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The Regularity and h-Polynomial of Cameron-Walker Graphs

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Abstract: Fix an integer $n \geq 1$, and consider the set of all connected finite simple graphs on n vertices. For each G in this set, let $I(G)$ denote the edge ideal of G in the polynomial ring $R = K[x_1, \ldots, x_n]$. We initiate a study of the set $\mathcal{RD}(n) \subseteq \mathbb{N}^2$ consisting of all the pairs (r, d) where $r = \operatorname{reg}(R/I(G))$, the Castelnuovo-Mumford regularity, and $d = \deg h_{R/I(G)}(t)$, the degree of the h-polynomial, as we vary over all the connected graphs on n vertices. In particular, we identify sets $A(n)$ and $B(n)$ such that $A(n) \subseteq \mathcal{RD}(n) \subseteq B(n)$. When we restrict to the family of Cameron-Walker graphs on n vertices, we can completely characterize all the possible (r, d).

Keywords: Castelnuovo-Mumford regularity; Edge ideals; Hilbert Series; h-polynomials

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1. Introduction

Let $R = K[x_1, \ldots, x_n]$ with K a field, and let I be a homogeneous ideal of R. In this paper we are interested in comparing $r = \operatorname{reg}(R/I)$, the regularity of R/I, with $d = \deg h_{R/I}(t)$, the degree of the h-polynomial of R/I (formal definitions are postponed until the next section) for the class of edge ideals. The first and third authors [10, 11] first showed that for any integers $1 \leq r, d$, there exists a monomial ideal $I_{r,d}$ (and in fact, a lexsegment ideal) such that $\operatorname{reg}(R/I_{r,d}) = r$ and $\deg h_{R/I_{r,d}}(t) = d$. In collaboration with the last author [13], it was later shown that the ideal $I_{r,d}$ could in fact be an edge ideal.

Given these results, it may appear that there is no relationship between the regularity and the degree of the h-polynomial, even in the case that $I = I(G)$ is an edge ideal of a graph G. However, our starting point is the following inequality found in [13, Theorem 13]; namely, if G is a graph on n vertices, then

$$\operatorname{reg}(R/I(G)) + \deg h_{R/I(G)}(t) \leq n, \tag{1}$$

which gives a bound on the possible values of r and d. If we fix an n and compute

$$(r, d) = (\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t))$$

for all connected graphs G on $n = |V(G)|$ vertices, and plot the corresponding pairs, some interesting patterns appear. Using *Macaulay2* [4], we computed (r, d) for all connected graphs on nine or fewer vertices. Figure 1 shows all the possible (r, d) for graphs on 8, respectively, 9 vertices. In particular, it is tantalizing to ask if the set of all possible (r, d) for a fixed n can be described as the integer points of some convex lattice polytope.

To study this question, for each integer $n \geq 1$ we define:

$$\mathcal{RD}(n) \;=\; \left\{ (r, d) \;\middle|\; \begin{array}{c} \text{there exists a connected graph } G \text{ with } |V(G)| = n \\ \text{and } (r, d) = (\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t)) \end{array} \right\} \subseteq \mathbb{N}^2.$$

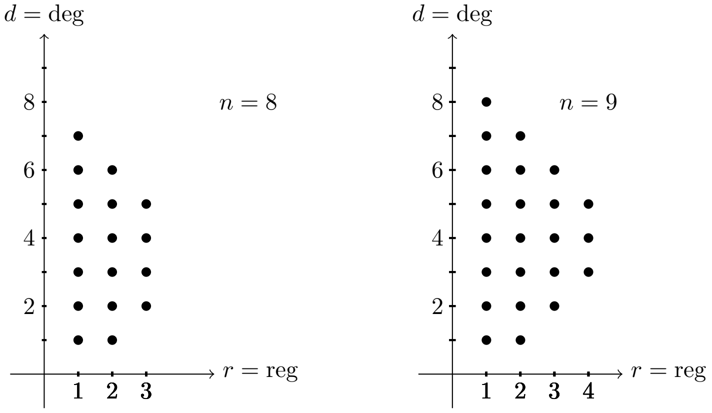


Figure 1: Possible $(r, d) = (\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t))$ for all connected graphs G on 8 and 9 vertices

One of our main results (see Theorem 3.2) describes finite subsets $A(n), B(n) \subseteq \mathbb{N}^2$ such that $A(n) \subseteq \mathcal{RD}(n) \subseteq B(n)$. Both $A(n)$ and $B(n)$ are the integer points of convex lattice polytopes.

Our results are stronger when we restrict to the connected graphs on n vertices that are also *Cameron-Walker graphs*. Cameron-Walker graphs are those graphs G which satisfy the property that the induced matching number of G equals the matching number of G; this family was first characterized by Cameron and Walker [2]. From a combinatorial commutative algebra point-of-view, these graphs are attractive since $\operatorname{reg}(R/I(G))$ is also equal to the induced matching number. In fact, a number of their algebraic properties have been developed, e.g., see [6, 7, 9]. The following classification is one of our main results:

Theorem 1.1 (Theorem 5.1)**.** *Fix an $n \geq 5$. Then there exists a Cameron-Walker graph G on n vertices with $\operatorname{reg}(R/I(G)) = r$ and $\deg h_{R/I(G)}(t) = d$ if and only if*

- $2 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$,
- $r \leq d \leq n - r$, *and*
- $d \geq -2r + n + 1$.

The pairs (r, d) in the above result form the integer points of a convex lattice polytope.

Our paper is structured as follows. In Section 2 we present the required background, including the undefined terminology from the introduction. In Section 3, we derive some properties about $\mathcal{RD}(n)$. In Section 4, we introduce Cameron-Walker graphs, and describe some of their relevant homological invariants. In Section 5, we give our proof to Theorem 5.1. This result is used to count the number of integer points in the lattice polytope defined by Theorem 5.1. Our final section includes some questions and observations about the ratio $|CW_{\mathcal{RD}}(n)|/|\mathcal{RD}(n)|$ as we vary n.

As a final comment, although our discussion in this introduction has been restricted to monomial ideals, some results are known about the pairs (r, d) for non-monomial ideals. In particular, the first and third authors [12] showed that for all $2 \leq r \leq d$, there is a binomial edge ideal J_G with regularity r and h-polynomial of degree d; Kahle and Krüsemann [14] have shown that for each integer $k \geq 0$, there exists a binomial edge ideal J_G with $r - d = k$. Finally, Favacchio, Keiper, and the last author [3] have shown that if $4 \leq r \leq d$, there is a toric ideal of a graph with regularity r and h-polynomial with degree d.

2. Background

In this section, we recall some of the relevant prerequisites about homological invariants, graph theory, and combinatorial commutative algebra. We have also included the formal definitions of the undefined terms from the introduction.

2.1 Homological Invariants

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with $\deg x_i = 1$ for all i. For any ideal I of R, the *dimension* of R/I, denoted $\dim R/I$, is the length of the longest chain of prime ideals in R/I.

If $I \subseteq R$ is a homogeneous ideal, then the *Hilbert series* of R/I is

$$H_{R/I}(t) = \sum_{i \geq 0} \dim_K [R/I]_i t^i$$

where $[R/I]_i$ denotes the i-th graded piece of R/I. If $\dim R/I = d$, then the Hilbert series of R/I is of the form

$$H_{R/I}(t) = \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_s t^s}{(1-t)^d} = \frac{h_{R/I}(t)}{(1-t)^d},$$

where each $h_i \in \mathbb{Z}$ ([1, Proposition 4.4.1]) and $h_{R/I}(1) \neq 0$. We say that

$$h_{R/I}(t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_s t^s$$

with $h_s \neq 0$ is the *h-polynomial* of R/I.

The (*Castelnuovo-Mumford*) *regularity* of R/I, with I homogeneous, is

$$\operatorname{reg}(R/I) = \max\{j - i \mid \beta_{i,j}(R/I) \neq 0\}$$

where $\beta_{i,j}(R/I)$ denotes an (i,j)-th graded Betti number in the minimal graded free resolution of R/I. (For more details see, for example, [16, Section 18].)

2.2 Graph theory

Let $G = (V(G), E(G))$ be a finite simple graph (i.e., a graph with no loops and no multiple edges) on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$.

A subset $S \subset V(G)$ is an *independent set* of G if $\{x_i, x_j\} \notin E(G)$ for all $x_i, x_j \in S$. In particular, the empty set $\emptyset$ is an independent set.

A subset $\mathcal{M} \subset E(G)$ is a *matching* of G if $e \cap e' = \emptyset$ for any $e, e' \in \mathcal{M}$ with $e \neq e'$. A matching $\mathcal{M}$ of G is called an *induced matching* of G if for $e, e' \in \mathcal{M}$ with $e \neq e'$, there is no edge $f \in E(G)$ with $e \cap f \neq \emptyset$ and $e' \cap f \neq \emptyset$. The *matching number* $\mathrm{m}(G)$ of G is the maximum cardinality of the matchings of G. Similarly, the *induced matching number* $\mathrm{im}(G)$ of G is the maximum cardinality of the induced matchings of G. Because an induced matching is also a matching, we always have $\mathrm{im}(G) \leq \mathrm{m}(G)$.

The S-suspension ([8, p.313]) of a graph G plays an important role in our results; we recall this construction. If $G = (V(G), E(G))$ is a finite simple graph, then for any independent set $S \subset V(G) = \{x_1, \ldots, x_n\}$, we construct the graph G^S with the vertex and the edge sets given by:

- $V(G^S) = V(G) \cup \{x_{n+1}\}$, where x_{n+1} is a new vertex, and
- $E(G^S) = E(G) \cup \{\{x_i, x_{n+1}\} \mid x_i \notin S\}$.

That is, we add a new vertex x_{n+1} and join it to every vertex *not* in S. The graph G^S is called the *S-suspension* of G. Note that this construction still holds if $S = \emptyset$.

2.3 Combinatorial commutative algebra

Graphs can be studied algebraically by employing the edge ideal construction. If $G = (V(G), E(G))$ is a finite simple graph on $V(G) = \{x_1, \ldots, x_n\}$, we associate with G the quadratic square-free monomial ideal

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E(G) \rangle \subseteq R = K[x_1, \ldots, x_n].$$

The ideal $I(G)$ is the *edge ideal* of the graph G. We sometimes write $K[V(G)]$ for the polynomial ring $K[x \mid x \in V(G)]$.

Under this construction, invariants of G and homological invariants of $I(G)$ are then related. For example, it is known that

$$\dim R/I(G) = \max\{|S| \mid S \text{ is an independent set of } G\}.$$

Another relevant example of this behaviour is the following lemma.

Lemma 2.1. *For any finite simple graph $G = (V(G), E(G))$ on n vertices, we have*

$$\mathrm{im}(G) \leq \operatorname{reg}(R/I(G)) \leq \mathrm{m}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. The first inequality is [15, Lemma 2.2], and the second inequality is [5, Theorem 6.7]. The last inequality follows from the observation that $\mathrm{m}(G)$ edges in G contain $2\mathrm{m}(G)$ distinct vertices, so $2\mathrm{m}(G) \leq n$. ☐

If G is a graph with an S-suspension G^S, then by virtue of [8, Lemma 1.5], we have some relationships between the homological invariants of $I(G)$ and $I(G^S)$.

Lemma 2.2. *Let G be a finite simple graph on $V(G) = \{x_1, \dots, x_n\}$, and suppose that G^S is the S-suspension of G for some independent set S of $V(G)$. If $I(G) \subseteq R = K[x_1, \dots, x_n]$ and $I(G^S) \subseteq R' = K[x_1, \dots, x_n, x_{n+1}]$ are the respective edge ideals, then*

1. $\operatorname{reg}(R'/I(G^S)) = \operatorname{reg}(R/I(G))$ *if G has no isolated vertices.*
2. $$H_{R'/I(G^S)}(t) = H_{R/I(G)}(t) + \frac{t}{(1-t)^{|S|+1}}.$$
   In particular, $\deg h_{R'/I(G^S)}(t) = \deg h_{R/I(G)}(t)$ *if* $|S| = \dim R/I(G) - 1$.
3. $\dim R'/I(G^S) = \dim R/I(G)$ *if* $|S| \leq \dim R/I(G) - 1$.

Let H_1 and H_2 be finite simple graphs, and let $H = H_1 \cup H_2$ the disjoint union of H_1 and H_2. Then one has the following identities.

Lemma 2.3. *Under the above situation, we have*

1. $\operatorname{reg}(K[V(H)]/I(H)) = \operatorname{reg}(K[V(H_1)]/I(H_1)) + \operatorname{reg}(K[V(H_2)]/I(H_2))$.
2. $\deg h_{K[V(H)]/I(H)}(t) = \deg h_{K[V(H_1)]/I(H_1)}(t) + \deg h_{K[V(H_2)]/I(H_2)}(t)$.

Proof. The result follows from the fact that $K[V(H)]/I(H)$ is the tensor product of $K[V(H_1)]/I(H_1)$ and $K[V(H_2)]/I(H_2)$. □

3. Properties of the set $\mathcal{RD}(n)$

Recall from the introduction that for each $n \geq 1$, the set $\mathcal{RD}(n)$ compares the regularity and the degree of the h-polynomial over all connected graphs on n vertices. The purpose of this section is to derive some basic properties of this set. We begin with the following observations, which rely heavily on the S-suspension construction.

Lemma 3.1. *For all $n \geq 1$, we have $\mathcal{RD}(n) \subseteq \mathcal{RD}(n+1)$.*

Proof. Let $(r, d) \in \mathcal{RD}(n)$. Then there exists a connected graph G with n vertices such that $\operatorname{reg}(R/I(G)) = r$ and $\deg h_{R/I(G)}(t) = d$. Take an independent set S of G with $|S| = \dim R/I(G) - 1$. This is possible since there is an independent set W with $|W| = \dim R/I(G)$, so we can take $S = W \setminus \{w\}$ for any $w \in W$. By virtue of Lemma 2.2 (1) and (2), we have $\operatorname{reg}(R'/I(G^S)) = r$ and $\deg h_{R'/I(G^S)}(t) = d$. Since G^S is a graph on $n+1$ vertices, we have $(r, d) \in \mathcal{RD}(n+1)$. □

Lemma 3.2. *Let $n_1, \dots, n_p \geq 2$ be integers. Suppose that $(r_i, d_i) \in \mathcal{RD}(n_i)$ for all $i = 1, \dots, p$. Then $(r_1 + \cdots + r_p, d_1 + \cdots + d_p) \in \mathcal{RD}(n_1 + \cdots + n_p + 1)$.*

Proof. Let G_i denote a connected graph with n_i vertices such that

$$\operatorname{reg}(K[V(G_i)]/I(G_i)) = r_i \text{ and } \deg h_{K[V(G_i)]/I(G_i)}(t) = d_i$$

for all $i = 1, \dots, p$. Let us consider the disjoint union $G = G_1 \cup \cdots \cup G_p$. By virtue of Lemma 2.3, one has $\operatorname{reg}(K[V(G)]/I(G)) = r_1 + \cdots + r_p$ and $\deg h_{K[V(G)]/I(G)}(t) = d_1 + \cdots + d_p$. Let $S \subset V(G)$ be an independent set of G with $|S| = \dim K[V(G)]/I(G) - 1$. Then the S-suspension G^S is a connected graph, since there is at least one vertex in G_i that is not in S for each i. Furthermore, G^S has $n_1 + \cdots + n_p + 1$ vertices and $\operatorname{reg}(K[V(G^S)]/I(G^S)) = r_1 + \cdots + r_p$ and $\deg h_{K[V(G^S)]/I(G^S)}(t) = d_1 + \cdots + d_p$ by Lemma 2.2. Hence we have the desired conclusion. □

We now focus on the elements of $\mathcal{RD}(n)$. Our starting point is the next lemma which identifies some elements of this set. To prove this lemma, we require the following two graphs. The ribbon graph denoted G_{ribbon}, is the graph on five vertices as given in Figure 2. The regularity and the degree of the h-polynomial for $K[V(G_{\text{ribbon}})]/I(G_{\text{ribbon}})$ are computed in [13, Example 10] (or can be computed via a computer algebra system):

$$\operatorname{reg}(K[V(G_{\text{ribbon}})]/I(G_{\text{ribbon}})) = 2 \text{ and } \deg h_{K[V(G_{\text{ribbon}})]/I(G_{\text{ribbon}})}(t) = 1.$$

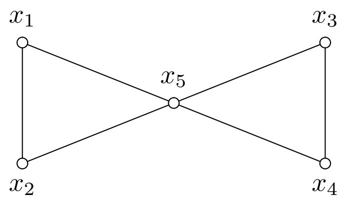
$G_{\text{ribbon}} =$ 

Figure 2: The graph G_{ribbon}

Our second family is D_r, where D_r is a graph on $2r$ vertices consisting of the disjoint union of r paths of length 1. In this case $I(D_r)$ is a complete intersection since $I(D_r) = \langle x_1x_2, x_3x_4, \ldots, x_{2r-1}x_{2r}\rangle$ is generated by r monomials which have pairwise disjoint support. So, by properties of complete intersections,

$$H_{K[V(D_r)]/I(D_r)}(t) = \frac{(1+t)^r}{(1-t)^r},$$

and consequently, $h_{K[V(D_r)]/I(D_r)}(t) = (1+t)^r$ and $\dim K[V(D_r)]/I(D_r) = r$. Moreover, since the Koszul complex gives a minimal free resolution of $K[V(D_r)]/I(D_r)$, we have $\operatorname{reg}(K[V(D_r)]/I(D_r)) = r$.

Lemma 3.3. *Let $r \geq 1$, $d \geq 1$ be integers.*

1. *Then $(r, 1) \in \mathcal{RD}(2^r + r - 1)$.*
2. *If $r < d$, then $(r, d) \in \mathcal{RD}(r + d)$.*
3. *If $r \geq 2$, then $(r, d) \notin \mathcal{RD}(2r)$. In particular, if $(r, d) \in \mathcal{RD}(n)$, then $r \leq \lfloor \frac{n-1}{2} \rfloor$.*
4. *If $r = d \geq 2$, then $(r, d) = (r, r) \in \mathcal{RD}(2r + 1)$.*
5. *If $r = d + 1$ and r is even (respectively, r is odd), then $(r, d) = (r, r-1) \in \mathcal{RD}(2r + 1)$ (respectively, $(r, d) = (r, r-1) \in \mathcal{RD}(2r + 2)$).*
6. *Let c be an integer with $c \geq 1$. If $r \geq d + 2$ and $cd < r \leq (c + 1)d$, then*

$$(r, d) \in \mathcal{RD}\left((2^c + 1)r - ((c-1)2^c + 1)d + 1\right).$$

Proof. Statement (1) follows from [13, Lemma 12] which constructs a connected graph G on $2^r + r - 1$ vertices that has $\operatorname{reg}(K[V(G)]/I(G)) = r$ and $\deg h_{K[V(G)]/I(G)}(t) = 1$.

To prove (2), let D_r be the graph defined prior to this lemma. Let S_1 be an independent set of D_r with $|S_1| = r$ (for example, take one vertex from each path of length one). The S-suspension graph $B_1 = D_r^{S_1}$ has $2r + 1$ vertices, and by Lemma 2.2 (1) $\operatorname{reg}(K[V(B_1)]/I(B_1)) = r$ and by Lemma 2.2 (2)

$$H_{K[V(B_1)]/I(B_1)}(t) = \frac{(1+t)^r}{(1-t)^r} + \frac{t}{(1-t)^{r+1}} = \frac{(1+t)^r(1-t) + t}{(1-t)^{r+1}}$$

and so $\deg h_{K[V(B_1)]/I(B_1)}(t) = r + 1$.

We now reiterate this process. Let S_i be the independent set of B_{i-1} of size $r + i - 1$ that contains the r independent elements of S_1 and $y_1, \ldots, y_{i-1}$ where y_j was the new vertex we added when we constructed $B_j = B_{j-1}^{S_j}$ by forming the S-suspension of B_{j-1} with S_j. Each set S_i is independent because each new y_j is only adjoined to the vertices not in S_1 in D_r. By induction on i, Lemma 2.2 implies that the graph B_i satisfies $\operatorname{reg}(K[V(B_i)]/I(B_i)) = r$ and $\deg h_{K[V(B_i)]/I(B_i)} = r + i$. It then follows that B_{d-r} has $2r + d - r = r + d$ vertices, $\operatorname{reg}(K[V(B_{d-r})]/I(B_{d-r})) = r$ and $\deg h_{K[V(B_{d-r})]/I(B_{d-r})}(t) = d$. So $(r, d) \in \mathcal{RD}(r + d)$.

For the proof of (3), we assume that $(r, d) \in \mathcal{RD}(2r)$. Then there exists a connected simple graph G with $\operatorname{reg}(K[V(G)]/I(G)) = r \geq 2$ and $|V(G)| = 2r$. By Lemma 2.1, we have $r = \operatorname{reg}(K[V(G)]/I(G)) \leq \operatorname{m}(G) \leq \lfloor \frac{2r}{2} \rfloor$, that is, $\operatorname{reg}(K[V(G)]/I(G)) = \operatorname{m}(G) = r$. If $\operatorname{im}(G) = r$, then $G = D_r$, a contradiction for the connectivity of G. Hence $\operatorname{im}(G) < \operatorname{reg}(K[V(G)]/I(G)) = \operatorname{m}(G)$, which implies that G is not a Cameron-Walker graph (see Definition 4.1 in the next section). But [17, Theorem 11] states if $\operatorname{reg}(K[V(G)]/I(G)) = \operatorname{m}(G)$, then G is a Cameron-Walker graph or a pentagon. So G is a pentagon. But this contradictions the fact that G has an even number of vertices. Thus $(r, d) \notin \mathcal{RD}(2r)$.

For the proof of (4), again consider the graph D_r, and let S be an independent set with $|S| = r - 1 = \dim K[V(D_r)]/I(D_r) - 1$. Then by Lemma 2.2 (1) and (2), the ring $K[V(D_r^S)]/I(D_r^S)$ has regularity r and $\deg h_{K[V(D_r^S)]/I(D_r^S)}(t) = \deg h_{K[V(D_r)]/I(D_r)]}(t) = r$. Since D_r^S has $2r + 1$ vertices, $(r, r) \in \mathcal{RD}(2r + 1)$.

To prove (5), first assume that r is even. Let D_r be as above, and consider the S-suspension with $S = \emptyset$. By Lemma 2.2 (1), the regularity of $K[V(D_r^{\emptyset})]/I(D_r^{\emptyset})$ equals r, while

$$H_{K[V(D_r^{\emptyset})]/I(D_r^{\emptyset})}(t) = H_{R/I(D_r)}(t) + \frac{t}{1-t} = \frac{(1+t)^r}{(1-t)^r} + \frac{t}{1-t} = \frac{(1+t)^r + t(1-t)^{r-1}}{(1-t)^r}.$$

Because r is even, when we simplify the h-polynomial we find $\deg h_{K[D_r^{\emptyset}]/I(D_r^{\emptyset})}(t) = r-1$. So $(r, r-1) \in \mathcal{RD}(2r+1)$.

If we instead assume that r is odd, consider the graph G which is the disjoint union of $D_{r-1}^{\emptyset}$ and D_1. Then $|V(G)| = 2r+1$, $\operatorname{reg}(K[V(G)]/I(G)) = \dim K[V(G)]/I(G) = r$, and $\deg h_{K[V(G)]/I(G)}(t) = r-1$. Let S be an independent set of G with $|S| = r-1$. By Lemma 2.2 the S-suspension of G creates a graph with $(r, r-1) \in \mathcal{RD}(2r+2)$.

Finally, we give proof of (6). We set $i = r - d \geq 2$. Note that

$$(r, r-i) = (cr - (c+1)i) \cdot (c, 1) + (ci - (c-1)r) \cdot (c+1, 1).$$

By virtue of (1), one has $(c, 1) \in \mathcal{RD}(2^c + c - 1)$ and $(c+1, 1) \in \mathcal{RD}(2^{c+1} + c)$. Then, since $i = r - d$, it follows that

$$\begin{aligned}(r, d) = (r, r-i) &\in \mathcal{RD}\left((cr - (c+1)i)(2^c + c - 1) + (ci - (c-1)r)(2^{c+1} + c) + 1\right) \\ &= \mathcal{RD}\left((2^c + 1)r - ((c-1)2^c + 1)d + 1\right)\end{aligned}$$

by virtue of Lemma 3.2. We now have the desired conclusion. ☐

By virtue of Lemmas 3.1 and 3.3, we have the following theorem. Recall that if $(r, d) \in \mathcal{RD}(n)$, then $r \leq \lfloor \frac{n-1}{2} \rfloor$ by Lemma 3.3 (3) and $r + d \leq n$ by (1).

Theorem 3.1. *Let $r \geq 1$, $d \geq 1$, and $n \geq 3$ be integers. Assume that $r \leq \lfloor \frac{n-1}{2} \rfloor$ and $r + d \leq n$. Then*

1. *If $r < d$, then $(r, d) \in \mathcal{RD}(n)$.*
2. *If $r = d \geq 2$ and $r + d = r + r < n$, then $(r, d) = (r, r) \in \mathcal{RD}(n)$.*
3. *If $r = d + 1$ and $r < \lfloor \frac{n-1}{2} \rfloor$, then $(r, d) = (r, r-1) \in \mathcal{RD}(n)$.*

Proof. For the proof of (1), assume that $r < d$. Since $r + d \leq n$, we have $(r, d) \in \mathcal{RD}(r+d) \subseteq \mathcal{RD}(n)$ by virtue of Lemmas 3.1 and 3.3 (2). Statement (2) follows from Lemmas 3.1 and 3.3(4).

For statement (3), if $r < \frac{n-1}{2}$, we have $2r + 1 < n$, or equivalently, $2r + 2 \leq n$. If $r = d + 1$, then by Lemma 3.3 (5), we have $(r, r-1) \in \mathcal{RD}(2r+1)$ or $\mathcal{RD}(2r+2)$, depending upon the parity of r. The result now follows from Lemma 3.1 since $2r + 1 < 2r + 2 \leq n$. ☐

Remark 3.1. *We can improve the above result slightly in the case that $r = d + 1$, $r = \lfloor \frac{n-1}{2} \rfloor$, and if n is even. In this case we have $r < \frac{n-1}{2}$, and so by the same argument as Theorem 3.1 (3), $(r, r-1) \in \mathcal{RD}(n)$. On the other hand, if n is odd, and if $r = \frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$, then we may or may not have $(r, r-1) \in \mathcal{RD}(n)$. For example, the graph G_{ribbon} implies that $(2, 1) \in \mathcal{RD}(5)$ where $2 = r = \frac{5-1}{2}$. However, a computer search over all graphs on seven vertices reveals that $(3, 2) \notin \mathcal{RD}(7)$ where $3 = r = \frac{7-1}{2}$. For this reason, we exclude the point $(\lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor)$ in the definition of $A(n)$ below.*

For a positive integer n, we define

$$\begin{aligned} A(n) &= \left\{ (r, d) \in \mathbb{N}^2 \;\middle|\; 1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor, 1 \leq d \leq n - r, r - d \leq 1 \right\} \setminus \left\{ \left(\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lfloor \frac{n-3}{2} \right\rfloor \right) \right\}, \\ B(n) &= \left\{ (r, d) \in \mathbb{N}^2 \;\middle|\; 1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor, 1 \leq d \leq n - r \right\}. \end{aligned}$$

Both $A(n)$ and $B(n)$ are convex lattice polytopes. As an example, Figure 3 illustrates $A(11)$ and $B(11)$. Specifically, the filled in points represent the elements of $A(11)$ and the filled in points and the empty points represent all the points of $B(11)$.

The following theorem is one of our main theorems, and it follows directly from Theorem 3.1.

Theorem 3.2. *Let $n \geq 3$ be an integer. Let $A(n)$ and $B(n)$ be sets of integer points as above. Then*

$$A(n) \subseteq \mathcal{RD}(n) \subseteq B(n).$$

Proof. For all $(r, d) \in A(n)$ except $(r, d) = (1, 1)$, the first inclusion follows from Theorem 3.1. For $(1, 1)$, note that the graph D_1 has $\operatorname{reg}(R/I(D_1)) = \deg h_{R/I(D_1)}(t) = 1$. So by Lemma 3.1, one has $(1, 1) \in \mathcal{RD}(n)$. The second inclusion follows from Lemma 3.3 (3) and (1). ☐

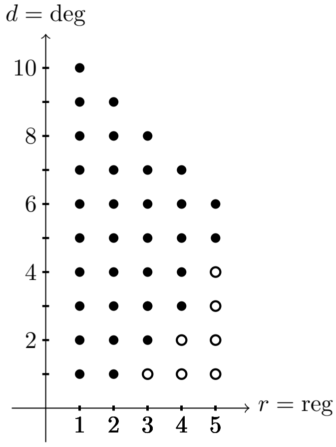


Figure 3: The sets $A(11)$ (all the black points) and $B(11)$ (all the points)

We end this section with a question inspired by our results and computer experiments.

Question 3.1. *For all $n \geq 1$, is the set $\mathcal{RD}(n)$ a convex set? That is, if (r, d) and (r, d') with $d < d'$, respectively (r', d) with $r < r'$, are in $\mathcal{RD}(n)$, is $(r, s) \in \mathcal{RD}(n)$ for all $d < s < d'$, respectively is $(s, d) \in \mathcal{RD}(n)$ for all $r < s < r'$?*

4. Cameron-Walker graphs: relevant properties

For the remainder of this paper, we will focus on describing all possible pairs $(r, d) = (\text{reg}(R/I(G)), \deg h_{R/I(G)}(t))$ when G is a Cameron-Walker graph, a family of connected graphs. Towards this end, we introduce the following subset of $\mathcal{RD}(n)$:

$$CW_{\mathcal{RD}}(n) = \left\{ (r,d) \in \mathbb{N}^2 \;\middle|\; \begin{array}{c} \text{there exists a Cameron-Walker graph } G \text{ with} \\ |V(G)| = n \text{ and } (r,d) = (\text{reg}(R/I(G)), \deg h_{R/I(G)}(t)) \end{array} \right\}.$$

In this section, we review the relevant background on Cameron-Walker graphs so that in the next section we can completely describe $CW_{\mathcal{RD}}(n)$ for all $n \geq 1$.

Recall from Lemma 2.1 the following inequalities:

$$\text{im}(G) \leq \text{reg}\,(K[V(G)]/I(G)) \leq \text{m}(G).$$

By virtue of [2, Theorem 1] together with [6, Remark 0.1], we have that the equality $\text{im}(G) = \text{m}(G)$ holds if and only if G is one of the following graphs:

- a star graph, i.e., a graph joining some paths of length 1 at one common vertex (see Figure 4);
- a star triangle, i.e., a graph joining some triangles at one common vertex (see Figure 4); or
- a finite graph consisting of a connected bipartite graph with vertex partition $\{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_p\}$ such that there is at least one leaf edge attached to each vertex v_i and that there may be possibly some pendant triangles attached to each vertex w_j; see Figure 5 where $s_i \geq 1$ for all $i = 1, \ldots, m$ and $t_j \geq 0$ for all $j = 1, \ldots, p$. Note that a leaf edge is an edge meeting a vertex of degree 1 and a pendant triangle is a triangle where two vertices have degree 2 and the remaining vertex has a degree more than 2.

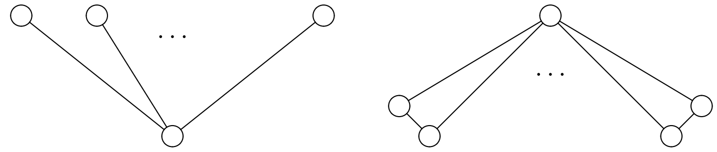


Figure 4: The star graph (left) and the star triangle (right)

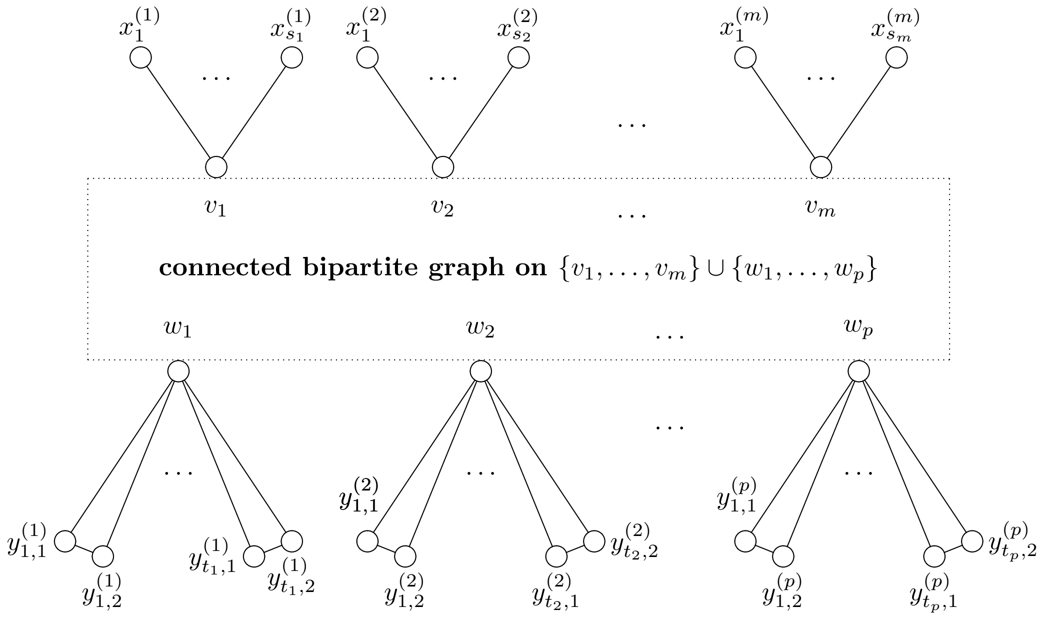


Figure 5: Cameron-Walker graph

Definition 4.1. *A finite connected simple graph G is* a Cameron-Walker graph *if* $\mathrm{im}(G) = \mathrm{m}(G)$ *and if G is neither a star graph nor a star triangle.*

Some invariants of Cameron-Walker graphs were computed in [9]:

Theorem 4.1. *Let G be a Cameron-Walker graph with notation as in Figure 5. Then*

1. $|V(G)| = m + p + \sum_{i=1}^{m} s_i + 2\sum_{j=1}^{p} t_j$;
2. $\deg h_{R/I(G)}(t) = \dim R/I(G) = \sum_{i=1}^{m} s_i + \sum_{j=1}^{p} \max\{t_j, 1\}$; *and*
3. $\mathrm{reg}(R/I(G)) = m + \sum_{j=1}^{p} t_j$.

Proof. (1) follows from the definition of a Cameron-Walker graph. See [9, Proposition 1.3] for (2). Statement (3) is easy to see by computing $\mathrm{im}(G)$. ☐

The following class of Cameron-Walker graphs plays an important role in Section 5.

Construction 4.1. *Fix $a, b \geq 1$ and $0 \leq c \leq b$. Let $G = G_{a,b,c}$ be the Cameron-Walker graph whose bipartite part is the complete bipartite graph $K_{a,b}$, and $s_1 = \cdots = s_a = 1$, $t_1 = \cdots = t_c = 1$, and $t_{c+1} = \cdots = t_b = 0$ (see Figure 6).*

Example 4.1. *Let $a = 2, b = 3$, and $c = 2$. Then the graph $G_{2,3,2}$ is as in Figure 7.*

As a direct application of Theorem 4.1, we can compute some invariants of $G_{a,b,c}$.

Lemma 4.1. *Let $G = G_{a,b,c}$ be the Cameron-Walker graph as in Construction 4.1. Then* $\mathrm{reg}(R/I(G)) = a + c$ *and* $\deg h_{R/I(G)}(t) = a + b$.

5. The regularity and h-polynomials of Cameron-Walker graphs

In this section, we prove our second main result, namely, a characterization of the elements of $CW_{\mathcal{RD}}(n)$. We then use this characterization to compute $|CW_{\mathcal{RD}}(n)|$.

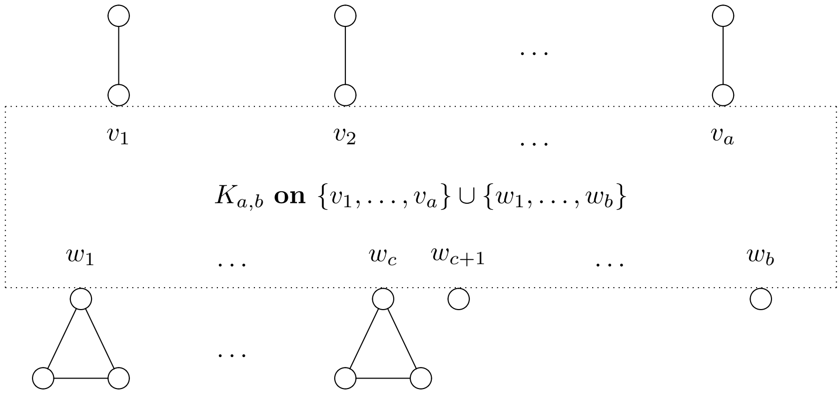


Figure 6: The Cameron–Walker graph $G_{a,b,c}$

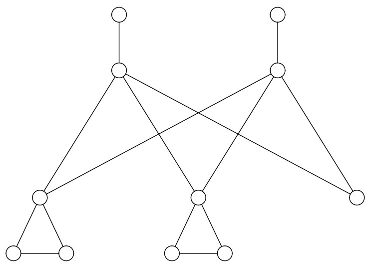


Figure 7: The Cameron–Walker graph $G_{2,3,2}$

Theorem 5.1. *For all $n \geq 5$, $(r, d) \in CW_{\mathcal{RD}}(n)$ if and only if*

- $2 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$,
- $r \leq d \leq n - r$, *and*
- $d \geq -2r + n + 1$.

Proof. The hypothesis $n \geq 5$ allows us to assume the conditions are not vacuous.

Suppose (r, d) satisfy all the above conditions. Let $G = G_{d+2r-n,n-2r,n-r-d}$ be the graph of Construction 4.1 and $R = K[V(G)]$. The conditions on (r, d) imply $d + 2r - n, n - 2r \geq 1$ and $0 \leq n - r - d \leq n - 2r$, so the graph G is defined. Then $|V(G)| = n$ and Lemma 4.1 says that

- $\operatorname{reg}(R/I(G)) = (d + 2r - n) + (n - r - d) = r$,
- $\deg h_{R/I(G)}(t) = (d + 2r - n) + (n - 2r) = d$.

Thus one has $(r, d) \in CW_{\mathcal{RD}}(n)$.

We will now verify that all the $(r, d) \in CW_{\mathcal{RD}}(n)$ satisfy the given inequalities. We know that $r + d \leq n$ (which is equivalent to $d \leq n - r$) holds for all graphs by [13, Theorem 13] (also see (1)). For Cameron-Walker graphs, it was shown that $d \geq r$ in [9, Theorem 3.1]. Consequently, $r \leq d \leq n - r$, as desired.

We now show that $r \geq 2$ for any Cameron-Walker graph. Suppose that $r = 1$. Then by Theorem 4.1 (3), we must have $m = 1$ and $t_j = 0$ for all j. But this then forces the graph to be the star graph $K_{1,n-1}$, which is not considered as a Cameron-Walker graph. So $r \geq 2$.

To show that $r \leq \lfloor \frac{n-1}{2} \rfloor$, it suffices to show that $r < \frac{n}{2}$ (if n is even $\lfloor \frac{n-1}{2} \rfloor = \frac{n}{2} - 1$, and if n is odd, $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$). Suppose for a contradiction that $r \geq \frac{n}{2}$. Since $n = m+p+2\sum_{j=1}^{p} t_j + \sum_{i=1}^{m} s_i$ and $\sum_{i=1}^{m} s_i \geq m$, we have $n \geq 2m + 2\sum_{j=1}^{p} t_j + p$. Thus

$$r \geq \frac{n}{2} \geq m + \sum_{j=1}^{p} t_j + \frac{p}{2} > r$$

where the last inequality follows from Theorem 4.1 (3). This gives the desired contradiction. This paragraph and the previous paragraph now show $2 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$.

Finally, we show that $d \geq -2r + n + 1$. We first note that we can rewrite d as

$$d = \sum_{i=1}^{m} s_i + \sum_{j=1}^{p} \max\{t_j, 1\} = \sum_{i=1}^{m} s_i + \sum_{j=1}^{p} t_j + |\{j \mid t_j = 0\}|.$$

We then have

$$\begin{aligned}
&d + 2r - n - 1 \\
= \;& \sum_{i=1}^{m} s_i + \sum_{j=1}^{p} t_j + |\{j \mid t_j = 0\}| + 2\left(m + \sum_{i=1}^{p} t_j\right) - \left(m + \sum_{i=1}^{m} s_i + p + 2\sum_{j=1}^{p} t_j\right) - 1 \\
= \;& \left(\sum_{i=1}^{p} t_j + |\{j \mid t_j = 0\}| - p\right) + (m-1) \geq 0
\end{aligned}$$

because $\sum_{i=1}^{p} t_j + |\{j \mid t_j = 0\}| \geq p$ and $m \geq 1$. Thus we have $d \geq -2r + n + 1$, as desired. □

When $(r, d) \in CW_{\mathcal{RD}}(n)$, we have $r + d \leq n$ by (1) and so $r + d = n - e$ for some integer $e \geq 0$. As an interesting consequence, the following theorem gives a graph-theoretical interpretation of this integer e.

Theorem 5.2. *Suppose that G is a Cameron-Walker graph on n vertices with $(r, d) = (\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t))$. If $r + d = n - e$, then G has at least e pendant triangles. In particular, if $r + d = n$, then G has no pendant triangles.*

Proof. We have

$$\begin{aligned}
e &= n - r - d \\
&= \left(m + \sum_{i=1}^{m} s_i + p + 2\sum_{j=1}^{p} t_j\right) - \left(m + \sum_{j=1}^{p} t_j\right) - \left(\sum_{i=1}^{m} s_i + \sum_{j=1}^{p} t_j + |\{j \mid t_j = 0\}|\right) \\
&= p - |\{j \mid t_j = 0\}|.
\end{aligned}$$

So e is the number of $j \in \{1, \ldots, p\}$ with $t_j \geq 1$, i.e., the vertices $w_j \in \{w_1, \ldots, w_p\}$ that have a pendant triangle attached to it. So, e is a lower bound on the number of pendant triangles in G.

Moreover if $e = 0$, then $p = |\{j \mid t_j = 0\}|$. This means that G has no pendant triangles. □

It is natural to ask how many elements (r, d) belong to $CW_{\mathcal{RD}}(n)$. Using Theorem 5.1 we can answer this question.

Theorem 5.3. *Fix an integer $n \geq 5$. Then*

$$|CW_{\mathcal{RD}}(n)| = \begin{cases} \frac{1}{12}(n+6)(n-4) & \text{if } n = 6k \text{ or } n = 6k+4, \\ \frac{1}{12}(n-3)(n+5) & \text{if } n = 6k+1 \text{ or } n = 6k+3, \\ \frac{1}{12}(n+1)^2 - \frac{7}{4} & \text{if } n = 6k+2, \\ \frac{1}{12}(n+1)^2 - 1 & \text{if } n = 6k+5. \end{cases}$$

Proof. By Theorem 5.1, we have inequalities:

$$2 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad r \leq d \leq n - r, \quad d \geq -2r + n + 1.$$

We fix an integer r with $2 \leq r \leq \lfloor (n-1)/2 \rfloor$. When $r \leq n - 2r + 1$, namely $r \leq (n+1)/3$, the number of d satisfying $(r, d) \in CW_{\mathcal{RD}}(n)$ is r. Indeed, if $r \leq n - 2r + 1$, we have $n - 2r + 1 \leq d \leq n - r$, so there are r possibilities for d. When $r > n - 2r + 1$, namely $r > (n+1)/3$, the number of d satisfying $(r, d) \in CW_{\mathcal{RD}}(n)$ is $n - 2r + 1$. To see this, in this range, we must have $r \leq d \leq n - r$, so $d = r + i$ with $i = 0, \ldots, n - 2r$. Summing up d for all r, we can compute $|CW_{\mathcal{RD}}(n)|$.

Note the number of elements will thus depend upon knowing the exact value of $\frac{n+1}{3}$. In particular, from the previous paragraph

$$|CW_{\mathcal{RD}}(n)| = \sum_{r=2}^{\lfloor \frac{n+1}{3} \rfloor} r + \sum_{r=\lfloor \frac{n+1}{3} \rfloor + 1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2r + 1). \tag{2}$$

If $n = 6k$, then $\frac{n+1}{3} = \frac{6k+1}{3}$ and $\frac{n-1}{2} = \frac{6k-1}{2}$. Consequently, $\lfloor \frac{n+1}{3} \rfloor = 2k$ and $\lfloor \frac{n-1}{2} \rfloor = 3k - 1$. Plugging this information into (2), we get

$$\begin{aligned}
|CW_{\mathcal{RD}}(n)| &= \sum_{r=2}^{2k} r + \sum_{r=2k+1}^{3k-1} (6k - 2r + 1) \\
&= [2 + 3 + \cdots + 2k] + [(2k-1) + (2k-3) + \cdots + 3]
\end{aligned}$$

$$= \left[\frac{2k(2k+1)}{2} - 1\right] + [k^2 - 1] = 3k^2 + k - 2.$$

Using the fact that $k = \frac{n}{6}$, now gives the desired formula

$$|CW_{\mathcal{RD}}(n)| = 3\frac{n^2}{36} + \frac{n}{6} - 2 = \frac{1}{12}(n+6)(n-4).$$

The other cases are computed in a similar fashion, so we have omitted the details. ☐

The next result is an immediate corollary of Theorem 5.3.

Corollary 5.1. *We have*

$$\lim_{n\to\infty} \frac{|CW_{\mathcal{RD}}(n)|}{n^2} = \frac{1}{12}.$$

6. Future directions

We conclude this paper with some questions inspired by the results of this paper. It would be interesting to compare the number of integer points in $CW_{\mathcal{RD}}(n)$ to the number of integer points in $\mathcal{RD}(n)$. In particular, one might wish to know what percentage of possible $(r,d) = (\mathrm{reg}(R/I(G)), \deg h_{R/I(G)}(t))$ can be realized by Cameron-Walker graphs. Thus, an answer to the following question would be of interest:

Question 6.1. *What is the value of*

$$\lim_{n\to\infty} \frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|}?$$

It is not clear that this limit exists due, in part, to the fact that we can only bound $|\mathcal{RD}(n)|$ (see Theorem 3.2). Observe that to show that this limit exists, it is enough to show that $\frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|} \leq \frac{|CW_{\mathcal{RD}}(n+1)|}{|\mathcal{RD}(n+1)|}$ for all n since $\frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|} \leq 1$, and then one can use the fact that we have a bounded monotic increasing sequence.

If we assume that the limit exists, we can give a partial answer to Question 6.1.

Theorem 6.1. *Suppose that* $\lim_{n\to\infty} \frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|}$ *exists. Then*

$$\frac{2}{9} \leq \lim_{n\to\infty} \frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|} \leq \frac{1}{3}.$$

Proof. Note that by Theorem 5.3, we always have $|CW_{\mathcal{RD}}(n)| = \frac{1}{12}(n+a)(n+b)+c$ for some a, b and c that satisfy $-4 \leq a, b \leq 6$ and $-\frac{7}{4} \leq c \leq 0$. Thus, for all $n \geq 5$,

$$\frac{1}{12}(n-4)(n-4) - \frac{7}{4} \leq |CW_{\mathcal{RD}}(n)| \leq \frac{1}{12}(n+6)(n+6).$$

Using the fact that if $(r,d) \in \mathcal{RD}(n)$, then $r + d \leq n$ and $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, we get an upper bound

$$|\mathcal{RD}(n)| \leq \binom{n}{2} - \binom{\lceil \frac{n+1}{2} \rceil}{2},$$

where we use the fact that $n - 1 < \lfloor \frac{n-1}{2} \rfloor + \lceil \frac{n+1}{2} \rceil < n + 1$. Combining this bound with the lower bound for $|CW_{RD}(n)|$ above gives

$$\frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|} \geq \frac{\frac{1}{12}(n-4)(n-4) - \frac{7}{4}}{\binom{n}{2} - \binom{\lceil \frac{n+1}{2} \rceil}{2}}.$$

Letting $n \to \infty$ on the right hand side gives $\frac{2}{9}$.

Moreover, by Theorem 3.2,we get a lower bound

$$|\mathcal{RD}(n)| \quad \geq \quad \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2.$$

Hence we have the bound

$$\frac{|CW_{\mathcal{RD}}(n)|}{|\mathcal{RD}(n)|} \leq \frac{\frac{1}{12}(n+6)(n+6)}{\left(\lfloor \frac{n}{2} \rfloor\right)^2}.$$

Letting $n \to \infty$ on the right hand side gives $\frac{1}{3}$. ☐

Instead of comparing $CW_{\mathcal{RD}}(n)$ to $\mathcal{RD}(n)$, another variation is to ask for the frequency of (r, d). We phrase this as a specific question:

Question 6.2. *Fix an integer $n \geq 1$ and suppose that $(r, d) \in \mathcal{RD}(n)$. What percentage of all the connected graphs on n vertices have $(\text{reg}(R/I(G)), \deg h_{R/I(G)}(t)) = (r, d)$?*

The above question would also be interesting if we only consider $(r, d) \in CW_{\mathcal{RD}}(n)$.

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References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings (Revised Edition)*, Cambridge University Press, 1998.

[2] K. Cameron and T. Walker, *The graphs with maximum induced matching and maximum matching the same size*, Discrete Math. 299 (2005), 49–55.

[3] G. Favacchio, G. Keiper, and A. Van Tuyl, *On the regularity and h-polynomial of toric ideals of graphs*, Proc. Amer. Math. Soc. 148 (2020), 4665–4677.

[4] D. Grayson and M. Stillman, *Macaulay2, a software system for research in algebraic geometry.* Available at `http://www.math.uiuc.edu/Macaulay2/`.

[5] H.T. Hà and A. Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. 27 (2008), 215–245.

[6] T. Hibi, A. Higashitani, K. Kimura, and A. B. O'Keefe, *Algebraic study on Cameron–Walker graphs*, J. Algebra 422 (2015), 257–269.

[7] T. Hibi, H. Kanno, K. Kimura, K. Matsuda, and A. Van Tuyl, *Homological invariants of Cameron-Walker graphs*, Trans Amer. Math. Soc. 374 (2021), 6559–6582.

[8] T. Hibi, H. Kanno, and K. Matsuda, *Induced matching numbers of finite graphs and edge ideals*, J. Algebra 532 (2019), 311–322.

[9] T. Hibi, K. Kimura, K. Matsuda, and A. Tsuchiya, *Regularity and a-invariant of Cameron–Walker graphs*, J. Algebra 584 (2021), 215–242.

[10] T. Hibi and K. Matsuda, *Regularity and h-polynomials of monomial ideals*, Math. Nachr. 291 (2018), 2427–2434.

[11] T. Hibi and K. Matsuda, *Lexsegment ideals and their h-polynomials*, Acta. Math. Vietnam. 44 (2019), 83–86.

[12] T. Hibi and K. Matsuda, *Regularity and h-polynomials of binomial edge ideals*, Acta. Math. Vietnam., to appear.

[13] T. Hibi, K. Matsuda, and A. Van Tuyl, *Regularity and h-polynomials of edge ideals*, Electron. J. Combin. 26 (2019), #P1.22.

[14] T. Kahle and J. Krüsemann, *Binomial edge ideals of cographs*, arXiv:1906.05510.

[15] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A 113 (2006), 435–454.

[16] I. Peeva, *Graded syzygies*, Algebra and Applications, **14**, Springer-Verlag London, Ltd., London, 2011.

[17] T. N. Trung, *Regularity, matchings and Cameron–Walker graphs*, Collect. Math. 71 (2020), 83–91.