On the Activities and Partitions of the Vertex Subsets of Graphs

Kristina Dedndreaj and Peter Tittmann

Faculty of Applied Computer Sciences and Biosciences, University of Applied Sciences Mittweida, Technikumplatz 17, 09648 Mittweida, Germany
Email: dedndrea@hs-mittweida.de, peter@hs-mittweida.de

ABSTRACT: Crapo introduced a construction of interval partitions of Boolean lattice for sets equipped with matroid structure. This construction, in the context of graphic matroids, is related to the notion of edge activities introduced by Tutte. This implies that each spanning subgraph of a connected graph can be constructed from edges of exactly one spanning tree by deleting a unique subset of internally active edges and adding a unique subset of externally active edges. The family of vertex independent sets does not give rise to a matroid structure. Therefore, we cannot apply Crapo's construction on the vertex set when using the family of independent sets as generating sets. In this paper, we introduce the concept of vertex activities to tackle the problem of generating interval partitions of the Boolean lattice of the vertex set. We show how to generate a cover, present some properties related to vertex activities of some special maximal independent sets and consider some special graphs. Finally, we will show that level labellings in pruned graphs always generate a partition.

1. Introduction

1.1 Definitions and Notions

Here we will mention briefly the main concepts which are used in this paper. The terminology which is not specified here conforms to that of standard textbooks in Graph Theory.

We will deal with simple, connected and undirected graphs G with edge set $E(G)$ and vertex set $V_\leq(G)$, where $V_\leq$ denotes the fact that V is equipped with some linear ordering $\leq$. For notation and reading convenience, vertex sets will be subsets of natural numbers equipped with the natural ordering. For example, for a graph $G = (V_\leq, E)$, where $V \subset \mathbf{N}$ and $\leq$ is the natural ordering of natural numbers, we will simply write $G = (V, E)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is spanning if $V(H) = V(G)$. If H is a subgraph of G, then G is said to be a supergraph of H. If $S \subseteq V(G)$, then the *induced subgraph* $G[S]$ is the graph whose vertex set is S and its edge set consists of edges of $E(G)$ which have both endpoints in S. If $S = \{v\}$ for some $v \in V$, then we write $G[v]$.

A subset of vertices of G is called *independent* if no two vertices are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any independent set. The family of maximal independent sets of a graph G is denoted by $\mathscr{M}(G)$.

Let $v \in V$, the *open neighbourhood* of the vertex v, denoted by $N(v)$, is the set of vertices which are adjacent to v in G,

$$N(v) = \{u \in V : \{u, v\} \in E\}.$$

The *closed neighbourhood* of v, denoted $N[v]$, is the set of vertices adjacent to v together with v. That is, $N[v] = N(v) \cup \{v\}$. Analogously, for $S \subseteq V$ we define

$$N(S) = \bigcup_{v \in S} N(v)$$

and $N[S] = N(S) \cup S$. A set $S \subseteq V$ is called *dominating* if $N[S] = V$. If $A \subseteq B \subseteq V$, the set $[A; B] = \{X : A \subseteq X \subseteq B\}$ is called an *interval*. A *cover* of V is a set of non-empty intervals $[A_1; B_1], \ldots, [A_k; B_k]$ such that

$$2^V = \bigcup_{i=1}^{k} [A_i; B_i].$$

If the intervals $[A_1; B_1], \ldots, [A_k; B_k]$ are pairwise disjoint they form a *partition* or *interval partition* of V.

1.2 Background and Motivation

Crapo introduced in [2] a construction of the interval partitions of the Boolean lattice of a set equipped with matroid structure. This construction is related to the notion of activities as defined by Tutte in [8] which he used to state the polynomial which is now called after his name. The Tutte polynomial of a graph G is denoted by $T(G; x, y)$ and defined as

$$T(G; x, y) = \sum_{\text{T is a spanning tree of G}} x^{|i(T)|} y^{|e(T)|}, \tag{1}$$

where $i(T)$ is the set of internal active edges of the tree T and $e(T)$ is the set of external active edges of the tree T. This notion of activities assumes a linear order on the edge set. As shown in [8], the Tutte polynomial is independent of the linear order defined on the edge set.

Since the family of trees of a simple, connected, undirected graph gives rise to a matroid structure on the edge set, we can use this result to construct the family of spanning subgraphs of such graphs. This means that each spanning subgraph of a connected graph can be constructed from edges of exactly one spanning tree by deleting a unique subset of internally active edges and adding a unique subset of externally active edges. That is,

$$2^E = \dot{\bigcup}_{\text{T is a spanning tree of } G} [T \setminus i(T); T \cup e(T)]. \tag{2}$$

For background on the Tutte polynomial, the reader is referred to [1], [4] and [6].

The family of vertex independent sets does not give rise to a matroid structure in the vertex set because it does not satisfy the augmentation axiom. Therefore, we cannot rely on the result of Crapo [2] to generate interval partitions of the Boolean lattice using the family of vertex independent sets as generating sets. In this paper, we will introduce the notion of vertex activities with the intention of using it to generate interval partitions of the Boolean lattice of the vertex set.

2. Vertex Activities

Definition 2.1. *Let G be a graph and let A be an independent set of G. A vertex $v \in V \setminus A$ is called externally active with respect to A if there exists $a \in A$ such that $v \in N(a)$ and $v > a$. We denote the set of all externally active vertices with respect to A by $Ext(A)$. So we have*

$$Ext(A) = \{v \in A^{\complement} : \exists a \in A : a \in N(v) \text{ with } v > a\}.$$

That is, a vertex outside an independent set A is externally active with respect to A if it is adjacent to and greater than some vertex in A.

Definition 2.2. *Let G be a graph, A be an independent set of G and $v \in A$. The subset of neighbors of v which can substitute v in A while maintaining independence is denoted by $Subs(v)$. So we have*

$$Subs(v) = \{u \in N(v) : (A \setminus \{v\}) \cup \{u\} \text{ is independent}\}. \tag{3}$$

Definition 2.3. *Let G be a graph and let A be an independent set of G. A vertex $v \in A$ is called internally active in A if*

1. $Subs(v) = \emptyset$ or

2. $v > \max\{Subs(v)\}$ when $Subs(v) \neq \emptyset$.

That is, a vertex v is internally active with respect to some independent set A if v is irreplaceable by any of its neighbors with greater label. We denote by $Int(A)$ the set of all internally active vertices of an independent set A of G.
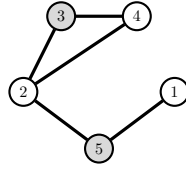
Figure 1: A graph on 5 vertices

Example 2.1. *In the graph on Figure 1, $Ext(\{3,5\}) = \{4\}$ and $Int(\{3,5\}) = \{5\}$.*

For an independent set A we call $[A \setminus Int(A); A \cup Ext(A)]$ the interval generated by A. It turns out that the activities according to Definitions 2.1 and 2.3 always generate a cover. The proof of the following theorem is inspired by the proof of Theorem 2.1 in [7].

Theorem 2.1. *Let G be a graph with $V(G) = \{1, \ldots, n\}$ and let $\mathscr{M}$ be the family of maximal independent sets of G. The activities induce a cover of the Boolean lattice of the vertex set, i.e.,*

$$\bigcup_{A \in \mathscr{M}} [A \setminus Int(A); A \cup Ext(A)] = 2^V. \tag{4}$$

Proof. Since $[A \setminus Int(A); A \cup Ext(A)] \subseteq V$ for all $A \in \mathscr{M}$ it is clear that

$$\bigcup_{A \in \mathscr{M}} [A \setminus Int(A); A \cup Ext(A)] \subseteq 2^V.$$

Now we prove the converse relation. Let $A \subseteq V$, we will construct a maximal independent set B such that

$$A \in [B \setminus Int(B); B \cup Ext(B)].$$

We arrange the vertices V in a sequence $v_1, v_2, \ldots, v_n$ such that the vertices of A come first in increasing order and the vertices of $V \setminus A$ come after in decreasing order.

$$\underbrace{\overrightarrow{v_1, v_2, \ldots, v_{|A|}}}_{A}, \underbrace{\overleftarrow{v_{|A|+1}, \ldots, v_n}}_{V \setminus A}$$

We start with the empty set and add the vertices of V in the order they appear in the sequence if the set remains independent. We set $B^{(0)} = \emptyset$ and for $i = 1, \ldots, n$

$$B^{(i)} = \begin{cases} B^{(i-1)} \cup \{v_i\} & \text{if } B^{(i-1)} \cup \{v_i\} \text{ is independent} \\ B^{(i-1)} & \text{otherwise.} \end{cases}$$

$B^{(n)}$ is an independent dominating set and thus a maximal independent set. Now we put $B = B^{(n)}$. Let v be a vertex in $A \setminus B$. This vertex is not added to B because it is adjacent to one of the vertices added earlier with lower index. Therefore, v is an external active vertex of B, that is

$$A \setminus B = A_e \subseteq Ext(B). \tag{5}$$

A vertex $v \in B \setminus A$ is added to some $B^{(i)}$, where $|A| \le i \le n$, because $B^{(i)} \cup \{v\}$ is independent and v is the first and thus the greatest vertex of $V \setminus A$ which can be added to $B^{(i)}$ while maintaining independence. Therefore, the only vertices (if any) which are in $N(v)$ and that are greater than v are to be searched in $A \setminus B$. These vertices cannot substitute v because they are adjacent to some vertices added earlier in B. Therefore, v is an internally active vertex of B, that is

$$B \setminus A = A_i \subseteq Int(B). \tag{6}$$

Combining (5) and (6) we have

$$A = (B \setminus A_i) \cup A_e \in [B \setminus Int(B); B \cup Ext(B)]. \tag{7}$$

$\square$

The intervals of the cover constructed according to (4) depend on the labelling of the vertices V and it turns out that for some labellings some subsets may be generated by more than a maximal independent set.
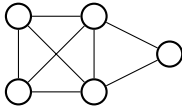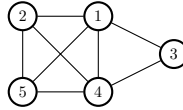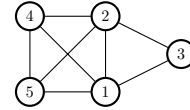
Figure 2: Graph G

Figure 3: Labelling 1

Figure 4: Labelling 2

Max. ind. set	Int. active vertices	Ext. active vertices	Generated interval
{1}	∅	{2,3,4,5}	[1; 12345]
{2,3}	{3}	{4,5}	[2; 2345]
{3,5}	{3,5}	∅	[∅; 345]
{4}	∅	{5}	[4; 45]

2 repeated sets

Table 1: Cover for labelling 1

Max. ind. set	Int. active vertices	Ext. active vertices	Generated interval
{1}	∅	{2,3,4,5}	[1; 12345]
{2}	∅	{3,4,5}	[2; 2345]
{3,4}	{3}	{5}	[4; 345]
{3,5}	{3,5}	∅	[∅; 35]

0 repeated sets

Table 2: Cover for labelling 2

Example 2.2. *We consider graph G in Figure 2 and two different labellings of its vertices as shown in Figure 3 and Figure 4. For notation convenience we will write, for example $[1; 12345]$ instead of $[\{1\}; \{1, 2, 3, 4, 5\}]$.*

In the cover constructed from Labelling 1, the sets $\{4\}$ and $\{4,5\}$ are generated by maximal independent set $\{3,5\}$ and maximal independent set $\{4\}$. This means that the cover constructed from Labelling 1 is not an interval partition. In the cover constructed form Labelling 2, all the sets are generated by exactly one maximal independent set, i.e., they belong to exactly one interval. This means that the cover constructed from Labelling 2 is an interval partition.

Definition 2.4. *Let G be a graph with vertex set $V = \{1, \ldots, n\}$. A maximal independent set S of G is called externally complete if $Ext(S) = S^{\complement}$.*

Theorem 2.2. *Let G be a graph with vertices $V = \{1, \ldots, n\}$. Then G has a unique externally complete set S.*

Proof. We will construct the required set S by using Algorithm 1.

Algorithm 1: Construction of an externally complete set

Input : A graph G with vertex set $V = \{1, \ldots, n\}$
Output : An externally complete set S of G
Initialize: $S := \emptyset$ and $A := V$
1 while *While $A \neq \emptyset$* **do**
2 | Select the vertex $v \in A$ with the smallest label and add it to S;
3 | Remove from A the vertex v and its neighbors;
4 end

According to Definition 2.4, we need to prove that the output set S of Algorithm 1 is a maximal independent set and that it satisfies $Ext(S) = S^{\complement}$. By the structure of Algorithm 2.4 it is evident that S is a maximal independent set. By definition of the external activity, it is clear that $Ext(S) \subseteq S^{\complement}$. Now we prove that $S^{\complement} \subseteq Ext(S)$. If some vertex a is not in S it means that it is adjacent to some vertex $b \in S$ with smaller label, thus $a \in Ext(S)$. To show uniqueness we assume the opposite, that the graph G has two different externally complete sets A and B. Let

$$a = \min\{A \setminus B\}$$
$$b = \min\{B \setminus A\}.$$

Without loss of generality assume that $a < b$. By external completeness of B we have

$$\forall u \in B^{\complement}, \exists v \in B : u \in N(v) \text{ and } u > v. \tag{8}$$

This means that if we put $u = a$ we should find $v \in B$ such that $a \in N(v)$ and $a > v$. Since $a \in A \setminus B$, $a \in N(v)$ and A is independent, then v cannot be in A. Thus we have $v \in B \setminus A$ and $a > v \geq b$, which is a contradiction. $\qquad\square$

Definition 2.5. *Let G be a graph with vertex set $V = \{1, \ldots, n\}$. A maximal independent set S of G is called internally complete if $Int(S) = S$.*

Theorem 2.3. *Let G be a graph with vertices $V = \{1, \ldots, n\}$. Then G contains an internally complete set S.*

Proof. We will construct the required set S by using Algorithm 2.

Algorithm 2: Construction of an internally complete set

Input   : A graph G with vertex set $V = \{1, \ldots, n\}$
Output : An internally complete set S of G
Initialize: $S := \emptyset$ and $A := V$
1 **while** *While $A \neq \emptyset$* **do**
2 $\quad$ Select the vertex $v \in A$ with the greatest label and add it to S;
3 $\quad$ Remove from A the vertex v and its neighbors;
4 **end**

According to Definition 2.5 we need to prove that the output set S of Algorithm 2 satisfies $Int(S) = S$ and that it is a maximal independent set. By the structure of Algorithm 2 it is evident that S is a maximal independent set. By the definition of internal activity, it is clear that $Int(S) \subseteq S$. Now we prove the converse. Let a be an element of S. If $Subs(a) = \emptyset$, then a is internally active. Now we consider the case when $Subs(a) \neq \emptyset$. Let $b \in Subs(a)$ such that $a < b$. Then $b \in N(c)$ for some c which has been added to S in some preceding step, otherwise the algorithm would have chosen b instead of a. As a result the set

$$(S \setminus \{a\}) \cup \{b\}$$

is not independent since b is adjacent to $c \in S \setminus \{a\}$, which is a contradiction. $\qquad\square$

Remark 2.1. *Internally complete sets are not unique. There can be constructed small graphs which have more than one internally complete set.*
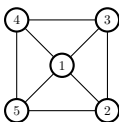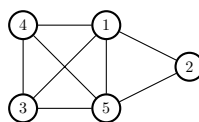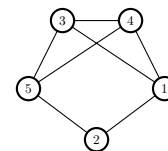

Figure 5: Figure 6: Figure 7:

Example 2.3. *The graph in Figure 5 has the internally complete sets $\{3, 5\}$ and $\{2, 4\}$. The graph in Figure 6 has the internally complete sets $\{5\}$ and $\{2, 4\}$. The graph in Figure 7 has the internally complete sets $\{1, 5\}$ and $\{2, 4\}$.*

Definition 2.6. *A maximal independent set A is called complete if it is both internally complete and externally complete.*

The following theorem is easy to see.

Theorem 2.4. *Let G be a graph with vertex set V, then the following hold:*

1. *If G has a complete maximal independent set then its cover does not generate a partition.*

2. *A graph with two internally complete sets cannot generate a partition.*

3. *If a graph G has a complete maximal independent set A then it is unique. Moreover, it generates all the subsets of V.*

Example 2.4. *The graph in Figure 8 has the complete maximal independent set $\{1, 4, 5, 6, 8, 10\}$ and the graph in Figure 9 has the complete maximal independent set $\{1, 2, 3, 7, 8, 9\}$.*
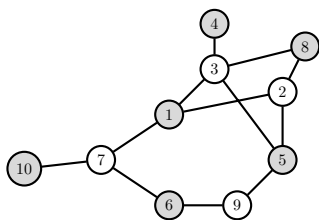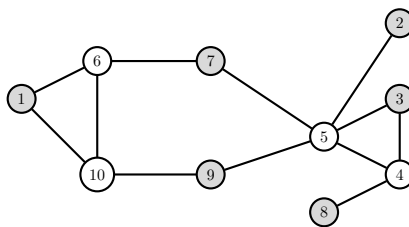
Figure 8:

Figure 9:

3.   Special Cases

In this section we will discuss classes of graphs for which we can generate an interval partition of the Boolean lattice of the vertex set with all maximal independent sets as generators.
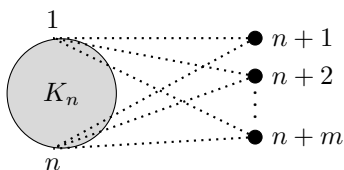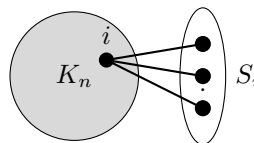

Figure 10:

Figure 11:

Definition 3.1 (Graph Join). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that V_1 and V_2 are disjoint. The join $G_1 + G_2$ of the graphs G_1 and G_2 is the graph $(V_1 \cup V_2, E_1 \cup E_2)$ together with all the edges formed by connecting all the vertices of G_1 with the vertices of G_2.*

The following results are also not difficult to see, hence we will state them without proof.

Theorem 3.1. *For the complete graph K_n the cover* (4) *always generates disjoint intervals.*

Theorem 3.2. *Let K_n be a complete graph on n vertices and E_m an empty graph on m vertices where $V(K_n) = \{1, \ldots, n\}$ and $V(E_m) = \{n+1, \ldots, n+m\}$. The cover of $K_n + E_m$ (Figure 10) always generates a partition.*

Theorem 3.3. *Let K_n be the complete graph on n vertices and let S_i be the set of vertices which are adjacent only to the vertex i such that for all $v \in S_i$ it holds $v > n$, $i = 1, \ldots, n$ (Figure 11). The cover of the graph formed by K_n together with its pendant vertices generates a partition if and only if*

1. *$|S_i| > 0$ for $i = 1, \ldots, n$ or*

2. *If there exists some set $B \subseteq V(K_n)$ such that $|S_i| = 0$ for $i \in B$ then $n \in B$.*

3.1   The Lex and Colex Graphs

Definition 3.2. *[3] Given $A, B \subset \mathbf{N}$ we say A precedes B in lexicographic (or lex) ordering, written $A <_L B$, if $\min\{A \triangle B\} \in A$.*

The lex graph, denoted $L(n, m)$, is the graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set the first m elements of $\binom{[n]}{2}$ under the lex ordering. The first few elements of the lex order on $\binom{[n]}{2}$ are

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots, \{1, n\}, \{2, 3\}, \{2, 4\}, \ldots, \{2, n\}, \{3, 4\}, \ldots$$
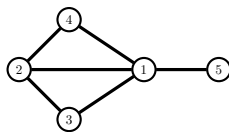

Figure 12: $L(5, 6)$

The class of the lex graphs is interesting because they are extremal for the number of independent sets. Among the graphs with n vertices and m edges the lex graph $L(n, m)$ maximizes both, the count of total independent sets and the count of independent sets of fixed size [3].

Theorem 3.4. *Let m, n be two natural numbers such that $1 \leq m \leq \binom{n}{2}$. Then there exists a unique sequence $\{p_i\}_{i=1}^k$ such that*

$$\sum_{i=1}^{k} p_i = m, \tag{9}$$

where

1. $p_i > 0$ and $1 \leq k \leq n-1$,

2. $p_i = n - i$ for $i < k$,

3. If $p_k \neq n - k$ then $p_k \in \{1, \ldots, n - k - 1\}$.

Proof. The proof can be done by double induction on m, n. $\qquad \square$

Definition 3.3. *Let m, n be two natural numbers such that $1 \leq m \leq \binom{n}{2}$. The expansion (9) satisfying conditions (1)-(3) in Theorem 3.4 is called subsequent decreasing summation decomposition of m with base n or shortly $sds(m, n)$. The number k is called depth of the summation (9) or the depth of $sds(m, n)$.*

Let $L(n, m)$ be a lex graph on n vertices and m edges such that $m \geq 1$. The depth of $sds(m, n)$ is called the depth of the lex graph $L(n, m)$. For $1 \leq i \leq n$, let $A(i) = \{v \in [n] : v < i\}$ and $B = \{k+1, \ldots, k + p_k\}$. By virtue of the definition of $L(n, m)$ we have

$$N(i) = \begin{cases} V \setminus \{i\} & \text{for } i \in \{1, \ldots, k-1\}, \\ A(i) \cup B & \text{for } i = k, \\ A(k+1) & \text{for } i \in \{k+1, \ldots, k + p_k\}, \\ A(k) & \text{for } i \in \{k + p_k + 1, \ldots, n\}. \end{cases}$$

Theorem 3.5. *Let $L(n, m)$ be an arbitrary lex graph. The cover (4) generates a partition.*

Proof. There are two cases:

1. $m < n - 1$. Then the lex graph $L(n, m)$ is composed of the star graph S_m where 1 is central and the set of isolated vertices $S = \{m+1, \ldots, n\}$. There are two maximal independent sets: the set $\{1\} \cup S$ which generates the interval $[\{1\}; \{1, \ldots, n\}]$ and the set $\{2, \ldots, n\}$ which generates the interval $[\emptyset; \{2, \ldots, n\}]$.

2. $m \geq n - 1$. Let k be the depth of the lex graph $L(n, m)$. There are two cases:

    (a) $p_k \neq n - k$. Then there are three types of maximal independent sets:

    - The sets $\{i\}$ for $i \in \{1, \ldots, k-1\}$ which generate the intervals $[\{i\}; \{i, i+1, \ldots, n\}]$, respectively.
    - The set $\{k, k + p_k + 1, \ldots, n\}$ which generates the interval $[\{k\}; \{k, \ldots, n\}]$.
    - The set $\{k+1, \ldots, n\}$ which generates the interval $[\emptyset; \{k+1, \ldots, n\}]$.

    (b) $p_k = n - k$. Then the maximal independent sets there are of the first and the third type presented in Case 2a.

The intervals are disjoint. This ends the proof. $\qquad \square$

Example 3.1. *We will construct the partition cover of the lex graph $L(5, 6)$ in Figure 12 which has depth 2 and $p_2 = 2$.*

$$\{1\} : [\{1\}; \{1, 2, 3, 4, 5\}$$
$$\{2, 5\} : [\{2\}; \{2, 3, 4, 5\}]$$
$$\{3, 4, 5\} : [\emptyset; \{3, 4, 5\}].$$

Definition 3.4. *[3] Given $A, B \subset \mathbf{N}$ we say A precedes B in colexicographic (or colex) ordering, written $A <_C B$, if $\max\{A \triangle B\} \in B$.*

The colex graph, denoted $C(n, m)$, is the graph with vertex set $[n]$ and edge set the first m elements of $\binom{[n]}{2}$ under the colex ordering. The first few elements of the colex order on $\binom{[n]}{2}$ are

$$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \ldots$$

The colex graph $C(n, m)$ has the fewest independent sets among threshold graphs on a given number of vertices and edges [5].
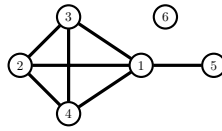
Figure 13: $C(6,7)$

Theorem 3.6. *Let m, n be two natural numbers such that $1 \leq m \leq \binom{n}{2}$. There exists a unique sequence $\{q_i\}_{i=1}^k$ such that*

$$\sum_{i=1}^k q_i = m, \tag{10}$$

where

1. *$q_i > 0$ and $1 \leq k \leq n-1$,*

2. *$q_i = i$ for $i \in \{1, \ldots, k-1\}$,*

3. *$q_k \in \{1, \ldots, k\}$.*

Proof. The proof can be done by double induction on m, n. □

Definition 3.5. *Let m, n be two natural numbers such that $1 \leq m \leq \binom{n}{2}$. The expansion (10) satisfying conditions (1)-(3) in Theorem 3.6 is called subsequent increasing summation decomposition of m with base n or shortly $sis(m, n)$. The number k is called depth of the summation (10) or the depth of $sis(m, n)$.*

Let $C(n, m)$ be a colex graph on n vertices and m edges. The depth of $sis(m, n)$ is called the depth of the colex graph $C(n, m)$. By virtue of the definition of $C(n, m)$ we have:

$$N(i) = \begin{cases} [k+1] \setminus \{i\} & \text{for } i \in \{1, \ldots, q_k\}, \\ [k] \setminus \{i\} & \text{for } i \in \{q_k+1, \ldots, k\}, \\ \emptyset & \text{for } i \in \{k+1, \ldots, n\}. \end{cases}$$

Theorem 3.7. *Let $C(n, m)$ be an arbitrary colex graph on n vertices and m edges with depth k. The cover (4) generates a partition.*

Proof. Let $S = \{v \in [n] : v > k+1\}$, this is the set of the isolated vertices of $C(n, m)$. We distinguish two cases:

1. $q_k \neq k$. In this scenario there are three possible types of maximal independent sets:

   - The sets $\{i\} \cup S$ for $i \in \{1, \ldots, q_k\}$ which generate the intervals $[\{i\}; \{i, \ldots, n\}]$, respectively.
   - The sets $\{i, k+1\} \cup S$ for $i \in \{q_k+1, \ldots, k-1\}$ which generate the intervals $[\{i\}; \{i, \ldots, n\}]$, respectively.
   - The set $\{k, k+1\} \cup S$ which generates the interval $[\emptyset; \{k, k+1\} \cup S]$.

2. $q_k = k$. There are two types of maximal independent sets:

   - The sets $\{i\} \cup S$ for $i \in \{1, \ldots, k\}$ which generate the intervals $[\{i\}; \{i, \ldots, n\}]$, respectively.
   - The set $\{k+1\} \cup S$ which generates the interval $[\emptyset; \{k+1\} \cup S]$.

The intervals are disjoint. This ends the proof. □

Example 3.2. *We will construct the partition cover of the colex graph $C(6, 7)$ in Figure 13 which has depth 4 and $q_4 = 1$.*

$$\{1, 6\} : [\{1\}; \{1, 2, 3, 4, 5, 6\}]$$
$$\{2, 5, 6\} : [\{2\}; \{2, 3, 4, 5, 6\}$$
$$\{3, 5, 6\} : [\{3\}; \{3, 4, 5, 6\}]$$
$$\{4, 5, 6\} : [\emptyset; \{4, 5, 6\}].$$

3.2   Pruned Graphs

In this section, we will deal with rooted trees and graphs which are closely related to them. Let T be a tree that is rooted in the vertex r, where $deg(r) \geq 2$. The *level* of a node v of the tree T, denoted $l(v)$, is the distance of the vertex v from the vertex r plus 1, that is, $l(v) = d(v,r) + 1$. The *k-level sets* of a tree T, denoted $L_k(T)$ or L_k when the tree T is known from the context, are the sets $\{v \in V(T) : l(v) = k\}$. As a consequence, the number of levels of T, denoted $l(T)$, can be written as

$$l(T) = \max\{k : L_k \text{ is a k-level set of } T\}. \tag{11}$$

Remark 3.1. *From now on we will assume that the trees are rooted in the center (or one of the centers, if there are two). The results hold also in the case when the trees are rooted in a vertex r such that $deg(r) \geq 2$.*

Let u,v be vertices of a tree T, the vertex u is called *child* of v if $u \in N(v)$ and $l(u) > l(v)$. The set of children of v is denoted $ch(v)$. Similarly, for $S \subseteq V(T)$, the children of S, denoted $ch(S)$, is the union of children of its elements. That is,

$$ch(S) = \bigcup_{v \in S} ch(v).$$

Definition 3.6. *Let T be a tree on n vertices. We say that T is a pruned tree if every node of every level (except the last one) has at least one leaf child.*

Definition 3.7 (Graph Union). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union $G_1 \cup G_2$ of the graphs G_1 and G_2 is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.*

Let $L(T)$ be the set of leafs of the tree T and let $v \in V(T) \setminus L(T)$. The graph $K(v)$ is the complete graph formed by v and all non-leaf vertices which are on the same level as v. By definition, $K(v) = K(u)$ when $l(v) = l(u)$. Let T be a pruned tree and H be a supergraph of T and a subgraph of

$$T \cup \bigcup_{v \in V(T) \setminus L(T)} \left((T[v] + \bigcup_{\substack{i \leq l(T) \\ i : i - l(v) \geq 2}} L_i) \cup K(v) \right). \tag{12}$$

We call such a graph a *pruned graph* of tree T. If we do not allow edges between the non-leaf vertices of the same level then we have an *inter level pruned graph* (Figure 14). All notions mentioned so far for pruned trees can be adapted for pruned graphs.
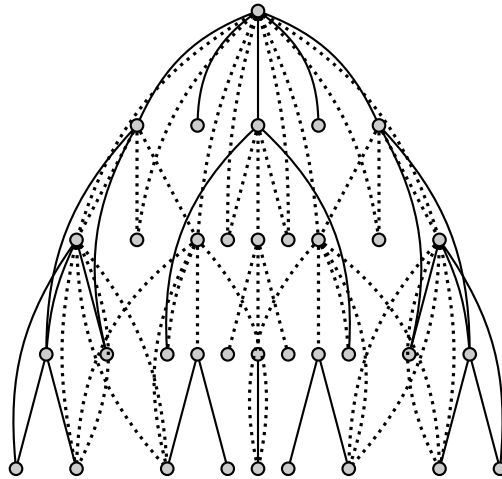

Figure 14: An inter level pruned graph

The graph in Figure 15 is a pruned tree and the graph in Figure 16 is one of its pruned graphs. Let T be a pruned tree with leaf set $L(T)$ and let H be as in (12). The nodes belonging to the same level have the same color. The colored areas are cliques. By definition, a pruned graph can have a lot of cycles and cliques. Let the set of leafs of H be $L(H)$. By (12), it is clear that $L(T)$ might be different from $L(H)$. By removing the leaves $L(H)$ from the graph H we form graph $H - L(H)$. Let the set of independent sets of $H - L(H)$ be $\mathscr{I}$ and the set of maximal independent sets of H be $\mathscr{M}$. We can describe the maximal independent sets of H by the independent sets of $H - L(H)$.

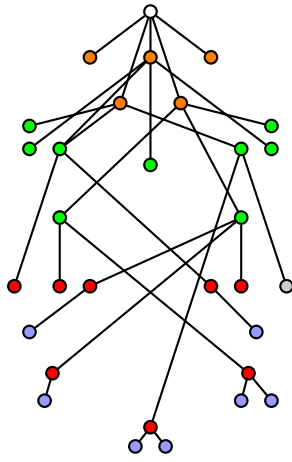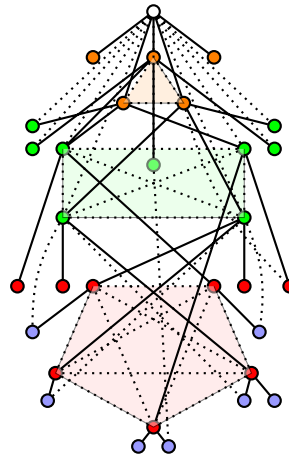In the following theorem, for notation convenience, we will write L instead of $L(H)$.

Figure 15:

Figure 16:

Theorem 3.8. *Let T be a pruned tree on n vertices and H defined as in (12). The function $f : \mathscr{I} \to \mathscr{M}$ with*

$$f(S) = S \cup \Big(L \cap \big(V \setminus ch(S) \big) \Big) \tag{13}$$

is a bijection.

Definition 3.8. *Let T be a pruned tree on n vertices. We say that T has a level labelling if for all $u, v \in V(T)$ it holds*

$$l(u) < l(v) \implies u < v. \tag{14}$$

If the pruned tree of a pruned graph has level labelling, then we say that a corresponding pruned graph has level labelling as well. The graph in Figure 17 is a pruned tree T with level labelling. The graph in Figure 18
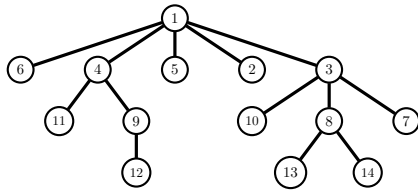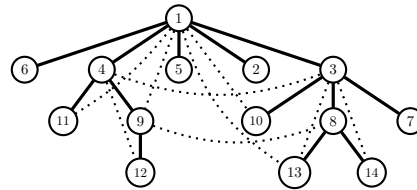

Figure 17:

Figure 18:

is a pruned graph of the tree T as defined in (12). The following is a property that can be observed on pruned graphs with level labelling.

Theorem 3.9. *Let T be a pruned tree on n vertices with level labelling, H a pruned graph of T and A a maximal independent set of H. The only internally active vertices of A are those which are leafs in T and any such vertices do not contribute to $Ext(A)$.*

Proof. Let $v \in A$ be a leaf in T. Since the only neighbour (neighbors) of v in H has (have) smaller label then $Ext(v) = \emptyset$. Also, since v is greater than all its neighbours in H which can substitute it while maintaining independence, then v is internally active in A. No non-leaf vertex $u \in A$ can be internally active in A because it can be substituted by its leaf child (children) with greater label. This ends the proof. $\qquad\square$

By combining Theorem 3.8 and Theorem 3.9 we express the non internally active elements of a set S by its pre-image.

Theorem 3.10. *Let T be a pruned tree on n vertices with level labelling, f defined as in Theorem 3.8 and H a pruned graph of T. Then a maximal independent set A of H generates the interval*

$$[f^{-1}(A); \; A \cup Ext(A)]. \tag{15}$$

Proof. In order to prove the theorem, we will show that $A \setminus Int(A) = f^{-1}(A)$. By Theorem 3.8, there exists a unique $S \in \mathscr{T}$ such that $f(S) = A$, which means

$$f(S) = S \cup \bigcup_{v \in L \cap (V \setminus ch(S))} \{v\} = A. \tag{16}$$

By Theorem 3.9, only leafs can be internally active. By construction, S contains no leaf. This implies

$$Int(f(S)) = \bigcup_{v \in L \cap (V \setminus ch(S))} \{v\}.$$

Therefore,

$$f(S) \setminus Int(f(S)) = S = A \setminus Int(A).$$

Since $f(S) = A$, then $S = f^{-1}(A)$ and $A \setminus Int(A) = f^{-1}(A)$. $\qquad\square$

Now we will show that the level labelling always generates a partition on pruned graphs.

Theorem 3.11. *If $T = (V, E)$ is a pruned tree with level labelling and H is a pruned graph of T. Then the set of intervals*

$$\{[f^{-1}(A); A \cup Ext(A)], A \in \mathscr{M}(H)\}, \tag{17}$$

is a partition of V.

Proof. We need to show that the intervals (17) are disjoint for different sets. Assume the opposite, that there exist $B, C \in \mathscr{M}$ and a nonempty set D such that

$$D \in [f^{-1}(B); B \cup Ext(B)] \text{ and } D \in [f^{-1}(C); C \cup Ext(C)]. \tag{18}$$

This means that $M = f^{-1}(B) \cup f^{-1}(C) \subset D$. Let $N = f^{-1}(B) \cap f^{-1}(C)$ and let

$$a = \min \{M \setminus N\} = \min \{f^{-1}(B) \triangle f^{-1}(C)\}, \tag{19}$$

that is, the smallest among elements which are in either $f^{-1}(B)$ or $f^{-1}(C)$ and not in their intersection. Without loss of generality assume $a \in f^{-1}(B)$. Consequently, $a \notin f^{-1}(C)$. Since $a \in D$ and D is contained in the interval generated by the set C, then $a \in Ext(C)$. By Theorem 3.9, only non-internally active vertices of C contribute to $Ext(C)$, therefore $a \in Ext(f^{-1}(C))$. Moreover, since $a \in f^{-1}(B)$ and $f^{-1}(B)$ is an independent set we have

$$a \in Ext(f^{-1}(C) \setminus N), \tag{20}$$

which means that a is adjacent to and greater than an element of $f^{-1}(C) \setminus N \subseteq M \setminus N$, a contradiction to (19). $\qquad\square$

Remark 3.2. *It is evident that Theorems 7-10 hold also in the special case of pruned trees.*

Example 3.3. *We will construct the partition cover of the pruned graph with level labelling in Figure 18 where $V = \{1, 2, \ldots, 14\}$.*

$$\{1, 7, 8, 12\} : [\{1, 8\}; V]$$
$$\{1, 7, 12, 14\} : [\{1\}; V \setminus \{8\}]$$

$$\{2, 3, 5, 6, 9, 11\} : [\{3, 9\}; V \setminus \{1\}]$$
$$\{2, 3, 5, 6, 11, 12\} : [\{3\}; V \setminus \{1, 9\}]$$

$$\{2, 4, 5, 6, 7, 8, 10\} : [\{4, 8\}; V \setminus \{1, 3\}]$$
$$\{2, 4, 5, 6, 7, 10, 13, 14\} : [\{4\}; V \setminus \{1, 3, 4\}]$$

$$\{2, 5, 6, 7, 8, 10, 11, 12\} : [\{8\}, V \setminus \{1, 3, 4\}]$$

$$\{2, 5, 6, 7, 9, 10, 11, 13, 14\} : [\{9\}; V \setminus \{1, 3, 4, 8\}]$$

$$\{2, 5, 6, 7, 10, 11, 12, 13, 14\} : [\{\emptyset\}; V \setminus \{1, 3, 4, 8, 9\}].$$

Acknowledgments

References

[1] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, pp. 123-225 in Matroid Applications (Encyclopedia of Mathematics and its Applications), ed. N. White, Cambridge University Press, Cambridge (1992).

[2] H. H. Crapo, *The Tutte polynomial*, Aequat. Math. 3:3 (1969), 211–229.

[3] J. Cutler and A. Radcliffe, *Extremal problems for independent set enumeration*, Electron. J. Combin. 18:1 (2011), #P169.

[4] J. A. Ellis-Monaghan and C. Merino, *Graph polynomials and their applications I: The Tutte polynomial*, pp. 219–255 in Structural analysis of complex networks, ed. M. Dehmer, Birkhäuser Boston (2011).

[5] L. Keough and A. Radcliffe, *Extremal threshold graphs for matchings and independent sets*, Graphs Combin. 34:6 (2018), 1519–1537.

[6] A. D. Sokal, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, pp. 173–226 in Surveys in Combinatorics (London Mathematical Society Lecture Note Series), ed. B. Webb, Cambridge University Press, Cambridge (2005).

[7] M. Trinks, *From spanning forests to edge subsets*, Ars Math. Contemp. 7:1 (2014), 141–151.

[8] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Can. J. Math. 6 (1954), 80–91.