0 = 0\}$ is called the *hyperplane at infinity*.

A *variety* is an affine or projective algebraic variety, reduced and possibly reducible. If $X \subseteq \mathbb{A}^N$ is affine, write $\overline{X}$ for its closure in $\mathbb{P}^N$. We denote by I_X the defining ideal of X, which is an ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$ or $\mathbb{C}[x_0, \ldots, x_N]$ depending on whether X is affine or projective. Write $\mathbb{C}[X]$ for the coordinate ring of X, that is, the quotient of the polynomial ring over I_X. When X is projective (resp. affine), the natural grading of the polynomial ring induces a grading (resp. filtration) on $\mathbb{C}[X]$.

If $X \subseteq \mathbb{A}^N$ (resp. $X \subseteq \mathbb{P}^N$) is an irreducible variety of dimension n, the degree of X, denoted $\deg(X)$, is the number of points of intersection of $X \cap L$ where L is a generic linear space of codimension n. If X is possibly reducible but equidimensional, then the degree of X is the sum of the degrees of its irreducible components. If X is possibly reducible and possibly not equidimensional, then the degree of X is the degree of the union of the components of the largest dimension. It is immediate that $\deg(X) = \deg(\overline{X})$.

Fix a projective variety X of codimension c and suppose I_X is generated by c homogeneous polynomials $f_1, \ldots, f_c$ of degree $d_1, \ldots, d_c$ respectively. Then $\deg(X) = d_1 \cdots d_c$ and X is called a *complete intersection*. More generally, for any variety X of codimension c, the ideal I_X is generated by at least c homogeneous polynomials: the product of their degrees is called the *Bézout bound* and always serves as an upper bound for $\deg(X)$.

The *Hilbert function* of X is the function $\mathrm{HF}_X : \mathbb{N} \to \mathbb{N}$, defined by $\mathrm{HF}_X(t) = \dim(\mathbb{C}[X])_{\leq t}$ or $\mathrm{HF}_X(t) = \dim \mathbb{C}[X]_t$ depending on whether X is affine or projective. The Hilbert function is eventually a polynomial: there exists a univariate polynomial $\mathrm{HP}_X(t)$, called the *Hilbert polynomial* of X, with the property that $\mathrm{HF}_X(t) = \mathrm{HP}_X(t)$ for $t \gg 0$. Moreover, the degree of HP_X is $\dim X$ and its leading coefficient is $\dfrac{\deg(X)}{\dim(X)!}$.

Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$, write $\widehat{f}$ for its homogenization via x_0, i.e. the unique homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_N]$ with $\deg(f) = \deg(\widehat{f})$ such that $\widehat{f}|_{x_0=1} = f$. If X is an affine variety and $f_1, \ldots, f_\ell$ are generators of its ideal I_X then $\widehat{f}_1, \ldots, \widehat{f}_\ell$ cut out a scheme in $\mathbb{P}^N$ which is possibly not reduced; we call this scheme the *naive homogenization* of X (with respect to the chosen generators). We have the following elementary fact.

Lemma 2.1. *Let $X \subseteq \mathbb{A}^N$ be an affine variety and let $f_1, \ldots, f_\ell$ be generators of I_X. Let $Y \subseteq \mathbb{P}^N$ be the naive homogenization of X. The irreducible components of $\overline{X}$ are irreducible components of Y and every other irreducible component of Y is supported on H_∞.*

Proof. Clearly $\overline{X} \subseteq Y$. It suffices to show that $Y \cap \{x_0 \neq 0\} \subseteq X$. But localizing the equations of Y at $x_0 \neq 0$, one obtains exactly $f_1, \ldots, f_\ell$, which are defining equations for X. □

Corollary 2.1. *Let $X \subseteq \mathbb{A}^N$ be an affine variety and let $f_1, \ldots, f_\ell$ be generators of I_X. Let $Y \subseteq \mathbb{P}^N$ be the naive homogenization of X. If all irreducible components of $Y \cap H_\infty$ have dimension strictly smaller than* $\dim X$*, then* $\deg(X) = \deg(Y)$.

2.2 Orbits, algebraic groups and semistable points

We state the Algebraic Peter–Weyl Theorem [17, Thm. 4.2.7] in full generality for a complex semisimple algebraic group and we describe the application to the special orthogonal group that will be needed in Section 4. Our references for this material are [14, 17].

Let G be a complex semisimple algebraic group. Fix a maximal torus $T \subseteq G$ and a Borel subgroup B. Denote by Λ the weight lattice of G with respect to T and by Λ_+ the cone of dominant weights with respect to B. In other words, $\Lambda_+ = \Lambda \cap W$ where W is the principal Weyl chamber. For a dominant weight λ, denote by V_λ the irreducible representation with highest weight λ. We point out that if G is not simply connected, then there are dominant weights not corresponding to an irreducible representation of G. Denote by Λ_+^G the set of integral dominant weights corresponding to the representation of G.

Fix a G-representation V (not necessarily irreducible). Given $w \in V$, let $G_w = \{g \in G : g \cdot w = w\}$ be the stabilizer of w in G, which is a closed subgroup of G. An element $w \in V$ is called *semistable* (for the action of G) if the orbit $G \cdot w \subseteq V$ is Zariski closed (equivalently Euclidean closed). The set $G \cdot w$ is naturally an abstract algebraic variety $G \cdot w \simeq G/G_w$, where G/G_w denotes the set of left cosets of G_w in G.

In this case, the affine coordinate ring of $G \cdot w$ can be written intrinsically in terms of the representation theory of G and G_w, via the Algebraic Peter–Weyl Theorem:

$$\mathbb{C}[G \cdot w] = \bigoplus_{\lambda \in \Lambda_+^G} V_\lambda \otimes [V_\lambda^*]^{G_w} \tag{1}$$

where $[V_\lambda^*]^{G_w}$ denotes the subspace of G_w-invariants in V_λ^*.

Our goal is to apply the Algebraic Peter–Weyl Theorem to compute the leading coefficient of the Hilbert polynomial of Stiefel manifolds. In general, it is not immediate how the grading of the polynomial ring $\mathbb{C}[V]$ descends to a filtration of $\mathbb{C}[G \cdot w]$. However, if G can be realized as a closed subgroup of the endomorphism space of V, we have the following result.

Lemma 2.2. *Let G be a semisimple algebraic group, let V be a faithful G-representation such that the image of G in* $\mathrm{End}(V)$ *is closed. Let $w \in V$ be a semistable point. For every dominant weight λ of G, the summand $V_\lambda \otimes [V_\lambda^*]^{G_w}$ appears in $\mathbb{C}[G \cdot w]_{\leq j}$ if and only if $\lambda \in j\mathcal{C}_V$ where $\mathcal{C}_V$ is the convex hull of the integral weights occurring in V.*

Proof. Since V is faithful, we may regard G as a closed subvariety of $\mathrm{End}(V)$. Regard $\mathrm{End}(V) \simeq V^{\oplus \dim V}$ as a G-representation with respect to the left-composition by elements of G: the integral weights occurring in V are the same as the integral weights occurring in $\mathrm{End}(V)$. The statement holds for $\mathbb{C}[G]$ regarded as a quotient of $\mathbb{C}[\mathrm{End}(V)]$ from the *Claim* in the proof of [9, Theorem 9.1].

Now, consider the linear map

$$\begin{aligned} \mathrm{End}(V) &\to V \\ L &\mapsto Lw. \end{aligned}$$

By linearity, the pullback map on coordinate rings $\mathbb{C}[V] \to \mathbb{C}[\mathrm{End}(V)]$ preserves the grading. A consequence is that the restricted map $G \to G \cdot w$ defined by $g \mapsto g \cdot w$ induces a pullback map on coordinate rings $\mathbb{C}[G \cdot w] \to \mathbb{C}[G]$ which preserves the filtration given by the grading of the polynomial ring. In particular, $\mathbb{C}[G \cdot w]_{\leq j}$ is mapped to $\mathbb{C}[G]_{\leq j}$. This concludes the proof. □

2.3 Representation theory of $\mathrm{SO}(n)$ and branching rules

We briefly review some basics of the representation theory of $\mathrm{SO}(n)$. We refer to [14, 17] for an exposition of the theory and to [4, LaPlanche II, IV] for the explicit numerical data.

When $n = 2r + 1$ is odd, then $\mathrm{SO}(n)$ has dimension $\binom{n}{2}$ and rank r. Let $e_1, \ldots, e_r$ be the simple weights. The fundamental weights are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, r-1$ and $\omega_r = \frac{1}{2}(e_1 + \cdots + e_r)$; in particular ω_r

does not provide a representation for $\mathrm{SO}(n)$. The integral cone $\Lambda_+^{\mathrm{SO}(n)}$ is given by the $\mathbb{Z}_+$-linear combinations of $\omega_1, \ldots, \omega_{r-1}$ and $2\omega_r$. Equivalently integral linear combinations of the fundamental weights are recorded as partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ where λ_j is the coefficient of e_j in the linear combination. In summary, the irreducible representations of $\mathrm{SO}(2r+1)$ are uniquely determined by a partition of length r, that is, an integer sequence $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$.

When $n = 2r$ is even, then $\mathrm{SO}(n)$ has dimension $\binom{n}{2}$ and rank r. Let $e_1, \ldots, e_r$ be the simple weights. The fundamental weights are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, r-2$, $\omega_{r-1} = \frac{1}{2}(e_1 + \cdots + e_{r-1} + e_r)$ and $\omega_r = \frac{1}{2}(e_1 + \cdots + e_{r-1} - e_r)$; in particular ω_{r-1} and ω_r do not provide representations for $\mathrm{SO}(n)$. The integral cone $\Lambda_+^{\mathrm{SO}(n)}$ is given by the $\mathbb{Z}_+$-linear combinations of $\omega_1, \ldots, \omega_{r-2}, \omega_{r-1} + \omega_r$ and $\omega_{r-1} - \omega_r$. Equivalently, integral linear combinations of the fundamental weights are recorded as non-increasing sequences $\lambda = (\lambda_1, \ldots, \lambda_r)$ with λ_r possibly negative. In summary, the irreducible representations of $\mathrm{SO}(2r)$ are uniquely determined by non-increasing integral sequences $\lambda_1 \geq, \ldots, \geq \lambda_{r-1} \geq |\lambda_r|$.

Moreover, it is immediate that for every dominant weight λ for $\mathrm{SO}(n)$ we have $V_\lambda \simeq V_\lambda^*$ as $\mathrm{SO}(n)$-representations and the identification is simply given via contraction with the quadratic form.

We describe the branching rules for the restriction of representations from $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$, realized as the subgroup stabilizing a fixed hyperplane. See [17, Section 8.3].

Lemma 2.3 (Branching Rules)**.** *Let $n = 2r$ be even. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a dominant integral weight for $\mathrm{SO}(2r)$. Then, as an $\mathrm{SO}(n-1)$-representation, V_λ reduces to $V_\lambda = \bigoplus_{\mu \in \mathcal{I}} W_\mu$ where $\mathcal{I}$ ranges over all dominant integral weights $\mu = (\mu_1, \ldots, \mu_{r-1})$ of $\mathrm{SO}(2r-1)$ such that*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{r-1} \geq \mu_{r-1} \geq |\lambda_r|.$$

Let $n = 2r+1$ be odd. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a dominant integral weight for $\mathrm{SO}(2r+1)$. Then as an $\mathrm{SO}(n-1)$-representation, V_λ reduces to $V_\lambda = \bigoplus_{\mu \in \mathcal{I}} W_\mu$ where $\mathcal{I}$ ranges over all dominant integral weights $\mu = (\mu_1, \ldots, \mu_r)$ of $\mathrm{SO}(2r)$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{r-1} \geq \mu_{r-1} \geq \lambda_r \geq |\mu_r|.$$

2.4 Stiefel manifolds

This section is devoted to classical results about Stiefel manifolds. For $k \leq n$, define

$$\mathrm{St}(k,n) = \{A \in \mathrm{Mat}_{k\times n}(\mathbb{C}) : AA^T = \mathrm{id}_k\}.$$

This is an affine variety whose defining equations are the $\binom{k+1}{2}$ quadrics given by the entries of the symmetric $k \times k$ matrix $AA^T - \mathrm{id}_k$. The special orthogonal group $\mathrm{SO}(n)$ acts on $\mathrm{St}(k,n)$ by right multiplication: indeed if $A \in \mathrm{St}(k,n)$ and $g \in \mathrm{SO}(n)$, we have $(Ag)(Ag)^T = Agg^TA^T = A\mathrm{id}_nA^T = \mathrm{id}_k$. Note that if $n = k$, then $\mathrm{St}(k,n)$ coincides with the orthogonal group $\mathrm{O}(n)$: in particular $\mathrm{St}(n,n)$ is reducible.

If $k < n$, then the action of $\mathrm{SO}(n)$ on $\mathrm{St}(k,n)$ is transitive making $\mathrm{St}(k,n)$ the orbit of $\mathrm{SO}(n)$ under this action: if $A, B \in \mathrm{St}(k,n)$ then the rows of A are orthonormal as well as the rows of B; since $k < n$, there exists an element $g \in \mathrm{SO}(n)$ sending the rows of A to the rows of B. Observe that the stabilizer of $A \in \mathrm{St}(k,n)$ under this action is the subgroup acting as the identity on the space spanned by the rows of A: this is a conjugate of the subgroup $\mathrm{SO}(n-k)$.

We deduce the following classical fact.

Lemma 2.4. *If $k < n$, then $\mathrm{St}(k,n)$ is irreducible and isomorphic to the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-k)$. In particular, $\mathrm{St}(k,n)$ is smooth, irreducible, reduced and*

$$\dim \mathrm{St}(k,n) = \binom{n}{2} - \binom{n-k}{2}.$$

Thus, the codimension of $\mathrm{St}(k,n)$ in $\mathrm{Mat}_{k\times n}$ is $nk - (\binom{n}{2} - \binom{n-k}{2}) = \binom{k+1}{2}$, the same as the number of quadrics defining it. As a consequence, we obtain that $\mathrm{St}(k,n)$ is affinely cut out by these $\binom{k+1}{2}$ quadrics.

2.5 Outline of the Proof of the Main Theorem

The proof of Theorem 1.1 is essentially divided into two parts.

The first part is purely geometric and pertains to the green entries in Table 1. We compute the degree of $\mathrm{St}(k,n)$ when $n \geq 2k-1$. In this case, the naive homogenization of $\mathrm{St}(k,n)$ coincides with its closure in projective space $\overline{\mathrm{St}(k,n)}$, so that $\overline{\mathrm{St}(k,n)}$ is a complete intersection and its degree equals the Bézout bound. The proof relies on a dimension argument, showing that the naive homogenization of $\mathrm{St}(k,n)$ does not have additional components at infinity in this range. This is the result of Theorem 3.1 and Theorem 3.2.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64	64	64
4	*	*	*	80	384	704	1024	1024	1024	1024
5	*	*	*	*	768	4768	14848	23808	32768	32768
6	*	*	*	*	*	9536	111616	420736	1064960	1581056
7	*	*	*	*	*	*	223232	3433600	22429696	66082816
8	*	*	*	*	*	*	*	6867200	196968448	1604859904
9	*	*	*	*	*	*	*	*	393936896	14994641408
10	*	*	*	*	*	*	*	*	*	29989282816

Table 1: Degrees of Stiefel manifolds: (light green) Theorem 3.2, (dark blue) $\deg(\mathrm{O}(n))$ computed in [2], (light blue) Theorem 4.3, (dark green) base of induction for proof of Theorem 4.3

As noticed in Section 2.4, when $n = k$, we have $\mathrm{St}(k, n) = \mathrm{O}(n)$. The degrees of the orthogonal groups were determined in [2] and appear in Table 1 in dark blue.

The rest of the proof is aimed at determining the degrees of $\mathrm{St}(k, n)$ for $k + 1 \leq n \leq 2k - 2$ which appear in Table 1 as light blue. In this case, the degree of $\mathrm{St}(k, n)$ is determined by computing the leading coefficient of its Hilbert polynomial. We apply a representation-theoretic argument, built on the Algebraic Peter–Weyl Theorem, to the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n - k)$. Determining the dimensions of the summands of (1) is difficult. Following the work of [6, 7, 9, 19] in the setting of spherical varieties and generic orbits, we reduce the calculation of $\deg \mathrm{St}(k, n)$ to an integral of certain alternating functions, arising from volumes of *Gelfand–Tsetlin polytopes* associated to the representations of the orthogonal group and its invariant spaces. The proof is performed by an inductive argument which allows us to compute volumes of Gelfand–Tsetlin polytopes as alternating polynomials in the entries of their top row, see Theorem 4.2. The base cases for induction are given by the entries of Table 1 in dark green and the induction step moves south-east in the table. The degree formula for the degree of $\mathrm{St}(k, n,)$ in this range is given in Theorem 4.3, and its expression in terms of the combinatorics of non-intersecting lattice paths is obtained in Corollary 4.2.

3. Degree of $\mathrm{St}(k, n)$ for $n \geq 2k - 1$

In this section, we prove the first part of Theorem 1.1 when $n \geq 2k - 1$. Regard the space $\mathrm{Mat}_{k\times n}$ as the open subset of $\mathbb{P}(\mathrm{Mat}_{k\times n} \oplus \mathbb{C})$ and let z_0 be a coordinate on the direct summand $\mathbb{C}$, so that $\mathrm{Mat}_{k\times n}$ is regarded as the principal open set $\{z_0 \neq 0\}$ and $H_\infty = \{z_0 = 0\}$ is the hyperplane at infinity.

Let $\overline{\mathrm{St}(k, n)}$ be the closure of $\mathrm{St}(k, n)$ in $\mathbb{P}(\mathrm{Mat}_{k\times n} \oplus \mathbb{C})$ and let

$$\mathcal{Z}(k, n) = \{(A, z_0) \in \mathbb{P}(\mathrm{Mat}_{k\times n} \oplus \mathbb{C}) : AA^T - z_0^2 \mathrm{Id}_k = 0\}$$

be the naive homogenization of $\mathrm{St}(k, n)$. Let

$$\mathcal{Z}_\infty(k, n) = \mathcal{Z}(k, n) \cap H_\infty = \{A \in \mathbb{P}\mathrm{Mat}_{k\times n} : AA^T = 0\}.$$

First, we compute $\dim \mathcal{Z}_\infty(k, n)$ following a standard argument via an incidence correspondence over the Fano scheme of the quadric hypersurface. This is similar to the classical argument for determinantal varieties as in [1, II.2].

Given a variety $X \subseteq \mathbb{P}V$, denote the *Fano scheme of s-planes in X* is

$$\mathcal{F}_s(X) = \{E \in \mathbb{G}(s, V) : \mathbb{P}E \subseteq X\},$$

where $\mathbb{G}(s, V)$ denotes the Grassmannian of s-planes in V. Let $q_n = x_1^2 + \cdots + x_n^2$ and let $Q_n = \{q_n = 0\} \subseteq \mathbb{P}^{n-1}$ be the corresponding quadric hypersurface.

Lemma 3.1. *Let $A \in \mathrm{Mat}_{k\times n}$, then*

$$AA^T = 0 \quad \textit{if and only if} \quad \mathrm{Im}\, A^T \subseteq Q_n.$$

In particular, $\mathcal{Z}_\infty(k, n) = \{A \in \mathrm{Mat}_{k\times n} : \mathrm{Im}\, A^T \in \mathcal{F}_{\mathrm{rk}(A)}(Q_n)\}$.

Proof. Suppose $AA^T = 0$ and let $v \in \mathrm{Im}\, A^T$, with $v = A^T c$ for some $c \in \mathbb{C}^k$. Then $q_n(v) = v^T v = c^T AA^T c = 0$. Conversely, suppose $q_n(v) = 0$ for every $v \in \mathrm{Im}\, A^T$, so that $0 = q_n(A^T c) = c^T AA^T c$ for every $c \in \mathbb{C}^k$. This implies that the quadratic form associated to AA^T is identically 0 or equivalently $AA^T = 0$. □

If $s \leq n/2$, then the dimensions of the Fano schemes associated to the quadric are given by

$$\dim \mathcal{F}_s(Q_n) = ns - \frac{1}{2}(3s^2 + s).$$

If $s > n/2$ then $\mathcal{F}_s(Q_n) = \emptyset$. We refer to [15, §6.1] for the proof and additional information on the geometry of the Fano scheme.

Theorem 3.1. *For every k, n, we have*

$$\dim \mathcal{Z}_\infty(k,n) = \begin{cases} \frac{1}{8}(n^2 + 4kn - 2n - 8) & \text{if } n < 2k-1 \text{ and } n \text{ is odd} \\ \frac{1}{8}(n^2 + 4kn - 4k - 9) & \text{if } n < 2k-1 \text{ and } n \text{ is even} \\ \binom{n}{2} - \binom{n-k}{2} - 1 & \text{if } n \geq 2k-1 \text{ and } n \text{ is even} \end{cases}$$

In particular, if $n \geq 2k-1$, then $\dim \mathcal{Z}_\infty(k,n) = \dim \mathrm{St}(k,n) - 1$.

Proof. Let $s_{max} = \min\{k, \lfloor n/2 \rfloor\}$. For every $s \leq s_{max}$ define

$$\begin{tikzcd} & \mathcal{Y}_s = \{(A,E) \in \mathbb{P}\mathrm{Mat}_{k\times n} \times \mathcal{F}_s(Q_n) : \mathrm{Im}\, A^T \subseteq E\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}\mathrm{Mat}_{k\times n} & & \mathcal{F}_s(Q_n) \end{tikzcd}$$

where π_1, π_2 are the natural projections on the first and second factor. The generic fiber of π_2 over E is

$$\mathcal{Y}_{sE} := \{A \in \mathbb{P}\mathrm{Mat}_{k\times n} : \mathrm{Im}\, A^T \subseteq E\} \subseteq \mathbb{P}\mathrm{Mat}_{k\times n}$$

which is a (projective) linear space of dimension $ks-1$. The Theorem of the Dimension of the Fibers [27, Section I.6.3] provides

$$\dim \mathcal{Y}_s = \dim \mathcal{F}_s(Q_n) + \dim \mathcal{Y}_{sE} = ns - \frac{1}{2}(3s^2 + s) + ks - 1.$$

By Lemma 3.1, $\mathcal{Z}_\infty(k,n) = \bigcup_{s=1}^{s_{max}} \pi_1(\mathcal{Y}_s)$ and the projection π_1 is generically one-to-one. This shows that

$$\dim \mathcal{Z}_\infty(k,n) = \max\left\{ns - \frac{1}{2}(3s^2+s) + ks - 1 : s = 1, \ldots, s_{max}\right\}.$$

Rewrite $\dim \mathcal{Y}_s = s(n + k - \frac{1}{2} - \frac{3}{2}s) - 1$. As a function of s, $\dim \mathcal{Y}_s$ is increasing between 0 and $\frac{n+k-1/2}{3}$. In particular, $\dim \mathcal{Y}_s$ is increasing on $0 \leq s \leq s_{max}$ whenever $n > 2k$ or $n < 2k-1$, therefore the maximum value (on an integer) of $\dim \mathcal{Y}_s$ in this range is attained at s_{max}. For the remaining two cases of $(k, 2k-1)$ and $(k, 2k)$, one can check that the same conclusion holds. We obtain

$$\dim \mathcal{Z}_\infty(k,n) = \dim \mathcal{Y}_{s_{max}} = \begin{cases} \frac{1}{8}(n^2 + 4kn - 2n - 8) & \text{if } n < 2k-1 \text{ and } n \text{ is odd} \\ \frac{1}{8}(n^2 + 4kn - 4k - 9) & \text{if } n < 2k-1 \text{ and } n \text{ is even} \\ \binom{n}{2} - \binom{n-k}{2} - 1 & \text{if } n \geq 2k-1 \text{ and } n \text{ is even} \end{cases}$$

which concludes the proof. □

A consequence of Theorem 3.1 is that when $n \geq 2k-1$, $\mathcal{Z}_\infty(k,n)$ does not contain irreducible components of $\mathcal{Z}(k,n)$ of dimension as large as $\dim \mathrm{St}(k,n)$. In fact, $\mathcal{Z}_\infty(k,n)$ does not contain irreducible components of $\mathcal{Z}(k,n)$ at all. As a consequence, we obtain,

Theorem 3.2. *If $n \geq 2k-1$, then $\overline{\mathrm{St}(k,n)} = \mathcal{Z}(k,n)$ is a complete intersection of $\binom{k+1}{2}$ quadrics. In particular,*

$$\deg \mathrm{St}(k,n) = 2^{\binom{k+1}{2}}.$$

Proof. The equations defining $\mathcal{Z}(k,n)$ are the entries of $AA^T - z_0^2 \mathrm{Id}_k = 0$. Since $AA^T - z_0 \mathrm{Id}_k$ is symmetric, there are at most $\binom{k+1}{2}$ linearly independent equations. Therefore, every irreducible component of $\mathcal{Z}(k,n)$ has codimension at most $\binom{k+1}{2}$.

Since $\dim \mathrm{St}(k,n) = \dim \mathrm{SO}(n) - \dim \mathrm{SO}(n-k) = \binom{n}{2} - \binom{n-k}{2}$, by Theorem 3.1, we have $\dim \mathcal{Z}(k,n) = \binom{n}{2} - \binom{n-k}{2}$ as well, so that $\mathrm{codim}\, \mathcal{Z}(k,n) = nk - \left[\binom{n}{2} - \binom{n-k}{2}\right] = \binom{k+1}{2}$.

This shows that $\mathcal{Z}(k,n) = \overline{\mathrm{St}(k,n)}$ and in particular it is a complete intersection of the quadrics defined by $AA^T - z_0 \mathrm{Id}_k$. By Bézout's theorem, we conclude $\deg \mathcal{Z}(k,n) = \deg \mathrm{St}(k,n) = 2^{\binom{k+1}{2}}$. □

4. Degree of $\mathrm{St}(k,n)$ when $n \leq 2k-1$

Theorem 3.1 shows that when $k \leq n < 2k-1$, the variety $\mathcal{Z}(k,n)$ has components at infinity of dimension at least as large as $\dim \overline{\mathrm{St}(k,n)}$. Therefore, $\deg \mathrm{St}(k,n)$ is not equal to the Bézout bound in these cases.

In this range, we compute the degree by computing the leading coefficient of the Hilbert polynomial of $\mathrm{St}(k,n)$ via the Algebraic Peter–Weyl Theorem. More precisely, we use

$$\deg \mathrm{St}(k,n) = N! \lim_{j\to\infty} \frac{\dim \mathbb{C}[\mathrm{St}(k,n)]_{\leq j}}{j^N} \tag{2}$$

where $N = \dim \mathrm{St}(k,n) = \binom{n}{2} - \binom{n-k}{2}$.

The values of $\dim \mathbb{C}[\mathrm{St}(k,n)]_{\leq j}$ will be computed via Lemma 2.2. Indeed, (1) provides

$$\mathbb{C}[\mathrm{St}(k,n)] = \bigoplus_{\lambda \in \Lambda_+^{\mathrm{SO}(n)}} V_\lambda \otimes [V_\lambda^*]^{\mathrm{SO}(n-k)}.$$

The homogeneous space $\mathrm{St}(k,n) = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ is embedded in $\mathrm{Mat}_{k\times n} \simeq \mathbb{C}^k \otimes \mathbb{C}^n$, therefore the integral weights occurring in $\mathrm{Mat}_{k\times n}$ are the same as the integral weights occurring in the defining $\mathrm{SO}(n)$-representation $\mathbb{C}^n$. Since $\mathbb{C}^n = V_{(1)}$, the integral weights occurring in $\mathbb{C}^n$ are all the simple weights $\pm e_1, \dots, \pm e_r$, where $n = 2r$ or $n = 2r+1$ depending on the parity. Denote by $\mathcal{C}$ the convex hull of $\pm e_1, \dots, \pm e_r$, that is, the cross-polytope in the weight space $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$. By Lemma 2.2, we deduce

$$\mathbb{C}[\mathrm{St}(k,n)]_{\leq j} = \bigoplus_{\lambda \in j\mathcal{C} \cap \Lambda_+^{\mathrm{SO}(n)}} V_\lambda \otimes [V_\lambda^*]^{\mathrm{SO}(n-k)}. \tag{3}$$

To determine the dimensions of the direct summands, we introduce the formalism of Gelfand–Tsetlin polytopes.

4.1 Gelfand–Tsetlin polytopes and invariants

Definition 4.1. *For $m \leq n$, define*

$$\mathcal{B}(m,n) = \{\lambda^{\mathrm{SO}(i)} : i = m, \dots, n,\ \lambda^{\mathrm{SO}(i)} \textit{ an integral dominant weight for } \mathrm{SO}(i)\}.$$

The Bratteli poset *is the poset structure on $\mathcal{B}(m,n)$ where $\lambda^{\mathrm{SO}(i)} \preceq \mu^{\mathrm{SO}(j)}$ if and only if $i \leq j$ and $V_{\lambda^{\mathrm{SO}(i)}}$ appears in the decomposition of $V_{\mu^{\mathrm{SO}(j)}}$ as a $\mathrm{SO}(i)$-representation.*

This notion was introduced in [5]. We refer to [10] for some information on the underlying combinatorial structure.

Lemma 4.1. *Let λ be a dominant integral weight for $\mathrm{SO}(n)$. Let $m \leq n$. Then the dimension of the space of $\mathrm{SO}(m)$-invariants $\dim[V_\lambda]^{\mathrm{SO}(m)}$ equals the number of chains from $(0)^{\mathrm{SO}(m)}$ to $\lambda^{\mathrm{SO}(n)}$ in the Bratteli poset $\mathcal{B}(m,n)$.*

Proof. This is a direct consequence of the branching rules described in Lemma 2.3. Indeed the restriction of an irreducible representation from $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$ is multiplicity free, implying that every chain from $(0)^{\mathrm{SO}(m)}$ to $\lambda^{\mathrm{SO}(n)}$ gives a unique invariant and all these invariants are linearly independent. ☐

A useful combinatorial picture for recording chains in the Bratteli poset $\mathcal{B}(m,n)$ is a Gelfand–Tsetlin pattern of shape $(\mathrm{SO}(m), \mathrm{SO}(n))$. This is a diagram of boxes placed in $n-m+1$ rows, indexed by integers $m, \dots, n$. The number of boxes in the i-th row equals the rank of $\mathrm{SO}(i)$ and the left border of the diagram is an overlapping descending staircase. The boxes are labeled by the integer coefficients of a dominant weight in terms of the simple weights and these labels interlace along each row according to the branching rules. More precisely, the labels have to satisfy the inequalities:

$$\begin{array}{ccccc} \cdots & \boxed{\mu_{i,j}} & \boxed{\mu_{i+1,j}} & \cdots & \\ & \cdots & \boxed{\mu_{i,j+1}} & \cdots & \end{array} \quad \Longleftrightarrow \quad \mu_{i,j} \geq \mu_{i,j+1} \geq \mu_{i+1,j} \tag{4}$$

$$\begin{array}{ccc} \cdots & \boxed{\mu_{i,j}} & \\ & \cdots & \boxed{\mu_{i,j+1}} \end{array} \quad \Longleftrightarrow \quad \mu_{i,j} \geq \mu_{i,j+1} \geq -\mu_{i,j} \tag{5}$$

$$\begin{array}{cc} \cdots & \boxed{\mu_{i,j}}\,\boxed{\mu_{i+1,j}} \\ \cdots & \boxed{\mu_{i,j+1}} \end{array} \quad \iff \quad \mu_{i,j} \geq \mu_{i,j+1} \geq |\mu_{i+1,j}| \tag{6}$$

These inequalities ensure that a filling of the Gelfand–Tsetlin pattern corresponds to a chain in the Bratteli poset. Conversely, any chain in the Bratteli poset will correspond to a filling.

In Figure 1, we give an example of a Gelfand–Tsetlin pattern of shape $(\mathrm{SO}(3), \mathrm{SO}(7))$. Notice that the zero in the row corresponding to SO(4) is forced by the third inequality in (4)–(6).

6	2	2		SO(7)
5	2	-1		SO(6)
5	1			SO(5)
4	0			SO(4)
0				SO(3)

Figure 1: A chain from $\lambda^{\mathrm{SO}(7)} = (6, 2, 2)$ to $(0)^{\mathrm{SO}(3)}$ in the Bratteli poset $\mathcal{B}(3, 7)$ given by a Gelfand–Tsetlin pattern of shape $(\mathrm{SO}(3), \mathrm{SO}(7))$

In general, the shape of a Gelfand–Tsetlin diagram depends on the parity of n and m because the row corresponding to $\mathrm{SO}(i)$ has $\lfloor \frac{i}{2} \rfloor$ boxes. For reference, in Figure 2, we give the shape when $n = 2r + 1$ and $m = 2r' - 1$ are both odd, from the weight (0) for $\mathrm{SO}(m)$ to the weight λ for $\mathrm{SO}(n)$.

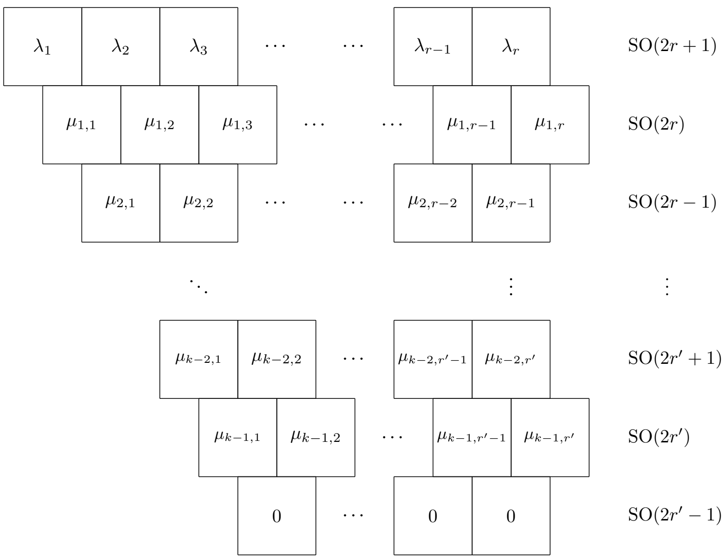


Figure 2: Gelfand–Tsetlin pattern of shape $(\mathrm{SO}(m), \mathrm{SO}(n))$ with n, m both odd

Definition 4.2. *Let $n = 2r$ or $n = 2r + 1$ depending on the parity. Let $\lambda \in \mathbb{R}^r$ be an r-tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ if n is odd and $\lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|$ if n is even. The Gelfand–Tsetlin polytope $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(m)}(\lambda)$ is the set of all fillings of the Gelfand–Tsetlin pattern of shape $(\mathrm{SO}(m), \mathrm{SO}(n))$ with λ in the top row, (0) in the bottom row and filled by real numbers subject to the inequalities of (4)–(6).*

When λ is a dominant integral weight for $\mathrm{SO}(n)$, then the integral points of the Gelfand–Tsetlin polytope $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(m)}(\lambda)$ correspond to chains in the Bratteli poset and therefore via Lemma 4.1 to $\mathrm{SO}(m)$-invariants in the $\mathrm{SO}(n)$-representation V_λ.

We establish the dimension of these polytopes in the range of interest.

Lemma 4.2. *Fix $n \leq 2k-1$ with $n = 2r$ or $n = 2r+1$ depending on the parity. Let $\lambda \in \mathbb{R}^r$ have distinct coefficients and let $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)$ be the corresponding Gelfand–Tsetlin polytope. Then*

$$\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda) = \begin{cases} r(2k-r) - \binom{k+1}{2} & \text{if } n = 2r \text{ is even;} \\ r(2k-r-1) - \binom{k}{2} & \text{if } n = 2r+1 \text{ is odd.} \end{cases}$$

Proof. The dimension of the Gelfand–Tsetlin polytope equals the number of labels of the Gelfand–Tsetlin pattern which are not forced to be 0 by the inequalities (4)–(6). Write $m = n-k$. Since $n \leq 2k-1$, we have $2m+1 \leq n$.

Suppose m is odd, so $m+1$ is even and the row labeled m of the Gelfand–Tsetlin pattern has $\frac{m+1}{2}$ boxes. From Figure 2, observe that all but the first label in the second row from the bottom are forced to be 0; similarly, all but the leftmost i labels in the $(i+1)$-th row from the bottom are forced to be 0 for $i = 1, \ldots, m$. This gives $1 + 2 + \cdots + m = \binom{m+1}{2}$ nonzero labels in the bottom $m+1$ rows of the Gelfand–Tsetlin pattern: indeed, observe that the $(m+1)$-th row from the bottom corresponds to $\mathrm{SO}(2m)$ and all its labels are nonzero. Now consider the rows from $2m$ to n: the last row is fixed and its labels do not contribute to the dimension; the remaining $n-2m-1$ rows contribute with a total of $2\left[\binom{r}{2} - \binom{m+1}{2}\right] + m$ labels if $n = 2r$ is even and $2\left[\binom{r}{2} - \binom{m+1}{2}\right] + m + r - 1$ if $n = 2r+1$ is odd. Expanding the binomial coefficients, we obtain the result. If m is even, the calculation is similar. □

We point out that a result similar to Lemma 4.2 holds in the range $n \geq 2k$, that is when $\deg(\mathrm{St}(k,n))$ equals the Bézout bound. However, in this case, the inequalities are more complicated and the statement is more involved. Although in principle one can compute $\deg(\mathrm{St}(k,n))$ using this approach in the Bézout range, we prefer the geometric argument of Section 3 and do not provide additional details on the representation-theoretic approach in these cases.

We now characterize the degree of $\mathrm{St}(k,n)$ in terms of volumes of Gelfand–Tsetlin polytopes, where *volume* means the Euclidean volume in the real dimensional space given by Lemma 4.2.

Theorem 4.1. *Fix k, n with $n \leq 2k-1$. Then*

$$\deg(\mathrm{St}(k,n)) = N! \int_{\mathcal{C}\cap W} \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)\right) \cdot \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)\right) d\lambda,$$

where $N = \dim \mathrm{St}(k,n) = \binom{n}{2} - \binom{n-k}{2}$.

Proof. From equation (3), via Lemma 4.1,

$$\deg(\mathrm{St}(k,n)) = N! \cdot \lim_{j\to\infty} \frac{1}{j^N} \sum_{\lambda \in j\mathcal{C}\cap\Lambda_+^{\mathrm{SO}(n)}} (\dim V_\lambda) \cdot \left(\dim[V_\lambda]^{\mathrm{SO}(n-k)}\right). \tag{7}$$

Now, $\dim[V_\lambda]^{\mathrm{SO}(n-k)}$ equals the number of lattice points in $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)$. Similarly, $\dim V_\lambda$ is the number of invariants for the trivial group $\mathrm{SO}(1) \subseteq \mathrm{SO}(n)$, therefore it equals the number of lattice points in $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)$.

Using Lemma 4.2, whenever λ has distinct coefficients, we obtain

$$N - \left[\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda) + \dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)\right] = r.$$

This allows us to rewrite (7) as

$$\begin{aligned} \deg(\mathrm{St}(k,n)) &= N! \cdot \lim_{j\to\infty} \sum_{\lambda\in j\mathcal{C}\cap\Lambda_+^{\mathrm{SO}(n)}} \frac{\dim V_\lambda}{j^{\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)}} \cdot \frac{\dim[V_\lambda]^{\mathrm{SO}(n-k)}}{j^{\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)}} \\ &= N! \cdot \lim_{j\to\infty} \sum_{\lambda\in\mathcal{C}\cap\frac{1}{j}\Lambda_+^{\mathrm{SO}(n)}} \frac{\dim V_{j\lambda}}{j^{\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)}} \cdot \frac{\dim[V_{j\lambda}]^{\mathrm{SO}(n-k)}}{j^{\dim GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)}}. \end{aligned}$$

As $j \to \infty$ this summation converges to an integral and the number of rescaled lattice points converges to the volume of the Gelfand–Tsetlin polytope. We conclude

$$\deg(\mathrm{St}(k,n)) = N! \int_{\mathcal{C}\cap W} \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)\right) \cdot \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)\right) d\lambda.$$

□

The volumes of the Gelfand–Tsetlin polytopes can be computed via straightforward integrals, using their definitions via the inequalities (4)–(6) which explicitly determine the range of each variable:

$$\begin{aligned}
\text{inequality (4)} &\longleftrightarrow \int_{\mu_{i+1,j}}^{\mu_{i,j}} 1 d\mu_{i,j+1},\\
\text{inequality (5)} &\longleftrightarrow \int_{-\mu_{i,j}}^{\mu_{i,j}} 1 d\mu_{i,j+1},\\
\text{inequality (6)} &\longleftrightarrow \int_{|\mu_{i+1,j}|}^{\mu_{i,j}} 1 d\mu_{i,j+1}.
\end{aligned}$$

In fact, we perform an additional reduction: for the integral associated to (5), we have $\int_{-\mu_{i,j}}^{\mu_{i,j}} 1 d\mu_{i,j+1} = 2\int_0^{\mu_{i,j+1}} 1 d\mu_{i,j+1}$. This allows us to assume that the rightmost label of every row of the Gelfand–Tsetlin pattern is nonnegative and simplifies the integral associated to (6) as well, providing $\int_{|\mu_{i+1,j}|}^{\mu_{i,j}} 1 d\mu_{i,j+1} = \int_{\mu_{i+1,j}}^{\mu_{i,j}} 1 d\mu_{i,j+1}$.

After this simplification, the volume of the Gelfand–Tsetlin polytope is provided by a series of nested integrals, where one counts twice every integral whose integration variable is the label of the rightmost box of a row corresponding to $\mathrm{SO}(i)$ with i even.

Example 4.1. *Consider the general pattern of shape* $(\mathrm{SO}(3), \mathrm{SO}(7))$*:*

$$\begin{array}{cccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & & \mathrm{SO}(7)\\
 & \mu_{1,1} & & \mu_{1,2} & & \mu_{1,3} & & \mathrm{SO}(6)\\
 & & \mu_{2,1} & & \mu_{2,2} & & & \mathrm{SO}(5)\\
 & & & \mu_{3,1} & & \mu_{3,2} & & \mathrm{SO}(4)\\
 & & & & 0 & & & \mathrm{SO}(3)
\end{array}$$

Note that the inequality (6) implies that $\mu_{3,2} = 0$ *and so there are only* 6 *free variables. The volume of* $GT_{\mathrm{SO}(3)}^{\mathrm{SO}(7)}(\lambda)$ *is given by*

$$\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} 2\int_0^{\lambda_3}\int_{\mu_{1,2}}^{\mu_{1,1}}\int_{\mu_{1,3}}^{\mu_{1,2}}\int_{\mu_{2,2}}^{\mu_{2,1}} 1 d\mu_{3,1}d\mu_{2,2}d\mu_{2,1}d\mu_{1,3}d\mu_{1,2}d\mu_{1,1}$$

which evaluates to

$$\mathrm{vol}\left(GT_{\mathrm{SO}(3)}^{\mathrm{SO}(7)}(\lambda)\right) = \frac{1}{6}(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)\lambda_1\lambda_2\lambda_3.$$

In particular, note the factor of 2 *arising in the integration with respect to* $\mu_{1,3}$ *between* 0 *and* λ_3*. We point out that this volume is an alternating function in the* λ_j*'s, evident from the outermost two integrals. Moreover, it is divisible by* λ_3 *(and thus* λ_1 *and* λ_2 *by the alternating property) evident from the third outermost integral.*

4.2 Alternating functions and volumes of Gelfand–Tsetlin polytopes

In this section, we use an induction argument to determine the volumes of the Gelfand–Tsetlin polytopes relevant to the calculation of $\deg \mathrm{St}(k, n)$.

In Example 4.1, we saw that the volume of $GT_{\mathrm{SO}(3)}^{\mathrm{SO}(7)}(\lambda)$ is an alternating polynomial in λ. It is clear that this is a general fact, because of the last sequence of integrals in $\mathrm{vol}\left(GT_{\mathrm{SO}(3)}^{\mathrm{SO}(7)}(\lambda)\right)$.

We record some facts about alternating polynomials referring to [21, Ch. I]. Given an integer partition $\mu = (\mu_1, \dots, \mu_r)$, define the alternating polynomial

$$a_\mu(\lambda_1, \dots, \lambda_r) = \det\left[\lambda_j^{\mu_i+r-i}\right].$$

We remark that our notation differs from the usual notation which uses the subscript $\mu+(r-1,\dots,1,0)$ instead of μ for the alternating polynomial a_μ.

We record two useful results on the integration of alternating functions. The first gives the result of the integral of a product of two alternating functions on the standard simplex.

Lemma 4.3. *Let Δ_r be the convex hull of the origin and the standard $r-1$-simplex in $\mathbb{R}^r$. Let $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_r)$ be two partitions. Then*

$$\int_{\Delta_r} a_\mu(\lambda) a_\nu(\lambda) d\lambda = \frac{r!}{(r^2 + |\mu| + |\nu|)!} \det \left(\left[(\nu_i + \mu_j + 2r - i - j)! \right]_{i,j=1}^r \right)$$

where $|\mu| = \sum \mu_i$ and $|\nu| = \sum \nu_i$.

Proof. The proof is an explicit calculation obtained by expanding the determinants defining $a_\mu(\lambda)$ and $a_\nu(\lambda)$. Given a permutation σ, write $(-1)^\sigma$ for its sign.

$$\begin{aligned} \int_{\Delta_r} a_\mu(\lambda) a_\nu(\lambda) d\lambda &= \int_{\Delta_r} \sum_{\sigma,\tau \in \mathfrak{S}_r} (-1)^{\sigma \circ \tau} \prod_{i=1}^r \lambda_i^{\mu_{\sigma(i)} + \nu_{\tau(i)} + 2r - \sigma(i) - \tau(i)} d\lambda \\ &= \sum_{\sigma,\tau \in \mathfrak{S}_r} (-1)^{\sigma \circ \tau} \left(\int_{\Delta_r} \prod_{i=1}^r \lambda_i^{\mu_{\sigma(i)} + \nu_{\tau(i)} + 2r - \sigma(i) - \tau(i)} d\lambda \right). \end{aligned}$$

The integral of a monomial over a simplex is given by [23, Lemma 4.23]. Applying this to our expression gives

$$\begin{aligned} &\sum_{\sigma,\tau \in \mathfrak{S}_r} (-1)^{\sigma \circ \tau} \frac{\prod_{i=1}^r (\mu_{\sigma(i)} + \nu_{\tau(i)} + 2r - \sigma(i) - \tau(i))!}{(r + (\sum_{i=1}^r 2r) + \mu_{\sigma(i)} + \nu_{\tau(i)} - \sigma(i) - \tau(i))!} \\ =& \frac{1}{(r^2 + |\mu| + |\nu|)!} \sum_{\sigma,\tau \in \mathfrak{S}_r} (-1)^{\sigma \circ \tau} \prod_{i=1}^r (\mu_{\sigma(i)} + \nu_{\tau(i)} + 2r - \sigma(i) - \tau(i))! \\ =& \frac{r!}{(r^2 + |\mu| + |\nu|)!} \sum_{\tau \in \mathfrak{S}_r} (-1)^{\tau} \prod_{i=1}^r (\mu_i + \nu_{\tau(i)} + 2r - i - \tau(i))! \\ =& \frac{r!}{(r^2 + |\mu| + |\nu|)!} \det \left(\left[(\mu_i + \nu_j + 2r - i - j)! \right] \right)_{i,j=1}^r . \end{aligned}$$

□

The second result provides a formula for the integral of alternating functions in terms of the integration bounds.

Lemma 4.4. *Let π be a partition $\pi = (\pi_1, \ldots, \pi_r)$. Then*

$$\int_{\lambda_2}^{\lambda_1} \cdots \int_{\lambda_{r+1}}^{\lambda_r} a_\pi(\mu_1, \ldots, \mu_r) d\mu_r \cdots d\mu_1 = \frac{1}{\prod_1^r (\pi_j + r - j + 1)} \cdot a_{(\pi,0)}(\lambda_1, \ldots, \lambda_{r+1}).$$

Proof. Consider the determinant representation of $a_\pi(\mu)$ and notice that each variable appears only in a single column of the corresponding matrix. By linearity, this implies that the integration can be performed directly on the entries of the matrix:

$$\begin{aligned} &\int_{\lambda_2}^{\lambda_1} \cdots \int_{\lambda_{r+1}}^{\lambda_r} a_\pi(\mu_1, \ldots, \mu_r) d\mu_r \cdots d\mu_1 \\ &= \det \begin{bmatrix} \int_{\lambda_2}^{\lambda_1} \mu_1^{\pi_1 + r - 1} d\mu_1 & \cdots & \int_{\lambda_2}^{\lambda_1} \mu_r^{\pi_1 + r - 1} d\mu_r \\ \vdots & & \vdots \\ \int_{\lambda_2}^{\lambda_1} \mu_1^{\pi_r} d\mu_1 & \cdots & \int_{\lambda_2}^{\lambda_1} \mu_r^{\pi_r} d\mu_r \end{bmatrix} \\ &= \frac{1}{\prod(\pi_j + r - j + 1)} \det \begin{bmatrix} \lambda_1^{\pi_1 + r} - \lambda_2^{\pi_1 + r} & \cdots & \lambda_r^{\pi_1 + r} - \lambda_{r+1}^{\pi_1 + r} \\ \vdots & & \vdots \\ \lambda_1^{\pi_r + 1} - \lambda_2^{\pi_r + 1} & \cdots & \lambda_r^{\pi_r + 1} - \lambda_{r+1}^{\pi_r + 1} \end{bmatrix} \\ &= \frac{1}{\prod(\pi_j + r - j + 1)} \det \begin{bmatrix} \lambda_1^{\pi_1 + r} - \lambda_2^{\pi_1 + r} & \cdots & \lambda_r^{\pi_1 + r} - \lambda_{r+1}^{\pi_1 + r} & \lambda_{r+1}^{\pi_1 + r} \\ \vdots & & \vdots & \vdots \\ \lambda_1^{\pi_r + 1} - \lambda_2^{\pi_r + 1} & \cdots & \lambda_r^{\pi_r + 1} - \lambda_{r+1}^{\pi_r + 1} & \lambda_{r+1}^{\pi_r + 1} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{\prod(\pi_j + r - j + 1)} \det \begin{bmatrix} \lambda_1^{\pi_1+r} & \lambda_2^{\pi_1+r} & \cdots & \lambda_{r+1}^{\pi_1+r} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{\pi_r+1} & \lambda_2^{\pi_r+1} & \cdots & \lambda_{r+1}^{\pi_r+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$= \frac{1}{\prod(\pi_j + r - j + 1)} \cdot a_{(\pi,0)}(\lambda_1, \ldots, \lambda_{r+1}).$$

□

Define recursively the following partitions. Let $\Omega_{k,2k-1} = \underbrace{(1, \ldots, 1)}_{k-1}$ and let

$$\Omega_{k,n} = \begin{cases} (\Omega_{k-1,n-1}, 0) & \text{if } n \text{ is even} \\ \Omega_{k-1,n-1} + (1, \ldots, 1) & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

A closed expression for $\Omega_{k,n}$ can be obtained by induction and it is given by

$$\Omega_{k,n} = \begin{cases} (\underbrace{k-r, \ldots, k-r}_{n-k}, k-r-1, \ldots, 0) & \text{if } n = 2r \text{ is even} \\ (\underbrace{k-r, \ldots, k-r}_{n-k}, k-r-1, \ldots, 1) & \text{if } n = 2r+1 \text{ is odd.} \end{cases}$$

Notice that the recursion reaches all pairs (k, n) with $n \leq 2k - 1$. For reference, Table 2 contains the first values of $\Omega_{k,n}$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1		(0)								
2	*		(1)							
3	*	*		(1,0)	(1,1)					
4	*	*	*		(2,1)	(1,1,0)	(1,1,1)			
5	*	*	*	*		(2,1,0)	(2,2,1)	(1,1,1,0)	(1,1,1,1)	
6	*	*	*	*	*		(3,2,1)	(2,2,1,0)	(2,2,2,1)	(1,1,1,1,0)
7	*	*	*	*	*	*		(3,2,1,0)	(3,3,2,1)	(2,2,2,1,0)
8	*	*	*	*	*	*	*		(4,3,2,1)	(3,3,2,1,0)
9	*	*	*	*	*	*	*	*		(4,3,2,1,0)
10	*	*	*	*	*	*	*	*	*	

Table 2: Partitions $\Omega_{k,n}$ from (8). The bases of the recursion are the dark green boxes; the recursive steps move southeast.

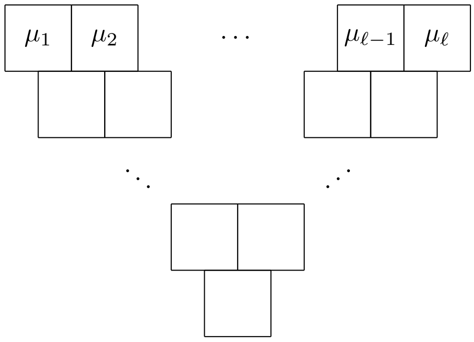
Proposition 4.1. *The volume of* $GT^{\mathrm{SO}(2k-1)}_{\mathrm{SO}(k-1)}(\lambda)$ *is*

$$\mathrm{vol}\left(GT^{\mathrm{SO}(2k-1)}_{\mathrm{SO}(k-1)}(\lambda)\right) = \frac{2}{\prod_{j=1}^{k-1} j!} a_{\Omega_{k,2k-1}}(\lambda).$$

Proof. Let $n = 2k-1$. As in the proof of Lemma 4.2, observe that only some of the labels on the Gelfand–Tsetlin pattern can be nonzero: only the i leftmost labels in the row corresponding to $\mathrm{SO}(k-1+i)$ are nonzero, for $i = 1, \ldots, 2k-2$. In particular, the row corresponding to $\mathrm{SO}(2k-2)$ has no labels identically equal to 0. This shows

$$\mathrm{vol}\left(GT^{\mathrm{SO}(2k-1)}_{\mathrm{SO}(k-1)}(\lambda)\right) = \int_{\lambda_2}^{\lambda_1} \cdots \int_{\lambda_{k-1}}^{\lambda_{k-2}} 2 \int_0^{\lambda_{k-1}} \mathrm{vol}(T_{k-1}(\mu_1, \ldots, \mu_{k-1})) d\mu_{k-1} \cdots d\mu_1, \tag{9}$$

where $T_\ell(\mu_1, \ldots, \mu_\ell)$ is the polytope defined by the same inequalities as in (4)–(6) and the triangular shape



Observe that $\dim T_\ell(\mu_1, \ldots, \mu_\ell) = \binom{\ell}{2}$ and its volume is an alternating polynomial in the variables $\mu_1, \ldots, \mu_\ell$. There is a unique, up to scale, alternating polynomial of degree $\binom{\ell}{2}$ in ℓ variables and it is the Vandermonde determinant. Therefore,

$$\mathrm{vol}(T_\ell(\mu_1, \ldots, \mu_\ell)) = \kappa_\ell a_{(0,\ldots,0)}(\mu).$$

for some constant κ_ℓ. We use induction on ℓ to determine $\kappa_\ell = \frac{1}{\prod_{j=1}^{\ell-1} j!}$. This holds when $\ell = 2$.

For $\ell \geq 3$, notice

$$\begin{aligned}
\mathrm{vol}(T_\ell(\mu_1, \ldots, \mu_\ell)) &= \int_{\mu_2}^{\mu_1} \cdots \int_{\mu_\ell}^{\mu_\ell} \mathrm{vol}(T_{\ell-1}(\nu_1, \ldots, \nu_{\ell-1})) d\nu_{\ell-1} \cdots d\nu_1 \\
&= \kappa_{\ell-1} \int_{\mu_2}^{\mu_1} \cdots \int_{\mu_\ell}^{\mu_{\ell-1}} a_{(0,\ldots,0)}(\nu_1, \ldots, \nu_{\ell-1}) d\nu_{\ell-1} \cdots d\nu_1 \\
&= \frac{1}{\prod_{j=1}^{\ell-2} j!} \cdot \frac{1}{\prod_1^{\ell-1}(\ell - j)} a_{(0,\ldots,0,0)}(\mu_1, \ldots, \mu_\ell),
\end{aligned}$$

where in the last line we used Lemma 4.4; since $\prod_1^{\ell-1}(\ell - j) = (\ell - 1)!$, we obtain the desired value of κ_ℓ.

It remains to evaluate the integral in (9). From (9), we see

$$\mathrm{vol}\left(GT^{\mathrm{SO}(2k-1)}_{\mathrm{SO}(k-1)}(\lambda)\right) = 2 \cdot \mathrm{vol}(T_k(\lambda_1, \ldots, \lambda_{k-1}, 0)).$$

This concludes the proof because

$$\begin{aligned}
2 \cdot \mathrm{vol}(T_k(\lambda_1, \ldots, \lambda_{k-1}, 0)) &= 2\left(\frac{1}{\prod_1^{k-1} j!} a_{(0,\ldots,0)}(\lambda_1, \ldots, \lambda_{k-1}, 0)\right) \\
&= \frac{2}{\prod_1^{k-1} j!} \lambda_1 \cdots \lambda_{k-1} a_{(0,\ldots,0)}(\lambda_1, \ldots, \lambda_{k-1}) \\
&= \frac{2}{\prod_1^{k-1} j!} a_{(1,\ldots,1)}(\lambda_1, \ldots, \lambda_{k-1}) \\
&= \frac{2}{\prod_1^{k-1} j!} a_{\Omega_{k,2k-1}}(\lambda_1, \ldots, \lambda_{k-1}).
\end{aligned}$$

□

Proposition 4.1 provides the base of the induction for the following result.

Theorem 4.2. *Let $n \leq 2k - 1$ with $n = 2r$ or $n = 2r + 1$ depending on its parity and let $\lambda = (\lambda_1, \ldots, \lambda_r)$. Then*

$$\begin{aligned}
\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)\right) &= \frac{2^{k-r}}{\prod_{j=1}^r ((\Omega_{k,n})_j + r - j)!} \cdot a_{\Omega_{k,n}}(\lambda) \\
&= \frac{2^{k-r}}{\prod_1^{n-k}(k-j)! \cdot \prod_{n-k+1}^r (n - 2j)!} \cdot a_{\Omega_{k,n}}(\lambda).
\end{aligned}$$

Proof. Since $n \leq 2k - 1$, there exists a nonnegative integer p such that $(k, n) = (\ell + p, 2\ell - 1 + p)$. We use induction on p. Notice that $n - k = \ell - 1$ does not depend on p. When $p = 0$, the statement is true by Proposition 4.1.

Notice that $\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda)\right)$ is obtained by integrating $\mathrm{vol}\left(GT^{\mathrm{SO}(n-1)}_{\mathrm{SO}(n-k)}(\lambda)\right)$ in the labels of the second (from the top) row of the Gelfand–Tsetlin pattern, see Figure 2.

We consider two cases depending on the parity of p.

Let p be odd. In this case, $n = 2\ell - 1 + p$ is even and if $\mathrm{SO}(n)$ has rank r then $\mathrm{SO}(n-1)$ has rank $r-1$. We have

$$\begin{aligned}\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(\ell-1)}(\lambda)\right) &= \int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_r}^{\lambda_{r-1}} \mathrm{vol}\left(GT^{\mathrm{SO}(n-1)}_{\mathrm{SO}(\ell-1)}(\mu_1,\ldots,\mu_{r-1})\right) d\mu_{r-1}\cdots d\mu_1\\ &= \frac{2^{k-1-(r-1)}}{\prod_{j=1}^{r-1}((\Omega_{k-1,n-1})_j + r-1-j)!}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_r}^{\lambda_{r-1}} a_{\Omega_{k-1,n-1}}(\mu)\mu_{r-1}\cdots d\mu_1\end{aligned}$$

where we use the inductive hypothesis for $p-1$ to compute $\mathrm{vol}\left(GT^{\mathrm{SO}(n-1)}_{\mathrm{SO}(\ell-1)}(\mu_1,\ldots,\mu_{r-1})\right)$.

Applying Lemma 4.4, we obtain

$$\begin{aligned}\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(\ell-1)}(\lambda)\right) =&\frac{2^{k-r}}{\prod_{j=1}^{r-1}((\Omega_{k-1,n-1})_j! + (r-1)-j)}\\ &\cdot\frac{1}{\prod_1^{r-1}((\Omega_{k-1,n-1})_j + (r-1)-j+1}a_{(\Omega_{k-1,n-1},0)}(\lambda).\end{aligned}$$

Since n is even, we have $(\Omega_{k-1,n-1},0) = \Omega_{k,n}$ so

$$\prod_{j=1}^{r-1}((\Omega_{k-1,n-1})_j + (r-1)-j)!\cdot\prod_1^{r-1}((\Omega_{k-1,n-1})_j + r-j) = \prod_1^{r}(\Omega_{k,n})_j + r-j)!.$$

This concludes the proof when p is odd.

Let p be even. In this case, $n = 2\ell - 1 + p$ is odd, so $\mathrm{SO}(n)$ and $\mathrm{SO}(n-1)$ have rank r. We have

$$\begin{aligned}\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(\ell-1)})(\lambda)\right) &= \int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_r}^{\lambda_{r-1}} 2\int_0^{\lambda_r}\mathrm{vol}\left(GT^{\mathrm{SO}(n-1)}_{\mathrm{SO}(\ell-1)}(\mu_1,\ldots,\mu_r)\right) d\mu_r\cdots d\mu_1\\ &= \frac{2^{k-1-r)}}{\prod_{j=1}^{r}((\Omega_{k-1,n-1})_j + r-j)!}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_r}^{\lambda_{r-1}} 2\int_0^{\lambda_r} a_{\Omega_{k-1,n-1}}(\mu)\mu_r\cdots d\mu_1\end{aligned}$$

Similarly to the proof of Proposition 4.1, we regard the last integration bound 0 as a variable λ_{r+1} and then evaluate it to 0. By Lemma 4.4, we deduce

$$\begin{aligned}\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(\ell-1)}(\lambda)\right) =&\frac{2^{k-1-r}}{\prod_{j=1}^{r}((\Omega_{k-1,n-1})_j + r-j)!}\\ &\cdot\frac{1}{\prod_1^{r}((\Omega_{k-1,n-1})_j + r+1-j)}\cdot 2\cdot a_{(\Omega_{k-1,n-1},0)}(\lambda,\lambda_{r+1})|_{\lambda_{r+1}=0}.\end{aligned}$$

From properties of alternating functions $a_{(\pi,0)}(\lambda_1,\ldots,\lambda_{r+1})|_{\lambda_{r+1}=0} = a_{(\pi+(1,\ldots,1))}(\lambda_1,\ldots,\lambda_r)$ for every partition π. Since n is odd, we have $\Omega_{k,n} = \Omega_{k-1,n-1} + (1,\ldots,1)$. This allows us to conclude:

$$\begin{aligned}\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(\ell-1)}(\lambda)\right) =&\frac{2\cdot 2^{k-1-r}}{\prod_1^{r}((\Omega_{k,n})_j + r-j-1)!}\cdot\frac{1}{\prod_1^{r}((\Omega_{k,n})_j + r-j)}a_{(\Omega_{k-1,n-1}+(1,\ldots,1))}(\lambda)\\ =&\frac{2^{k-r}}{\prod_1^{r}((\Omega_{k,n})_j + r-j)!}a_{\Omega_{k,n}}(\lambda).\end{aligned}$$

This concludes the proof for even p.

The second equality in the statement of the theorem is obtained by writing $\Omega_{k,n}$ explicitly in the denominator. □

We record separately the instance of Theorem 4.2 when $k = n-1$; by Lemma 4.1 and the discussion after that, these are the Gelfand–Tsetlin polytopes controlling the dimension of irreducible $\mathrm{SO}(n)$-representations. Indeed, when λ is a dominant integral weight for $\mathrm{SO}(n)$, the volume of $GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)$ can be recovered directly from the Weyl dimension formula, see e.g. [9].

Corollary 4.1. *Let n be a positive integer. Then*

$$\mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda)\right) = \begin{cases}\frac{2^{r-1}}{\prod_1^r(2(r-j))!}a_{(r-1,r-2,\ldots,0)}(\lambda) & \text{if } n = 2r \text{ is even}\\ \frac{2^{r}}{\prod_1^r(2(r-j)+1)!}a_{(r,r-1,\ldots,1)}(\lambda) & \text{if } n = 2r+1 \text{ is odd}\end{cases}$$

4.3 Degrees of Stiefel manifolds via volumes of Gelfand–Tsetlin polytopes

We have now completed all the preparatory work to determine the degree of $\mathrm{St}(k,n)$ when $n \leq 2k-1$.

Theorem 4.3. *Let $n \leq 2k-1$. Then*

$$\deg \mathrm{St}(k,n) = 2^k \det \left[\binom{(\Omega_{k,n})_i + (\Omega_{n-1,n})_j + 2r - i - j}{(\Omega_{k,n})_i + r - i} \right]_{1 \leq i,j \leq r}$$

Proof. From Theorem 4.1 we have

$$\deg(\mathrm{St}(k,n)) = N! \int_{\mathcal{C} \cap W} \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(n-k)}(\lambda) \right) \cdot \mathrm{vol}\left(GT^{\mathrm{SO}(n)}_{\mathrm{SO}(1)}(\lambda) \right) d\lambda,$$

and from Theorem 4.2 we write

$$\deg(\mathrm{St}(k,n)) = \frac{N! 2^{k+n-1-2r}}{\prod_{j=1}^r ((\Omega_{k,n})_j + r - j)! \cdot ((\Omega_{n-1,n})_j + r - j)!} \int_{\mathcal{C} \cap W} a_{\Omega_{k,n}}(\lambda) a_{\Omega_{n-1,n}}(\lambda) d\lambda.$$

When $n = 2r$ is even,

$$\mathcal{C} \cap W = \{(\lambda_1, \ldots, \lambda_r) | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|\}.$$

Since the integrand is alternating in λ, the integral over $\mathcal{C} \cap W$ is equal to $\frac{2}{r!}$ times the integral over Δ_r and so we may write

$$\deg(\mathrm{St}(k,n)) = \frac{N! \cdot 2^k}{r! \cdot \prod_{j=1}^r ((\Omega_{k,n})_j + r - j)! \cdot ((\Omega_{n-1,n})_j + r - j)!} \int_{\Delta_r} a_{\Omega_{k,n}}(\lambda) a_{\Omega_{n-1,n}}(\lambda) d\lambda.$$

We compute this integral using Lemma 4.3:

$$\int_{\Delta_r} a_{\Omega_{k,n}}(\lambda) a_{\Omega_{n-1,n}}(\lambda) d\lambda = \frac{r!}{(r^2 + |\Omega_{k,n}| + |\Omega_{n-1,n}|)!} \det M = \frac{r!}{N!} \det M$$

where M is the $r \times r$ matrix with (i,j)-th entry $M_{i,j} = ((\Omega_{k,n})_i + (\Omega_{n-1,n})_j + 2r - i - j)!$.

This yields

$$\deg(\mathrm{St}(k,n)) = 2^k \cdot \frac{1}{\prod_{j=1}^r ((\Omega_{k,n})_j + r - j)! ((\Omega_{n-1,n})_j + r - j)!} \cdot \det M.$$

Distributing the factor $1/((\Omega_{k,n})_j + r - j)!$ in the j-th column of the matrix and the factor $1/((\Omega_{n-1,n})_i + r - i)!$ in the i-th row provides the desired determinant when n is even.

When $n = 2r+1$ is odd, the proof is essentially the same. The only difference is that

$$\mathcal{C} \cap W = \{(\lambda_1, \ldots, \lambda_r) | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$$

therefore the integral over $\mathcal{C} \cap W$ equals $\frac{1}{r!}$ times the integral over Δ_r. Since in this case $2r+1 = n$, the power of 2 simplifies to 2^k as was the case when n was odd. $\square$

4.4 Non-intersecting lattice path interpretation

As is the case of the formula for $\deg(\mathrm{SO}(n))$ in [2], the result of Theorem 4.3 can be interpreted combinatorially in terms of non-intersecting lattice paths. We recall the Lindström–Gessel–Viennot Lemma (see e.g. [28, Thm. 2.7.1]):

Lemma 4.5 (Lindström–Gessel–Viennot [16,20])**.** *Let $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ be sets of points in $\mathbb{Z}^2$. Let $M_{i,j}$ denote the number of paths from a_i to b_j in the lattice $\mathbb{Z}^2$ using unit steps in only north and east directions. If the only way to connect all points in A to all points in B via non-intersecting paths is by connecting a_i to b_i then the number of ways to do this is given by* $\det([M_{i,j}]_{i,j=1,\ldots,r})$.

Example 4.2. *Consider the point configurations $A = \{(-3,0),(-2,0),(0,0)\}$, and $B = \{(0,4),(0,2),(0,0)\}$. Then the matrix M is given by*

$$M = \begin{bmatrix} \binom{7}{3} & \binom{5}{3} & \binom{3}{3} \\ \binom{6}{2} & \binom{4}{2} & \binom{2}{2} \\ \binom{4}{0} & \binom{2}{0} & \binom{0}{0} \end{bmatrix} = \begin{bmatrix} 35 & 10 & 1 \\ 15 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

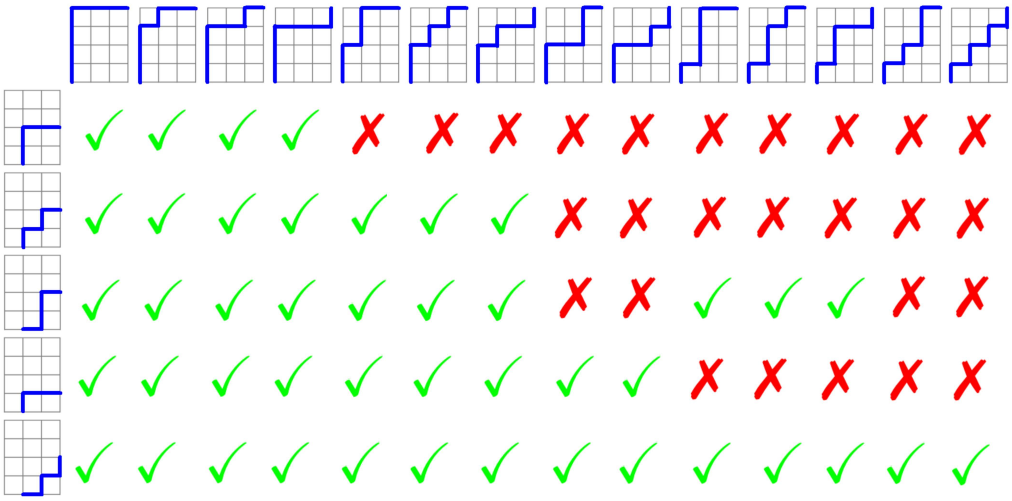


Figure 3: All non-intersecting lattice paths (green) from $A = \{(-3,0),(-2,0),(0,0)\}$ to $B = \{(0,4),(0,2),(0,0)\}$

Its determinant is 44. There is only one path from $A_3 = (0,0)$ to $B_3 = (0,0)$ and so a collection of non-intersecting lattice paths is uniquely determined by a pair of paths, one from A_1 to B_1 and another from A_2 to B_2, not passing through $(0,0)$.

Figure 3 displays paths from A_1 to B_1 in the first row and paths from A_2 to B_2 in the first column. A green ✓ indicates that the pair together with the stationary path at $(0,0)$ forms a collection of three non-intersecting lattice paths. Indeed, there are 44 green ✓'s.

Lemma 4.6. *Fix k, n with $k+1 \leq n \leq 2k-1$. Let*

$$A = \{(-((\Omega_{k,n})_j + r - j), 0) : j = 1, \dots, r\}$$
$$B = \{(0, n-2j) : j = 1, \dots, r\}.$$

The matrix in Theorem 4.3 is the matrix in the Lindström–Gessel–Viennot Lemma applied to A and B.

Proof. From the point $(-i,0)$ to $(0,j)$ there are $\binom{i+j}{i}$ paths. Notice that $n-2j = (\Omega_{n-1,n})_j + r - j$. These facts applied to A and B directly prove the result. □

Corollary 4.2. *For $k+1 \leq n \leq 2k-2$, let $L_{k,n}$ denote the number of non-intersecting lattice paths*

$$\text{from } A = \{(-(\Omega_{k,n})_j - r + j, 0)\}_{j=1}^{r} \text{ to } B = \{(0, n-2j)\}_{j=1}^{r}$$

in $\mathbb{Z}^2$ consisting of unit steps in north and east directions. The degree of $\mathrm{St}(k,n)$ is given by

$$\deg(\mathrm{St}(k,n)) = 2^k \cdot L_{k,n}.$$

Proof. By Lemma 4.6, the matrix in Theorem 4.3 is the matrix appearing in Lemma 4.5. Apply Lemma 4.5 to the sets of A and B to conclude. □

Example 4.3 (Degree of $\mathrm{St}(4,6)$). *Let $k=4$ and $n=6$. Example 4.2 calculated that $N(4,6) = 44$. Applying Corollary 4.2 computes the degree of $\mathrm{St}(4,6)$ to be*

$$\deg(\mathrm{St}(4,6)) = 2^4 \cdot 44 = 704.$$

5. Conclusions

The statements of Theorem 3.2 ($n \geq 2k-1$) and Corollary 4.2 ($n \leq 2k-1$) combine to produce the proof of Theorem 1.1. We write it explicitly for completeness.

Proof of Theorem 1.1. The first half of Theorem 1.1 is given directly by Theorem 3.2. The second half is given by writing $\Omega_{k,n}$ in the point configuration in Corollary 4.2 according to its expression in (4.2). □

Theorem 1.1 in the case $k = n-1$ gives the following corollary.

Corollary 5.1. *The degree of $\mathrm{SO}(n)$ is equal to the degree of $\mathrm{St}(n-1,n)$.*

We provide a geometric proof of this fact as well.

5.1 A geometric argument for the result of Corollary 5.1

Consider the rational map

$$\pi : \mathbb{P}(\mathrm{Mat}_{n\times n} \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathrm{Mat}_{(n-1)\times n} \oplus \mathbb{C})$$

sending an $n \times n$ matrix to the submatrix obtained by removing the first row. In other words, this is the projection with center $L = \{(A, z) : z = 0, A^{(i)} = 0 \text{ for } i > 1\}$, where $A^{(i)}$ denotes the i-th row of the $n \times n$ matrix A.

The restriction

$$\varphi : \overline{\mathrm{SO}(n)} \dashrightarrow \mathbb{P}(\mathrm{Mat}_{(n-1)\times n} \oplus \mathbb{C})$$

surjects onto $\overline{\mathrm{St}(n-1,n)}$. Since $\dim \mathrm{SO}(n) = \dim \mathrm{St}(n-1,n)$, φ is generically finite.

We show that φ is regular. To see this, it suffices to show that $\overline{\mathrm{SO}(n)}$ does not intersect the center of the projection L. Suppose $(A, z) \in L \cap \overline{\mathrm{SO}(n)}$. In particular, $z = 0$ and A is a matrix which is nonzero only in its first row and such that $AA^T = 0 \cdot \mathrm{id}_n = 0$. Notice that if $(A, z) \in \overline{\mathrm{SO}(n)}$, then $AA^T = A^T A$. This guarantees that if A is supported on a single row and $AA^T = 0$, then $A = 0$ and we conclude that $\overline{\mathrm{SO}(n)} \cap L = \emptyset$.

Moreover, φ is generically one-to-one. Indeed, let $B \in \mathrm{St}(k, n)$ and consider $(B, 1) \in \overline{\mathrm{St}(k, n)}$, so that $BB^T = \mathrm{id}_{n-1}$. The rows of B form a set of $n-1$ orthonormal vectors in $\mathbb{C}^n$; let u be the unique vector in $\mathbb{C}^n$ that is orthogonal to the vectors of B, has norm equal to 1, and forms a positively oriented basis together with the vectors of B. In particular, the matrix A obtained by placing the vector u above the matrix B is an $n \times n$ orthogonal matrix with determinant 1, and it is the unique preimage of B via φ. This shows $\deg \varphi = 1$.

Applying iteratively [24, Thm. 5.11(a)], we conclude

$$\deg \mathrm{SO}(n) = \deg \varphi(\mathrm{SO}(n)) = \deg \mathrm{St}(n-1, n).$$

5.2 A final connection to the combinatorics of domino tilings

The case $n = 2k - 1$ appearing as the overlap of Sections 3 and 4 produces the following simple combinatorial identity.

Corollary 5.2.

$$2^{\binom{r+1}{2}} = \det \left[\binom{2i}{j}\right]_{i,j=1,\ldots,r}$$

Proof. When $n = 2k-1$, the point configuration A, B given by Lemma 4.6 has the property that the first $r-j+1$ steps beginning at A_j must be vertical. Equivalently, the determinant of the path matrix associated with A and B is the same as the determinant of the path matrix associated with $\widetilde{A}, B$ where $\widetilde{A} = \{(-(r-j+1), (r-j+1)\}_{j=1}^{r}$. The new path matrix is

$$\mathcal{P} = \left[\binom{2i}{j}\right]_{i,j=1,\ldots,r}.$$

We can express $\deg(\mathrm{St}(k, 2k-1))$ by Theorem 3.2 as $2^{\binom{k+1}{2}}$ and by Theorem 4.3 as $2^k \det(\mathcal{P})$. We conclude

$$\det(\mathcal{P}) = 2^{-k} \cdot 2^{\binom{k+1}{2}} = 2^{\binom{k}{2}} = 2^{\binom{r+1}{2}}.$$

□

We could only find the result of Corollary 5.2 in a comment in the sequence `A006125` in OEIS [26]. The *Aztec diamond theorem* states that this power of two is the number of domino tilings of the Aztec diamond of order n. It was proved by Elkies, Kuperberg, Larsen, Propp in [13]. In [11], Eu and Fu provide a proof of the Aztec diamond theorem using non-intersecting lattice paths, but they do not seem to use the path matrix in Corollary 5.2.

Acknowledgements

F.G. acknowledges financial support from the VILLUM FONDEN via the QMATH Centre of Excellence (Grant no. 10059). T.B. acknowledges financial support from the National Science Foundation (DMS-1501370). This collaboration began while the authors were visiting the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during a semester-long program on *Nonlinear Algebra* in Fall 2018: we thank ICERM and the organizers for their support (NSF DMS-1439786) and for providing a wonderful research environment. We are grateful to Giorgio Ottaviani for his suggestions and comments on this project.

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