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Bijections for Dyck Paths with Colored Hills

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ABSTRACT: In a recent paper, Janjić enumerated Dyck paths of semilength $n-1$ having colored hills with $m \in \{2,3,4\}$ colors. For $m=2$, he showed that they are also enumerated by the n-th Catalan number C_n, which implies that they are in bijection with Dyck paths of semilength n. For $m=3$, he showed that they are enumerated by $\binom{2n-1}{n}$, which implies that they are in bijection with pairs of noncrossing paths of length $n-1$. In this paper, we present new bijections between Dyck paths with colored hills with m colors and various classes of paths, for $m \in \{2,3\}$, giving bijective proofs for the above results, as well as obtaining some new enumeration results for these classes of paths.

Keywords: Dyck path; Motzkin path; bijection

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1. Introduction

A (lattice) *path* of length $n \in \mathbb{N}^* := \{1,2,3,\ldots\}$ is a finite sequence of points $(x_i, y_i)_{0 \le i \le n}$ in $\mathbb{Z}^2$, starting at the origin, i.e., $(x_0, y_0) = (0,0)$. The vectors $(x_{i+1} - x_i, y_{i+1} - y_i)$, $0 \le i \le n-1$, are the *steps* of the path. The *length* of a path P, denoted by $|P|$, is the number of its steps. The *height* of the i-th point (x_i, y_i) of a path P, denoted by $h_i(P)$, is equal to y_i and $h(P) = h_n(P)$ is the height of the final point of P.

In this work, we are concerned with lattice paths having three kinds of steps: up-steps $u = (1,1)$, down-steps $d = (1,-1)$ and horizontal steps $h = (1,0)$. The set of these paths is denoted by $\{u,d,h\}^*$, since each such path can be identified by the sequence of its steps, i.e., a word in $\{u,d,h\}^*$. Given $\tau, P \in \{u,d,h\}^*$, we say that τ *occurs* in P whenever $P = R\tau Q$, for some $R, Q \in \{u,d,h\}^*$. The height of this occurrence is equal to the minimum height of its points. A low (resp. high) occurrence is an occurrence at height 0 (resp. greater than 0). A *hill* of a path is a low occurrence of ud (the starting point of the u step has zero y-coordinate). A *peak* (resp. *valley*) is an occurrence of ud (resp. du). The number of occurrences of τ in the path P is denoted by $|P|_\tau$. In particular, $|P|_u, |P|_d, |P|_h$ denote the number of u's, d's and h's in P respectively.

Next, we give the terminology and notation used for the sets of paths we are concerned within the sequel. A partial order $\le$ is defined in the set $\{u,d\}^n$ of binary paths of length $n \in \mathbb{N} := \{0,1,2,\ldots\}$ as follows: $P \le Q$ whenever the path P lies weakly below the path Q, i.e., whenever $h_i(P) \le h_i(Q)$, for all $0 \le i \le n$. A *pair of noncrossing paths* is a pair (P,Q) with $P \le Q$. A *Motzkin prefix* is a path in $\{u,d,h\}^*$ that stays weakly above the x-axis. A *Motzkin path* is a Motzkin prefix that ends on the x-axis. A *Dyck prefix* is a Motzkin prefix with no horizontal steps (also called a *ballot path*). A *Dyck path* is a Dyck prefix that ends on the x-axis. By coloring each horizontal step of a Motzkin prefix with one out of $m \in \mathbb{N}^*$ possible colors, we obtain an *m-Motzkin prefix.* We denote these colors by the integers $1,2,\ldots,m$ and the corresponding colored horizontal steps by $h_1, h_2, \ldots, h_m$. Similarly, we can color the hills of a Dyck path to obtain a *Dyck path with m-colored hills.* We denote these colored hills by $H_1, H_2, \ldots, H_m$. Below, we list the notation used in the rest of the paper:

- ε is the empty path, i.e., the path of length 0
- $\mathcal{MP}_n^{(m)}(h)$ is the set of m-Motzkin prefixes of length n ending at height h,
  $\mathcal{MP}_n^{(m)} := \bigcup_{h\ge 0} \mathcal{MP}_n^{(m)}(h)$, $\mathcal{MP}^{(m)}(h) := \bigcup_{n\ge 0} \mathcal{MP}_n^{(m)}(h)$ and $\mathcal{MP}^{(m)} := \bigcup_{n\ge 0} \mathcal{MP}_n^{(m)}$.

- $\mathcal{M}_n^{(m)} := \mathcal{MP}_n^{(m)}(0)$ is the set of m-Motzkin paths of length n, and $\mathcal{M}^{(m)} := \bigcup_{n\geq 0}\mathcal{M}_n^{(m)}$.
- $\mathcal{DP}_n := \mathcal{MP}_n^{(0)}$ is the set of Dyck prefixes of length n.
- $\mathcal{D}_n^{(m)}$ is the set of Dyck paths of length $2n$ with m-colored hills and $\mathcal{D}^{(m)} := \bigcup_{n\geq 0}\mathcal{D}_n^{(m)}$, $m \in \mathbb{N}$.

  The case $m = 0$ corresponds to the set $\mathcal{D}^{(0)}$ of Dyck paths with no hills (also called hill-free Dyck paths). The case $m = 1$ corresponds to the set $\mathcal{D} := \mathcal{D}^{(1)}$ of (ordinary) Dyck paths and we define $\mathcal{D}^+ := \mathcal{D} \setminus \{\varepsilon\}$.
- $\mathcal{W}_n(h)$ is the set of pairs of noncrossing binary paths of length n ending $2h$ units apart, $\mathcal{W}_n := \bigcup_{h\geq 0}\mathcal{W}_n(h)$, $\mathcal{W}(h) := \bigcup_{n\geq 0}\mathcal{W}_n(h)$ and $\mathcal{W} := \bigcup_{n\geq 0}\mathcal{W}_n$.

Recently, Janjić [6] proved, using recurrence relations, the following enumeration results:

i) $|\mathcal{D}_n^{(2)}| = C_{n+1}$, where $C_n = \binom{2n}{n}/(n+1)$ is the n-th Catalan number (seq. A000108 in the OEIS [9]). As Janjić notes, this interpretation of the Catalan numbers does not exist in Stanley's book "Catalan numbers" [10].

ii) $|\mathcal{D}_n^{(3)}| = \binom{2n+1}{n}$ (seq. A001700 in the OEIS).

Combining these two results with enumeration results known from the literature, we have the following equalities:

$$|\mathcal{W}_n(0)| = |\mathcal{D}_{n+1}| = |\mathcal{M}_n^{(2)}| = |\mathcal{D}_n^{(2)}| = C_{n+1}, \tag{1}$$

$$|\mathcal{W}_n| = |\mathcal{DP}_{2n+1}| = |\mathcal{MP}_n^{(2)}| = |\mathcal{D}_n^{(3)}| = \binom{2n+1}{n}. \tag{2}$$

For the first class of sets, i.e., those appearing in relation (1), there exist known bijections between $\mathcal{W}_n(0)$ and $\mathcal{D}_{n+1}$ (see Deutsch and Shapiro [5], Manes et al. [8]), as well as two bijections from $\mathcal{M}_n^{(2)}$ to $\mathcal{D}_{n+1}$. The first one, denoted here by χ, is given by Delest and Viennot [3], whereas the second one, denoted here by η, is given by Callan [1]. Moreover, a bijection between $\mathcal{W}_n(0)$ and $\mathcal{M}_n^{(2)}$ is given by Deutsch and Shapiro [5] (as the authors note, $W_n(0)$ is also in bijection with parallelogram polyominoes of perimeter $2(n+2)$). For the second class of sets in relation (2), there exists, to our knowledge, no bijection other than the one given in [8] between $\mathcal{W}_n$ and $\mathcal{DP}_{2n+1}$.

The purpose of this paper is to provide bijections that connect $\mathcal{D}_n^{(2)}$ and $\mathcal{D}_n^{(3)}$ with the rest of the sets in their class, thus also proving (i) and (ii) combinatorially, as well as to present new enumeration results that are derived from the properties of these bijections. The rest of the paper is organized as follows: In section 2, we introduce a simple new bijection $\phi : \mathcal{D}^{(2)} \to \mathcal{D}^*$ which proves (i) and we give a new enumeration result based on ϕ. In section 3, we describe bijections χ and η, both from $\mathcal{M}_n^{(2)}$ to $\mathcal{D}_{n+1}$, we give an equivalent recursive definition for η, which we exploit to obtain new properties for η, and we derive new enumeration results on Dyck and 2-Motzkin paths, based on these properties. In section 4, we introduce bijection $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$, also proving (i), and we study some of its properties, obtaining some known and some new results. For the second class of sets, the proof of (ii) is accomplished via the bijection $\varphi_3 : \mathcal{D}^{(3)} \to \mathcal{MP}^{(2)}$, presented in section 5. Moreover, in section 6, we describe a bijection $\psi : \mathcal{W} \to \mathcal{MP}^{(2)}$, extending the bijection of Deutsch and Shapiro [5] and finally, in section 7, using the properties of ϕ, χ and ψ, we introduce a bijection $\omega : \mathcal{W} \to \mathcal{D}^{(3)}$, also proving (ii). The connections between the aforementioned sets via existing and new bijections are depicted in Fig. 1. We finally note that some of the results of this paper were presented in [7].

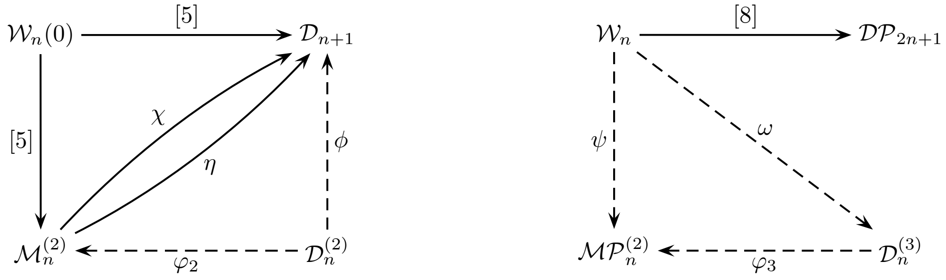

Figure 1: Arrows indicate a known bijection between corresponding sets. Dashed arrows indicate bijections introduced in this paper.

2. From Dyck paths with 2-colored hills to Dyck paths

We define a mapping $\phi : \mathcal{D}^{(2)} \to \mathcal{D}^+$ that has a simple non-recursive description; for every $\alpha \in \mathcal{D}^{(2)}$, the path $\phi(\alpha)$ is constructed in two steps as follows:

($\phi 1$) Transform each H_2 (hill with color 2) of α into a du (a valley at height -1).

($\phi 2$) Finally, add a u step at the beginning and a d step at the end of the path.

Obviously, the resulting path $\phi(\alpha)$ is a non-empty Dyck path such that $|\phi(\alpha)|_u = |\alpha|_u + 1$. The procedure is clearly reversible, so that we have a bijection, showing that $|\mathcal{D}_n^{(2)}| = |\mathcal{D}_{n+1}|$, for all $n \in \mathbb{N}$. Moreover, it is easy to check that ϕ satisfies the following properties:

i) The number of H_1's in α equals the number of ud's at height 1 in $\phi(\alpha)$.

ii) The number of H_2's in α equals the number of du's at height 0 (low valleys) in $\phi(\alpha)$.

Note that the first step ($\phi 1$) of the bijection transforms a path $a \in \mathcal{D}_n^{(2)}$ to a path $\beta \in \{u, d\}^n$ starting and ending on the x-axis and never falling below height -1. We will denote the set of such paths by $\mathcal{D}_n^-$, i.e., $\beta \in \mathcal{D}_n^- \Leftrightarrow u\beta d \in \mathcal{D}_{n+1}$. These paths appear as an intermediate product of bijections ϕ, χ, ψ, thus playing a key role in the sequel, in composing these bijections.

We close this section with a new enumeration result, derived from the properties of ϕ. The number of Dyck paths of length $2n$ with k hills is equal (e.g., see equation (6.16) in [4]) to $a_{n,k}$, where

$$a_{n,k} := \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \frac{i}{n-k-i} \binom{k+i}{k} \binom{2(n-k-i)}{n-k}, \text{ when } 0 \le k < n \quad \text{and} \quad a_{n,n} := 1.$$

Combining this result with properties (i) and (ii) of ϕ, we deduce the following:

Proposition 2.1. *The number of paths in $\mathcal{D}_n^{(2)}$ with k_1 H_1's and k_2 H_2's is equal to the number of paths in $\mathcal{D}_{n+1}$ with k_1 ud's at height 1 and k_2 du's at height 0 and equal to $\binom{k_1+k_2}{k_1} a_{n,k_1+k_2}$.*

3. From Dyck paths to 2-Motzkin paths

As noted before, there exists a folklore bijection $\chi : \mathcal{M}^{(2)} \to \mathcal{D}^+$, introduced by Delest and Viennot [3], which has a straightforward description; given a 2-Motzkin path α, the Dyck path $\chi(\alpha)$ is constructed in two steps as follows:

($\chi 1$) Replace in α each u by uu, each d by dd, each h_1 by ud, each h_2 by du.

($\chi 2$) Finally, add a u step at the beginning and a d step at the end of the path.

Obviously, bijections ϕ and χ can be combined to give a bijection $\chi^{-1} \circ \phi : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$. Note that the first step ($\chi 1$) transforms $\alpha \in \mathcal{M}_n^{(2)}$ into a path $\beta \in \mathcal{D}_n^-$. This implies that $\chi^{-1} \circ \phi$ is easily described in two steps: the step ($\phi 1$) followed by the inverse of step ($\chi 1$).

The bijection $\eta : \mathcal{M}^{(2)} \to \mathcal{D}^+$, introduced by Callan [1], is quite different from χ. Given a path $\alpha \in \mathcal{M}^{(2)}$, the path $\eta(\alpha) \in \mathcal{D}^*$ is obtained by applying the following steps:

($\eta 1$) append a d step, to obtain αd, so that every h_1 in αd has an associated d step (the first d step to the right of this h_1 that starts at the same height as h_1),

($\eta 2$) replace every d step by udd,

($\eta 3$) replace every h_2 step by ud,

($\eta 4$) replace every h_1 step by u and insert a d immediately before its associated d step and

($\eta 5$) delete the appended d, to obtain $\eta(\alpha) \in \mathcal{D}^+$.

Here, we present an equivalent recursive definition for η, based on the decompositions of the two sets: A path $\alpha \in \mathcal{M}^{(2)}$ is decomposed as

$$\alpha = \varepsilon \quad \text{or} \quad \alpha = h_1\beta \quad \text{or} \quad \alpha = h_2\beta \quad \text{or} \quad \alpha = u\beta d\gamma, \qquad \beta, \gamma \in \mathcal{M}^{(2)}.$$

On the other hand, a path $\alpha \in \mathcal{D}^+$ is decomposed as

$$\alpha = ud \quad \text{or} \quad \alpha = u\beta d \quad \text{or} \quad \alpha = ud\beta \quad \text{or} \quad \alpha = u\beta d\gamma, \qquad \beta, \gamma \in \mathcal{D}^+.$$

Then, η is equivalently defined by the following relations:

$$\eta(\varepsilon) = ud, \quad \eta(h_1\beta) = u\eta(\beta)d, \quad \eta(h_2\beta) = ud\eta(\beta), \quad \eta(u\beta d\gamma) = u\eta(\beta)d\eta(\gamma), \qquad \beta, \gamma \in \mathcal{M}^{(2)}. \tag{3}$$

Note that each replacement occurring in steps $(\eta 2) - (\eta 4)$ preserves the associated d step of each remaining h_1 and that the final result does not depend on the order in which h_1's and h_2's are replaced, as long as step $(\eta 2)$ is completed before steps $(\eta 3)$ and $(\eta 4)$ begin. Then, based on these observations, it is easy to verify that (3) indeed provides an equivalent definition of η.

We denote the inverse of η by φ_1, i.e., $\varphi_1 : \mathcal{D}^+ \to \mathcal{M}^{(2)}$ is a bijection mapping non-empty Dyck paths of length $2n+2$ to 2-Motzkin paths of length n, defined recursively as (see Fig. 2).

$$\varphi_1(ud) = \varepsilon, \quad \varphi_1(u\beta d) = h_1\varphi_1(\beta), \quad \varphi_1(ud\beta) = h_2\varphi_1(\beta), \quad \varphi_1(u\beta d\gamma) = u\varphi_1(\beta)d\varphi_1(\gamma), \qquad \beta, \gamma \in \mathcal{D}^+, \tag{4}$$

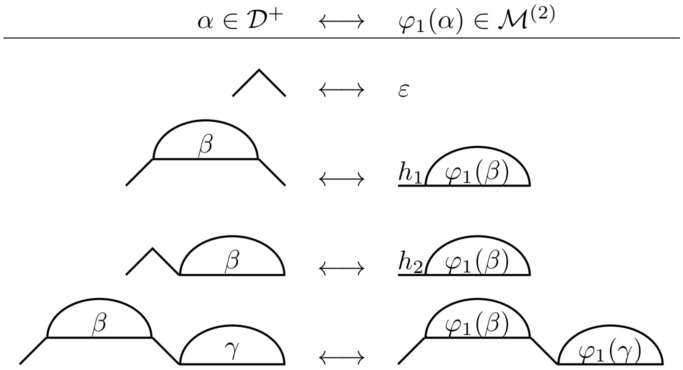

Figure 2: The bijection $\varphi_1 : \mathcal{D}^+ \to \mathcal{M}^{(2)}$.

Example For the path $\alpha = uududdduduudd \in \mathcal{D}_6$, we have

$$\varphi_1(\alpha) = u\varphi_1(udud)d\varphi_1(uduudd) = uh_2\varphi_1(ud)dh_2\varphi_1(uudd) = uh_2\varepsilon dh_2h_1\varphi_1(ud) = uh_2dh_2h_1 \in \mathcal{M}_5^{(2)}.$$

On the other hand, $\chi(uh_2dh_2h_1) = u\ uu\ du\ dd\ du\ ud\ d \neq \alpha$.

Using the recursive definition, it is easy to prove inductively that φ_1 has the following properties:

i) $|\varphi_1(\alpha)| = |\alpha|_u - 1$, i.e., $\varphi_1(\mathcal{D}_{n+1}) = \mathcal{M}_n^{(2)}$, for all $n \in \mathbb{N}$.

ii) $|\varphi_1(\alpha)|_{h_2} = |\alpha|_{udu}$,

iii) $|\varphi_1(\alpha)|_d = |\alpha|_{ddu}$,

iv) $|\varphi_1(\alpha)|_u + |\varphi_1(\alpha)|_{h_1} = |\alpha|_{uu} = |\alpha|_{dd}$,

v) $|\varphi_1(\alpha)|_{ud} = |\alpha|_{uuddu}$.

Remark Callan also derives properties (ii) and (iii) (in terms of η) and uses them to enumerate udu's and ddu's in Dyck paths (see Theorem 2 in [1]).

The recursive definitions of η and its inverse φ_1 have several advantages over the non-recursive definition consisting of steps $(\eta 1)$-$(\eta 5)$. First of all, it exposes and exploits the structural similarities of the two sets involved. Consequently, it can be modified easily to give bijections onto other sets of similar structure (e.g., ordered trees, binary trees, or any other combinatorial object counted by the Catalan numbers) which also translate statistics such as those involved in properties (ii)-(v) into equivalent statistics on these sets. Moreover, proving properties such as (i)-(v) reduces to a routine application of induction, when using such a recursive definition. Next, we give the proof of property (v), to demonstrate this advantage. The proofs of properties (i)-(iv) are similar and easier.

Proof of property (v) of φ_1. By induction on n. Let $\alpha \in \mathcal{D}_{n+1}$. The claim clearly holds for $n = 0$, i.e., when $\alpha = ud$. Assume that it holds for all Dyck paths in $\mathcal{D}_{k+1}$ and for all $k < n$. Since α is decomposed as

$$\alpha = u\beta d \quad \text{or} \quad \alpha = ud\beta \quad \text{or} \quad \alpha = u\beta' d\gamma, \quad \text{or} \quad \alpha = uudd\gamma, \qquad \beta, \gamma \in \mathcal{D}^+, \beta' \in \mathcal{D}^+ \setminus \{ud\},$$

and since, using the induction hypothesis, we have that

$$\begin{aligned}
|\varphi_1(u\beta d)|_{ud} &= |h_1\varphi_1(\beta)|_{ud} = |\varphi_1(\beta)|_{ud} = |\beta|_{uuddu} = |u\beta d|_{uuddu}, \\
|\varphi_1(ud\beta)|_{ud} &= |h_2\varphi_1(\beta)|_{ud} = |\varphi_1(\beta)|_{ud} = |\beta|_{uuddu} = |ud\beta|_{uuddu}, \\
|\varphi_1(u\beta' d\gamma)|_{ud} &= |u\varphi_1(\beta')d\varphi_1(\gamma)|_{ud} = |\varphi_1(\beta')|_{ud} + |\varphi_1(\gamma)|_{ud} = |\beta'|_{uuddu} + |\gamma|_{uuddu} = |u\beta' d\gamma|_{uuddu}, \\
|\varphi_1(uudd\gamma)|_{ud} &= |ud\varphi_1(\gamma)|_{ud} = 1 + |\varphi_1(\gamma)|_{ud} = 1 + |\gamma|_{uuddu} = |uudd\gamma|_{uuddu},
\end{aligned}$$

it follows that the claim also holds for α. □

We close this section with two new enumeration results that are derived from the properties of φ_1. The first result is immediately derived from properties (ii) and (v):

Proposition 3.1. *The number of Dyck paths of length $2n+2$ with k uuddu's and j udu's is equal to the number of 2-Motzkin paths of length n with k ud's (peaks) and j h_2's.*

This result introduces new combinatorial interpretations to several sequences in the OEIS:

- Seq. A097860, counting Motzkin paths of length n with k peaks, also counts paths Dyck paths of length $2n + 2$ with k *uuddu*'s and with no *udu*'s.

  In particular, seq. A004148, counting peakless Motzkin paths, is obtained by setting $k = 0$.
- Seq. A114848, counting Dyck paths of length $2n$ with k *uuddu*'s, also counts 2-Motzkin paths of length $n - 1$ with k peaks.

  In particular, seq. A187256, counting peakless 2-Motzkin paths, also counts Dyck paths with no *uuddu*'s.

The second result is derived from properties (i), (iii), (iv) of φ_1. For any $\alpha \in \mathcal{D}_{n+1}$, we have that

$$|\varphi_1(\alpha)|_{h_2} = |\varphi_1(\alpha)| - |\varphi_1(\alpha)|_u - |\varphi_1(\alpha)|_{h_1} - |\varphi_1(\alpha)|_d = |\alpha|_u - 1 - |\alpha|_{uu} - |\alpha|_{ddu} = n - |\alpha|_{uu} - |\alpha|_{ddu}.$$

Selecting only the paths $\alpha \in \mathcal{D}_{n+1}$ with $|\alpha|_{uu} + |\alpha|_{ddu} = n - k$, for some k such that $0 \le k \le n$, we obtain exactly the paths $\varphi_1(\alpha) \in \mathcal{M}_n^{(2)}$ with $|\varphi_1(\alpha)|_{h_2} = k$, counted by the number $\binom{n}{k} M_{n-k}$ (seq. A091869 in the OEIS), where M_n is the n-th Motzkin number M_n (seq. A001006 in the OEIS), thus deducing the following result:

Proposition 3.2. *The number of paths $\alpha \in \mathcal{D}_{n+1}$, $n \in \mathbb{N}$, with $|\alpha|_{uu} + |\alpha|_{ddu} = k$, $0 \le k \le n$, is equal to $\binom{n}{k} M_k$.*

4. From Dyck paths with 2-colored hills to 2-Motzkin paths

The mapping $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$ maps Dyck paths of length $2n$ with 2-colored hills to 2-Motzkin paths of length n. Its definition is based on the decompositions of the two sets: A non-empty path $\alpha \in \mathcal{D}^{(2)}$ is decomposed with respect to its first hill as

$$\alpha = u\alpha_1 d \cdots u\alpha_k d \quad \text{or} \quad \alpha = \beta H_1 \gamma \quad \text{or} \quad \alpha = \beta H_2 \gamma, \qquad \beta \in \mathcal{D}^{(0)}, \gamma \in \mathcal{D}^{(2)}, \alpha_i \in \mathcal{D}^+, i \in [k], k \in \mathbb{N}^*.$$

Note that the first equality corresponds to the case where $\alpha \in \mathcal{D}^{(0)}$, i.e., α is hill-free, whereas the second and third equalities correspond to the cases where $\alpha \in \mathcal{D}^{(2)} \setminus \mathcal{D}^{(0)}$, i.e., α has a hill. On the other hand, a nonempty path $\alpha \in \mathcal{M}^{(2)}$ is decomposed analogously, with respect to its first horizontal step at height 0, as

$$\alpha = u\alpha_1 d \cdots u\alpha_k d \quad \text{or} \quad \alpha = \beta h_1 \gamma \quad \text{or} \quad \alpha = \beta h_2 \gamma, \qquad \beta \in \overline{\mathcal{M}}^{(2)}, \gamma, \alpha_i \in \mathcal{M}^{(2)}, i \in [k], k \in \mathbb{N}^*,$$

where $\overline{\mathcal{M}}^{(m)}$ denotes the set of m-Motzkin paths with no horizontal steps at height 0. Then, φ_2 is defined recursively as

$$\varphi_2(\varepsilon) = \varepsilon, \quad \varphi_2(u\alpha_1 d \cdots u\alpha_k d) = u\varphi_1(\alpha_1)d \cdots u\varphi_1(\alpha_k)d, \quad \varphi_2(\beta H_i \gamma) = \varphi_2(\beta) h_i \varphi_2(\gamma), \qquad i \in \{1, 2\}, \tag{5}$$

where $a_1, \ldots, a_k \in \mathcal{D}^+$, $\beta \in \mathcal{D}^{(0)}$, $\gamma \in \mathcal{D}^{(2)}$, as depicted in Fig. 3.

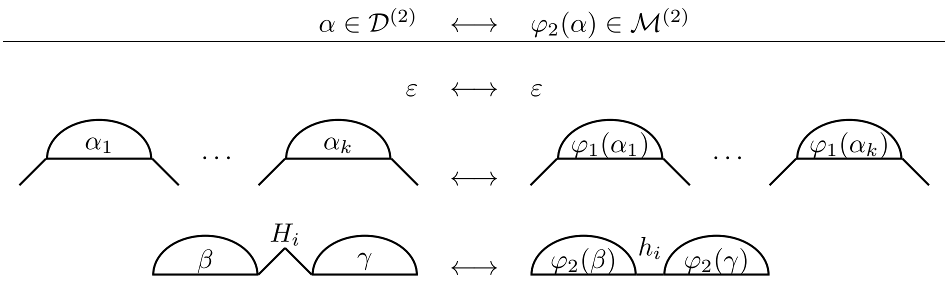

Figure 3: The bijection $\varphi_2 : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$.

Example For the path $\alpha = uududdH_1uuddH_2uuuddudd \in \mathcal{D}_{11}^{(2)}$ (recall that $|H_1| = |H_2| = 2$), we have

$$\begin{aligned}\varphi_2(\alpha) &= \varphi_2(uududd)h_1\varphi_2(uuddH_2uuuddudd) = u\varphi_1(udud)dh_1\varphi_2(uudd)h_2\varphi_2(uuuddudd)\\ &= uh_2\varphi_1(ud)dh_1u\varphi_1(ud)dh_2u\varphi_1(uuddud)d = uh_2\varepsilon dh_1u\varepsilon dh_2uu\varphi_1(ud)d\varphi_1(ud)d\\ &= uh_2dh_1udh_2uudd \in \mathcal{M}_{11}^{(2)}.\end{aligned}$$

On the other hand, $(\chi^{-1} \circ \phi)(\alpha) = \chi^{-1}(u\ uu\ du\ dd\ ud\ uu\ dd\ du\ uu\ ud\ du\ dd\ d) = uh_2dh_1udh_2uh_1h_2d \neq \varphi_2(\alpha)$, showing that φ_2 is different from $\chi^{-1} \circ \phi : \mathcal{D}^{(2)} \to \mathcal{M}^{(2)}$.

It is easy to prove inductively that φ_2 is a bijection having the following properties:

i) $|\varphi_2(\alpha)| = |\alpha|_u$.

ii) The number of hills of color 1 (resp. 2) in α equals the number of horizontal steps of color 1 (resp. 2) at height 0 in $\varphi_2(\alpha)$.

iii) The number of high udu's in α equals the number of high h_2's in $\varphi_2(\alpha)$ (according to property (ii) of φ_1).

iv) $|\varphi_2(\alpha)|_{ud} = |\alpha|_{uuddu} + [\alpha \text{ ends with } uudd] = |\alpha u|_{uuddu}$,

where $[S] := \begin{cases} 1, & \text{if } S \text{ is true,} \\ 0, & \text{if } S \text{ is false} \end{cases}$ is the Iverson bracket, applicable to any logical (true-false) statement S.

Next, we give the proof of property (iv), which is more subtle and is based on property (v) of φ_1. The proofs of properties (i)-(iii) are similar and easier.

Proof of property (iv) of φ_2. By induction on n. Let $\alpha \in \mathcal{D}_n^{(2)}$. The claim clearly holds for $n = 0$, i.e., when $\alpha = \varepsilon$. Assume that it holds for all paths in $\mathcal{D}_k^{(2)}$ and for all $k < n$. Since α is decomposed as

$$\alpha = u\alpha_1 d \cdots u\alpha_k d \quad \text{or} \quad \alpha = \beta H_1 \gamma \quad \text{or} \quad \alpha = \beta H_2 \gamma, \qquad \beta \in \mathcal{D}^{(0)}, \gamma \in \mathcal{D}^{(2)}, \alpha_i \in \mathcal{D}^+, i \in [k], k \in \mathbb{N}^*.$$

and since, using the induction hypothesis and property (v) of φ_1, we have that

$$\begin{aligned}|\varphi_2(u\alpha_1 d \cdots u\alpha_k d)|_{ud} &= |u\varphi_1(\alpha_1)d \cdots u\varphi_1(\alpha_k)d|_{ud} = \sum_{i=1}^{k} |\varphi_1(\alpha_i)|_{ud} + \sum_{i=1}^{k} [\varphi_1(\alpha_i) = \varepsilon]\\ &= \sum_{i=1}^{k} |\alpha_i|_{uuddu} + \sum_{i=1}^{k} [\alpha_i = ud] = |u\alpha_1 d \cdots u\alpha_k du|_{uuddu}\\ |\varphi_2(\beta h_i \gamma)|_{ud} &= |\varphi_2(\beta)|_{ud} + |\varphi_2(\gamma)|_{ud} = |\beta u|_{uuddu} + |\gamma u|_{uuddu} = |\beta H_i \gamma u|_{uuddu}, \qquad i \in \{1, 2\},\end{aligned}$$

it follows that the claim also holds for α. □

Remarks

- Property (ii) implies that the restriction of φ_2 on $\mathcal{D}^{(0)}$ is a bijection onto $\overline{\mathcal{M}}^{(2)}$. This verifies the well-known result that $|\mathcal{D}_n^{(0)}| = |\overline{\mathcal{M}}_n^{(2)}| = F_{n+1}$, where F_n is the n-th Fine number (seq. A000957 in the OEIS).

- Further restricting φ_2 on hill-free Dyck paths with no udu's, or equivalently udu-free Dyck paths of length $2n$ not ending with ud, we get a bijection onto $\overline{\mathcal{M}}^{(1)}$, i.e., Motzkin paths with no horizontal steps at height 0. It is known that $|\overline{\mathcal{M}}_n^{(1)}|$ is equal to the n-th Riordan number $R_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}C_k$ (seq. A005043 in the OEIS), therefore this class of Dyck paths is also enumerated by the Riordan numbers. Callan [1] refers to this class as Dyck paths with no short descents (a short descent is a d step preceded by a u step and not followed by a d step) and obtains the same result bijectively.

We close this section with a new enumeration result that is immediately derived from the properties of φ_2:

Proposition 4.1. *Let $\mathcal{D}_n^{(2)}(k,i,j)$ be the number of paths $\alpha \in \mathcal{D}_n^{(2)}$ with k hills of color 2, $|\alpha u|_{uuddu} = i$ and j high udu's and let $\mathcal{M}_n^{(2)}(k,i,j)$ be the number of 2-Motzkin paths of length n with k low h_2's, i ud's (peaks) and j high h_2's. Then, $|\mathcal{D}_n^{(2)}(k,i,j)| = |\mathcal{M}_n^{(2)}(k,i,j)|$, for all $n,k,i,j \in \mathbb{N}$.*

This result introduces new combinatorial interpretations to several sequences in the OEIS:

- Seq. A064189, counting Motzkin prefixes ending at height k, which are obtained from 2-Motzkin paths in $\bigcup_{i\geq 0}\mathcal{M}_n^{(2)}(k,i,0)$ by turning every low h_2 into a u, also counts paths in $\mathcal{D}_n^{(2)}$ with k H_2's and no high udu's.

  In particular, setting $k=0$, we deduce that Dyck paths of length $2n$ with no high udu's are counted by the n-th Motzkin number M_n, a result that was proved by Sun [11] using generating functions.

- Seq. A005773 (sums of seq. A064189 over k), counting Motzkin prefixes, also counts paths in $\mathcal{D}_n^{(2)}$ with no high udu's.

- Seq. A097724, counting peakless Motzkin prefixes ending at height k, which are obtained from paths in $\mathcal{M}_n^{(2)}(k,0,0)$ by turning every low h_2 into a u, also counts $\mathcal{D}_n^{(2)}(k,0,0)$.

  We also note that the numbers $|\mathcal{M}_n^{(2)}(k,0,0)|$ were also studied by Cameron and Sullivan [2] (using the notation $p_{n,k}^{(0)}$ for these numbers), from a different perspective.

- Seq. A004148, counting peakless Motzkin paths, i.e., paths in $\mathcal{M}_n^{(2)}(0,0,0)$, also counts $\mathcal{D}_n^{(2)}(0,0,0)$, i.e., Dyck paths of length $2n$ with no high udu's, no $uuddu$'s and not ending with $uudd$.

- Seq. A094148 (sums of seq. A097724 over k), counting peakless Motzkin prefixes, also counts paths in $\mathcal{D}_n^{(2)}$ with no high udu's, no $uuddu$'s and not ending with $uudd$.

- Seq. A187256, counting peakless 2-Motzkin paths, also counts paths in $\mathcal{D}_n^{(2)}$ with no $uuddu$'s and not ending with $uudd$.

5. From Dyck paths with 3-colored hills to 2-Motzkin prefixes

The mapping $\varphi_3 : \mathcal{D}^{(3)} \to \mathcal{MP}^{(2)}$ maps Dyck paths of length $2n$ with 3-colored hills to 2-Motzkin prefixes of length n. Its definition is based on the decompositions of the two sets. A path $\alpha \in \mathcal{D}^{(3)} \setminus \mathcal{D}^{(2)}$ is decomposed with respect to its first hill with color 3 as

$$\alpha = \beta H_3 \gamma, \qquad \beta \in \mathcal{D}^{(2)}, \gamma \in \mathcal{D}^{(3)}.$$

Analogously, a 2-Motzkin prefix $\alpha \in \mathcal{MP}^{(2)} \setminus \mathcal{M}^{(2)}$ is decomposed with respect to its last u step reaching height 1 as

$$\alpha = \beta u \gamma, \qquad \beta \in \mathcal{M}^{(2)}, \gamma \in \mathcal{MP}^{(2)}.$$

Then, φ_3 is defined recursively, using φ_2, as follows:

$$\varphi_3(\alpha) = \varphi_2(\alpha), \qquad \varphi_3(\beta H_3 \gamma) = \varphi_2(\beta) u \varphi_3(\gamma), \qquad \alpha, \beta \in \mathcal{D}^{(2)}, \gamma \in \mathcal{D}^{(3)}. \tag{6}$$

Equivalently, φ_3 can be defined as

$$\varphi_3(\alpha_0 H_3 \alpha_1 \cdots H_3 \alpha_k) = \varphi_2(\alpha_0) u \varphi_2(\alpha_1) \cdots u \varphi_2(\alpha_k), \qquad \alpha_i \in \mathcal{D}^{(2)}, 0 \leq i \leq k, k \in \mathbb{N}, \tag{7}$$

as depicted in Fig. 4. Here, the left-hand side corresponds to the decomposition of a path in $\mathcal{D}^{(3)}$ with $k \in \mathbb{N}$ H_3's, whereas the right-hand side corresponds to the decomposition of a path in $\mathcal{MP}^{(2)}(k)$. Note that, if $k=0$, then the last equality reduces to $\varphi_3(\alpha_0) = \varphi_2(\alpha_0)$, i.e., φ_3 coincides with φ_2, when restricted to $\mathcal{D}^{(2)}$.

Using this recursive definition, it is easy to prove inductively that φ_3 is a bijection having the following properties:

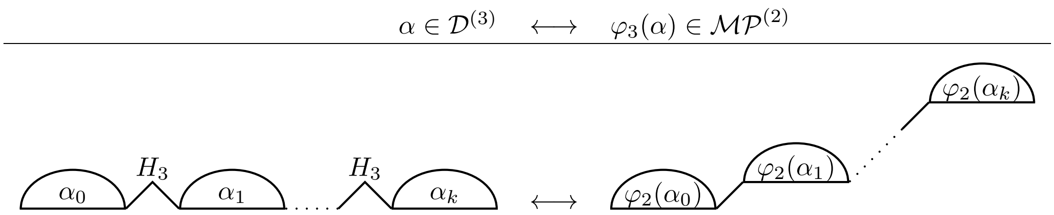

Figure 4: The bijection $\varphi_3 : \mathcal{D}^{(3)} \to \mathcal{MP}^{(2)}$.

i) $|\varphi_3(\alpha)| = |\alpha|_u$.

ii) $\varphi_3(\alpha) \in \mathcal{MP}^{(2)}(k) \Leftrightarrow |\alpha|_{H_3} = k$, i.e., the number of H_3's in α equals the ending height of $\varphi_3(\alpha)$.

iii) The number of low h_1's (resp. h_2's) in $\varphi_3(\alpha)$ equals the number of H_1's (resp H_2's) in α, before the first H_3.

In particular, $\varphi_3(\alpha)$ has no horizontal steps at height 0 iff α is either hill-free or its first hill has color 3 (according to property (ii) of φ_2).

iv) The number of high h_2's in $\varphi_3(\alpha)$ equals the number of high udu's in α plus the number of H_2's after the first H_3 (see property (iii) in φ_2).

v) $|\varphi_3(\alpha)|_{ud} = |\alpha|_{uuddu} + [\alpha \text{ ends with } uudd] = |\alpha u|_{uuddu}$ (according to property (iv) of φ_2).

Remarks

- If we choose to map each H_3 to an h_3 instead of a u, then we get a bijection onto 3-Motzkin paths where the h_3's occur only at height 0 (see a comment by Deutsch in seq. A001700 in the OEIS).
- According to the second property, the number of paths in $\mathcal{D}_n^{(3)}$ with exactly k hills of color 3 is equal to $|\mathcal{MP}^{(2)}(k)| = \dfrac{k+1}{n+1}\dbinom{2n+2}{n-k}$ (seq. A039598 in the OEIS).

We close this section with a new enumeration result that is immediately derived from the properties of φ_3:

Proposition 5.1. *The number of paths $\alpha \in \mathcal{D}_n^{(3)}$ with k hills of color 3 and $|\alpha u|_{uuddu} = i$ equals the number of paths in $\mathcal{MP}_n^{(2)}(k)$ with i ud's (peaks), for all $n, k, i \in \mathbb{N}$.*

As a special case, setting $i = 0$, we obtain the peakless 2-Motzkin prefixes ending at height k.

6. From pairs of noncrossing paths to 2-colored Motzkin prefixes

A pair $(P, Q) \in \mathcal{W}(0) \setminus \{(\varepsilon, \varepsilon)\}$ is decomposed according to the first reunion point of P, Q (a lattice point where the two paths meet after taking a step) as

$$(uP', uQ') \quad \text{or} \quad (dP', dQ') \quad \text{or} \quad (dP'uP'', uQ'dQ''), \qquad \text{where } (P', Q'), (P'', Q'') \in \mathcal{W}(0).$$

The first two cases occur whenever P and Q start with a joint step, so that their remaining parts P' and Q' clearly form a pair in $\mathcal{W}(0)$. The third case occurs whenever the initial step is not a joint step so that P must start with a d and Q with a u (since $P \leq Q$) and the paths must meet again, at their first reunion point, with an up-step for P and a down-step for Q. If $dP'u$ and $uQ'd$ are their initial parts until their first reunion point, then dP' and uQ' have no reunion points and the distance between their ending points is 2, so that $(P', Q') \in \mathcal{W}(0)$. Obviously, the remaining parts P'' and Q'' also form a pair in $\mathcal{W}(0)$.

Furthermore, a pair $(P, Q) \in \mathcal{W} \setminus \mathcal{W}(0)$ is decomposed uniquely with respect to the last reunion point as

$$(P'dP''', Q'uQ'''), \qquad \text{where } (P', Q') \in \mathcal{W}(0), (P''', Q''') \in \mathcal{W}.$$

Here, P' and Q' are the initial parts of P and Q until their last reunion point (these parts are empty if and only if no reunion point exists). The remaining parts dP''' and uQ''' have no common point so that $(P''', Q''') \in \mathcal{W}$.

Then, $\psi : \mathcal{W} \to \mathcal{MP}^{(2)}$ is defined recursively, based on the decompositions of the two sets, as (see Fig. 5)

$$\begin{aligned}&\psi(\varepsilon,\varepsilon)=\varepsilon, \quad \psi(uP',uQ')=h_1\psi(P',Q'), \quad \psi(dP',dQ')=h_2\psi(P',Q'),\\ &\psi(dP'uP'',uQ'dQ'')=u\psi(P',Q')d\psi(P'',Q''), \quad \psi(P'dP''',Q'uQ''')=\psi(P',Q')u\psi(P''',Q'''),\end{aligned} \tag{8}$$

where $(P',Q'),(P'',Q'') \in \mathcal{W}(0)$, $(P''',Q''') \in \mathcal{W}$.

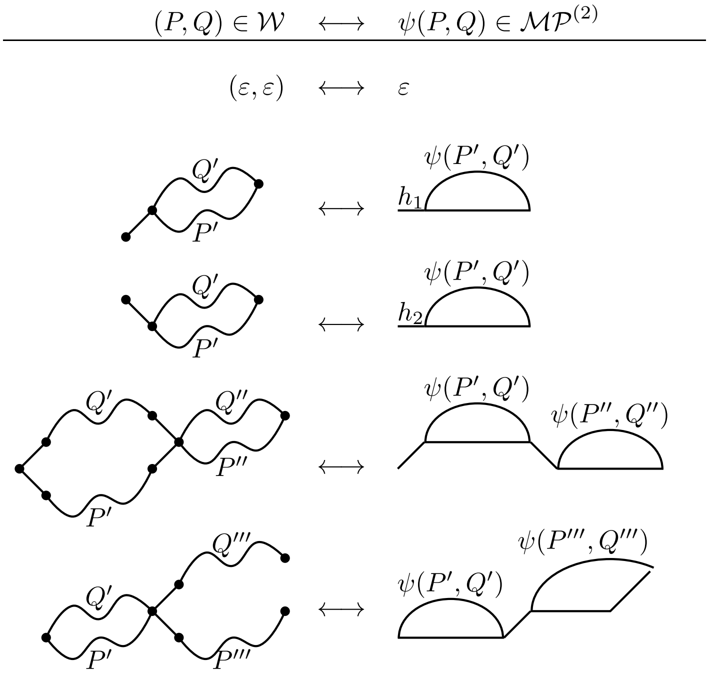

Figure 5: The bijection $\psi : \mathcal{W} \to \mathcal{MP}^{(2)}$.

Using the recursive definition of ψ, it is easy to prove inductively that ψ is a bijection having the following properties:

i) $|\psi(P,Q)| = |P|$.

ii) The restriction of ψ on $\mathcal{W}(0)$ is a bijection onto $\mathcal{M}^{(2)}$, showing that $|\mathcal{W}_n(0)| = |\mathcal{M}_n^{(2)}|$.

iii) The restriction of ψ on $\mathcal{W}(h)$ is a bijection onto $\mathcal{MP}^{(2)}(h)$.

iv) Joint u's of (P,Q) correspond to h_1's at height 0 of $\psi(P,Q)$.

v) Joint d's of (P,Q) correspond to h_2's at height 0 of $\psi(P,Q)$.

vi) ψ maps (u,u)'s to h_1's, (d,d)'s to h_2's, (d,u)'s to u's and (u,d)'s to d's.

Remarks

- The last property implies that ψ has a simple non-recursive description consisting of a single step:

  (ψ1) Read the pairs of steps of the pair (P,Q) and transform (u,u)'s into h_1's, (d,d)'s into h_2's, (d,u)'s into u's and (u,d)'s into d's.

- The restriction of ψ on $\mathcal{W}(0)$ coincides with the bijection given by Deutsch and Shapiro [5].

- The mapping $\psi^{-1} \circ \varphi_3 : \mathcal{D}^{(3)} \to \mathcal{W}$ is a bijection verifying that $|\mathcal{D}_n^{(3)}| = |\mathcal{W}_n|$.

7. From pairs of noncrossing paths to Dyck paths with 3-colored hills

Combining the steps of ϕ, χ and ψ, we define a bijection $\omega : \mathcal{W} \to \mathcal{D}^{(3)}$ with a simple description of three steps:

($\omega 1$) Each (u,u) is replaced by ud, each (d,d) by du, each (d,u) by uu and each (u,d) by dd.

($\omega 2$) Then, the uu's starting with an unmatched u at even height are replaced by H_3's. (A u step of a path is unmatched if the path contains no d step at the same height with this u step and to its right.)

($\omega 3$) Finally, each du at height -1 is turned into an H_2.

The first step ($\omega 1$) combines the steps ($\psi 1$) of ψ and ($\chi 1$) of χ: At first, the step ($\psi 1$) is applied and the pair $(P,Q) \in \mathcal{W}_n(h)$ is transformed into a 2-Motzkin prefix $\alpha \in \mathcal{MP}_n^{(2)}(h)$ of the form

$$\alpha = \alpha_0 u \alpha_1 \cdots u \alpha_h, \qquad \alpha_0, \ldots, \alpha_h \in \mathcal{M}^{(2)}, \quad |\alpha_0| + \cdots + |\alpha_h| = n - h,$$

and then ($\chi 1$) is applied to α so that each α_i, $0 \leq i \leq h$, is transformed to a path $\beta_i \in \mathcal{D}^-$, whereas the u's between the α_i's become uu's. Thus, the result of step ($\omega 1$) is a path

$$\beta = \beta_0 uu \beta_1 \cdots uu \beta_h, \qquad \beta_0, \ldots, \beta_h \in \mathcal{D}^-, \quad |\beta_0| + \cdots + |\beta_h| = 2(n-h),$$

ending at height $2h$. The uu's between the β_i's are exactly the uu's of β starting with an unmatched u at even height, so that they are clearly distinguishable (occurrences of uu starting with an unmatched u inside some β_i can only occur at heights $-1, 1, 3, \ldots$). The second step ($\omega 2$) replaces these uu's by H_3's to obtain a path $\beta' = \beta_0 H_3 \beta_1 \cdots H_3 \beta_h$ and the last step ($\omega 3$) transforms each β_i into a path in $\mathcal{D}^{(2)}$, so that β' is transformed into a path in $\mathcal{D}_n^{(3)}$ with h H_3's. The whole procedure is clearly reversible so that ω is a bijection.

A detailed example for bijection ω is given in Fig. 6. It is easy to check that the resulting path $\alpha = \omega(P,Q) \in \mathcal{D}_{18}^{(3)}$ of Fig. 6 is mapped via $\psi^{-1} \circ \varphi_3$ to a pair of noncrossing paths other than (P,Q), which shows that $\omega \neq \varphi_3^{-1} \circ \psi$.

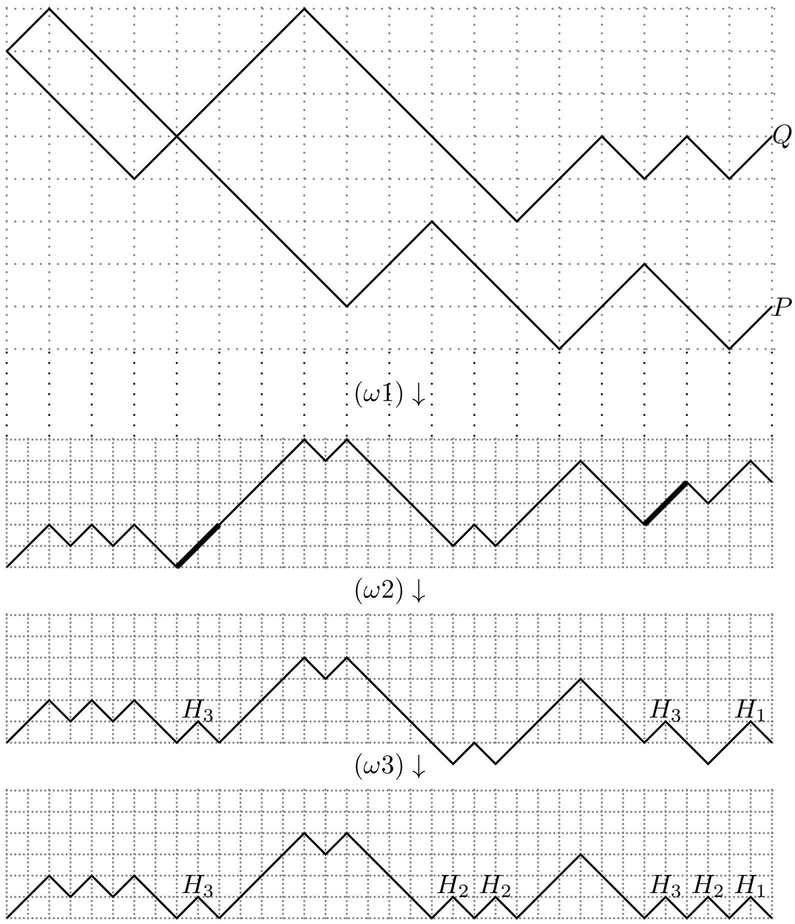

Figure 6: The bijection $\omega : \mathcal{W} \to \mathcal{D}^{(3)}$ mapping a pair $(P,Q) \in \mathcal{W}_{18}(h)$ to a path $\alpha \in \mathcal{D}_{18}^{(3)}$ with h H_3's, where $h = 2$. After the application of step ($\omega 1$), the uu's starting with an unmatched u at even height are drawn with a thicker line.

Remark The restriction of ω on $\mathcal{W}(0)$ is the bijection $\phi^{-1} \circ \chi \circ \psi : \mathcal{W}(0) \to \mathcal{D}^{(2)}$, described by omitting step ($\omega 2$).

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