ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **2:3** (2022) Article #S2R20
https://doi.org/10.54550/ECA2022V2S3R20

Hurwitz Numbers for Reflection Groups I: Generatingfunctionology

Theo Douvropoulos†, Joel Brewster Lewis‡, and Alejandro H. Morales†

†*Department of Mathematics and Statistics, University of Massachussets, Amherst, MA, USA*
Email: {ahmorales,douvropoulos}@math.umass.edu

‡*Department of Mathematics, George Washington University, Washington, DC, USA*
Email: jblewis@gwu.edu

Received: December 9, 2021, **Accepted**: April 12, 2022, **Published**: May 6, 2022
The authors: Released under the CC BY-ND license (International 4.0)

Abstract: The classical Hurwitz numbers count the fixed-length transitive transposition factorizations of a permutation, with a remarkable product formula for the case of minimum length (*genus* 0). We study the analogue of these numbers for reflection groups with the following generalization of transitivity: say that a reflection factorization of an element in a reflection group W is *full* if the factors generate the whole group W. We compute the generating function for full factorizations of arbitrary length for an arbitrary element in a group in the combinatorial family $G(m,p,n)$ of complex reflection groups in terms of the generating functions of the symmetric group $\mathfrak{S}_n$ and the cyclic group of order m/p. As a corollary, we obtain leading-term formulas which count minimum-length full reflection factorizations of an arbitrary element in $G(m,p,n)$ in terms of the Hurwitz numbers of genus 0 and 1 and number-theoretic functions. We also study the structural properties of such generating functions for any complex reflection group; in particular, we show via representation-theoretic methods that they can be expressed as finite sums of exponentials of the variable.

Keywords: Complex reflection groups; Full factorizations; Hurwitz numbers; Reflection factorizations; Transitive factorizations

2020 Mathematics Subject Classification: 05A15; 20B05; 20C15

1. Introduction

A factorization $t_1 \cdots t_N = \sigma$ of a given element σ in the symmetric group $\mathfrak{S}_n$ as a product of *transpositions* t_i is said to be *transitive* if the group $\langle t_i \rangle_{i=1}^N$ generated by the factors acts transitively on the set $\{1, \ldots, n\}$. Such factorizations arose initially in the work of Hurwitz, in connection with his study of Riemann surfaces. Hurwitz gave a sketch of an inductive argument, reproduced in detail in [37], for the following remarkable product formula.

Theorem 1.1 (Hurwitz formula [20]). *The minimum number of factors in a transitive transposition factorization of a permutation in $\mathfrak{S}_n$ of cycle type $\lambda := (\lambda_1, \ldots, \lambda_k)$ is $n + k - 2$. The number of such minimum-length factorizations is*

$$H_0(\lambda) = (n+k-2)! \cdot n^{k-3} \cdot \prod_{i=1}^{k} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}.$$

The formula in Theorem 1.1, rediscovered by Goulden and Jackson in [16], is for what are now called the *(single) Hurwitz numbers of genus* 0. These numbers also count certain connected planar graphs embedded in the sphere (the *planar maps*; see, e.g., [22]). In general, the *genus-g Hurwitz numbers* $H_g(\lambda)$ count transitive factorizations $t_1 \cdots t_N$ of a fixed element σ in $\mathfrak{S}_n$ of cycle type λ into $N = n + k + 2g - 2$ transpositions, as well as certain embedded maps in orientable surfaces of genus g.

In the 1980s, the work of Stanley [35] and Jackson [21] rekindled interest in the enumeration of factorizations in $\mathfrak{S}_n$, unrelated to the original topological context. Independently, the next few decades saw the emergence of *Coxeter combinatorics*; one of its main breakthroughs was the realization that theorems about $\mathfrak{S}_n$ are often shadows of more general results that hold for all reflection groups W. In the context of factorizations, this means replacing transpositions in $\mathfrak{S}_n$ with *reflections* in W.

The intersection of these two areas has witnessed a lot of research activity recently [4,6,23,27], especially for factorizations of *Coxeter elements* in W, generalizing the long cycle case $\lambda = (n)$. In general, however, analogues of Theorem 1.1 have been hard to find, not least because it is unclear how to define transitivity in reflection groups (see Section 2.3): in $\mathfrak{S}_n$, transitivity corresponds to the *connectedness* of the associated embedded map or Riemann surface, but these have no analogues for a general reflection group W.

An equivalent way to interpret the notion of transitivity is to require that the factorization cannot be realized in any *proper* Young subgroup (i.e., a proper subgroup generated by transpositions) of $\mathfrak{S}_n$. This interpretation makes sense for arbitrary reflection groups W, where we will thus say that $t_1 \cdots t_N = g$ is a *full reflection factorization* of an element $g \in W$ if the factors t_i are reflections and they generate the *full* group W.

This paper is the first of a series of three [9,10] in which we study the problem of enumerating full reflection factorizations in real and complex reflection groups – giving formulas for what one might call the *W-Hurwitz numbers.* In the present paper, we focus on the infinite family $G(m,p,n)$ of "combinatorial" complex reflection groups, whose elements may be viewed as colored permutations. Our main result provides a general formula for the (exponential) generating function for full reflection factorizations in $G(m,p,n)$ counted by length, expressed in terms of the corresponding series for the symmetric group $\mathfrak{S}_n$ and the cyclic group of order m/p. (For relevant notation and terminology, see Section 2.)

Theorem 4.1. *For an element $g \in G(m,p,n)$ with k cycles, of colors $a_1, \ldots, a_k$, let $d = \gcd(a_1, \ldots, a_k, p)$. We have*

$$\mathcal{F}^{\mathrm{full}}_{m,p,n}(g; z) = \frac{1}{m^{n-1}} \cdot \mathcal{F}^{\mathrm{full}}_{m,p,1}\big(\zeta_m^{\mathrm{col}(g)}; n \cdot z\big) \cdot \sum_{r\colon r|d} \big(\mu(r) \cdot r^{n+k-2} \cdot \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g); (m/r)\cdot z\big)\big),$$

where μ is the number-theoretic Möbius function.

(The series for the cyclic group is given in Example 3.1. The series for the symmetric group for the identity element is given by the Dubrovin–Yang–Zagier [11] recurrence – see Remark 3.4 – and for other elements is computable using characters – see Remark 3.2.)

From the generating function in Theorem 4.1 we can extract the leading coefficient, which is the number of *minimum-length* full factorizations of an arbitrary element g in $G(m,p,n)$. The answer in full generality (Theorem 5.2) involves the usual Hurwitz numbers of genus 0 and genus 1 and the *Jordan totient function* $J_2(m)$, which counts elements of order m in the group $(\mathbb{Z}/m\mathbb{Z})^2$. For the group $G(m,m,n)$ the leading terms are given by the following formulas.

Corollary (of Theorem 5.2). *For $g \in G(m,m,n)$ with k cycles, of lengths $\lambda_1, \ldots, \lambda_k$ and colors $a_1, \ldots, a_k$, let $d = \gcd(a_1, \ldots, a_k, m)$. We have that the number $F^{\mathrm{full}}_{m,m,n}(g)$ of minimum-length full reflection factorizations of g is*

$$F^{\mathrm{full}}_{m,m,n}(g) = \begin{cases} m^{k-1} \cdot H_0(\lambda_1, \ldots, \lambda_k), & \text{if } d = 1, \\ \dfrac{m^{k+1}}{d^2} J_2(d) \cdot H_1(\lambda_1, \ldots, \lambda_k), & \text{if } d \neq 1. \end{cases}$$

In the sequel [9] to this paper, we will study the *parabolic quasi-Coxeter elements* in a well generated complex reflection group W. This wide class of elements contains the parabolic Coxeter elements, and hence all elements in the symmetric group; we will establish a number of equivalent characterizations of these elements, including in terms of the good behavior of their full factorizations. In the climax [10], we will establish a uniform formula for the number of minimum-length full reflection factorizations of any parabolic quasi-Coxeter element in an arbitrary well-generated complex reflection group that very closely models the Hurwitz formula of Theorem 1.1.

We end this introduction with a brief outline of the present paper. In Section 2, we introduce the main actors: we give background on complex reflection groups, with an emphasis on the combinatorial family, introduce full reflection factorizations, and compare this latter notion with the existing generalizations of transitive factorizations in the literature. In Section 3, we discuss a general technique using character theory to count factorizations and study the structural properties of their generating functions. In Section 4, we use combinatorial methods to prove the main theorem (Theorem 4.1). Finally, as a corollary, in Section 5 we extract the lowest-order coefficients $F^{\mathrm{full}}_{m,p,n}(g)$ of the generating functions $\mathcal{F}^{\mathrm{full}}_{m,p,n}(g; z)$.

2. Reflection groups and reflection factorizations

Let V be a finite-dimensional complex vector space with a fixed Hermitian inner product. A *(unitary) reflection* t on V is a unitary map whose *fixed space* $V^t := \{v \in V \colon t(v) = v\}$ is a hyperplane (in other words, $\mathrm{codim}(V^t) =$

1). A finite subgroup W of the unitary group $U(V)$ is called a *complex reflection group* if it is generated by its subset $\mathcal{R}$ of reflections; we write $\mathcal{A}_W$ for its *reflection arrangement*, i.e., the arrangement of fixed hyperplanes of its reflections $t \in \mathcal{R}$. We say that W is *irreducible* if there is no nontrivial subspace of V stabilized by its action.

It is easy to see that if W_1 acts on V_1, with reflections $\mathcal{R}_1$, and W_2 acts on V_2, with reflections $\mathcal{R}_2$, then $W = W_1 \times W_2$ acts on $V = V_1 \oplus V_2$, with reflections $\mathcal{R} = (\mathcal{R}_1 \times \{\mathrm{id}_2\}) \cup (\{\mathrm{id}_1\} \times \mathcal{R}_2)$. In this case, W fails to be irreducible (as it stabilizes the subspace V_1 of V). Shephard and Todd [31] classified the complex reflection groups: every complex reflection group is a product of irreducibles, and every irreducible either belongs to an infinite three-parameter family $G(m, p, n)$ where m, p, and n are positive integers such that p divides m, or is one of 34 exceptional cases, numbered G_4 to G_{37}.

2.1 The combinatorial family $G(m, p, n)$

The infinite family may be described as

$$G(m,p,n) := \left\{ \begin{array}{c} n \times n \text{ monomial matrices whose nonzero entries are} \\ m\text{th roots of unity with product a } \frac{m}{p}\text{th root of unity} \end{array} \right\}. \tag{2.1}$$

Writing $\zeta_m := \exp(2\pi i/m)$ for the primitive mth root of unity, the reflections in $G(m, p, n)$ fall into two families: the *transposition-like reflections*, of the form

$$\begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & & & & \zeta_m^{-k} & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & \zeta_m^{k} & & & & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \tag{2.2}$$

for $1 \le i < j \le n$ and $k = 0, 1, \ldots, m-1$, which have order 2 and fix the hyperplanes $x_j = \zeta_m^k x_i$; and, when $p < m$, the *diagonal reflections*, of the form

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \zeta_m^{pk} & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{2.3}$$

for $i = 1, \ldots, n$ and $k = 1, 2, \ldots, \frac{m}{p} - 1$, which have various orders and fix the hyperplanes $x_i = 0$.

It is natural to represent such groups combinatorially. We may encode each element w of $G(m, 1, n)$ by a pair $[u; a]$ with $u \in \mathfrak{S}_n$ and $a = (a_1, \ldots, a_n) \in (\mathbb{Z}/m\mathbb{Z})^n$, as follows: for $k = 1, \ldots, n$, the nonzero entry in column k of w is in row $u(k)$, and the value of the entry is $\zeta_m^{a_k}$. With this encoding, it's easy to check that

$$[u; a] \cdot [v; b] = [uv; v(a) + b], \quad \text{where} \quad v(a) := \left(a_{v(1)}, \ldots, a_{v(n)}\right). \tag{2.4}$$

This shows that the group $G(m, 1, n)$ is isomorphic to the *wreath product* $\mathbb{Z}/m\mathbb{Z} \wr \mathfrak{S}_n$ of a cyclic group with the symmetric group $\mathfrak{S}_n$. If $w = [u; a]$, we say that u is the *underlying permutation* of w.

Two natural homomorphisms

For an element $w = [u; a]$ of $G(m, 1, n)$, by a *cycle* of w we mean a cycle of the underlying permutation u. For $S \subseteq \{1, \ldots, n\}$, we say that $\sum_{k \in S} a_k$ is the *color* of S; this notion will come up particularly when the elements of S form a cycle in w. When $S = \{1, \ldots, n\}$, we call $a_1 + \ldots + a_n$ the color of the element $[u; a]$, and we denote it $\mathrm{col}([u; a])$. In this terminology, $G(m, p, n)$ is the subgroup of $G(m, 1, n)$ containing exactly those elements whose color is a multiple of p.

One may think of the color as a surjective group homomorphism $\mathrm{col}\colon G(m, p, n) \to p\mathbb{Z}/m\mathbb{Z} \cong G(m, p, 1)$. In fact, this is one way to see that $G(m, p', n)$ is a *normal* subgroup of $G(m, p, n)$ whenever $p \mid p'$ (since $p'\mathbb{Z}/m\mathbb{Z}$ is a normal subgroup of $p\mathbb{Z}/m\mathbb{Z}$). For any $r \mid m$, there is also another important surjective group homomorphism $G(m, 1, n) \to G(r, 1, n)$ that we define now.

Definition 2.1. *For any $r \mid m$, let $\pi_{m/r} : G(m, 1, n) \to G(r, 1, n)$ be the group homomorphism defined at the level of matrices as replacing each copy of ζ_m with $\zeta_m^{m/r}$; equivalently, viewing $G(r, 1, n)$ as a subgroup of $G(m, 1, n)$, it is the map*

$$\pi_{m/r}([u; a]) = \left[u; \frac{m}{r} \cdot a\right].$$

When restricted to the subgroup $G(m, p, n)$, the map $\pi_{m/p}$ is a surjective homomorphism onto $G(p, p, n)$. In particular, for any $[u; a] \in G(m, p, n)$, applying $\pi_{m/1}$ recovers the underlying permutation $u \in \mathfrak{S}_n$ of w.

Conjugacy classes in $G(m, 1, n)$ are commonly indexed by tuples $(\lambda_0, \ldots, \lambda_{m-1})$ of integer partitions of total size n. In this indexing, λ_j is the partition composed of the lengths of the cycles of color j. In other words, two elements in $G(m, 1, n)$ are conjugate if and only if they have the same number of cycles of length k and color c for all $k = 1, \ldots, n$, $c \in \mathbb{Z}/m\mathbb{Z}$. (The description of the conjugacy classes in $G(m, p, n)$ is slightly more complicated [28], and we will not need it here.)

Graph interpretation and connected subgroups

It is natural to represent collections of reflections in $G(m, p, n)$ as graphs on the vertex set $\{1, \ldots, n\}$: a transposition-like reflection whose underlying permutation is (ij) can be represented by an edge joining i to j, while a diagonal reflection whose nonzero color occurs at position i can be represented by a loop at vertex i.

We say that a collection of reflections in $G(m, p, n)$ is *connected* if the associated graph is connected. It is a basic fact of graph theory that any connected graph contains a spanning tree; consequently, any connected set of reflections in $G(m, p, n)$ contains a connected subset of $n - 1$ transposition-like reflections. Such a subset always generates a subgroup of $G(m, p, n)$ isomorphic to the symmetric group $\mathfrak{S}_n$ [32, Lem. 2.7].

More generally, if H is a subgroup of $G(m, p, n)$ generated by a connected collection of reflections, then H is isomorphic to $G(m', p', n)$ for some integers m', p' such that $m' \mid m$, $p' \mid m'$, and $\frac{m'}{p'} \mid \frac{m}{p}$. The isomorphism can be concretely realized as conjugation by an appropriate diagonal element of $G(m, 1, n)$.*

2.2 Reflection factorizations and reflection length

Starting with an arbitrary group G and some generating set $S \subset G$, the Cayley graph of G with respect to S determines a natural length function on the elements of the group. Indeed, we may define the length of an element $g \in G$ as the size of the shortest path in the Cayley graph from the identity element to g.

In the case of a complex reflection group W, we pick the set $\mathcal{R}$ of reflections as the distinguished generating set. Then the *reflection length* $\ell_W^{\text{red}}(g)$ of an element $g \in W$ is the smallest number k such that there exist reflections $t_1, \ldots, t_k$ that form a factorization $g = t_1 \cdots t_k$. (Here the superscript "red" stands for "reduced".)

For an arbitrary number k, if $t_1, \ldots, t_k$ are reflections such that $t_1 \cdots t_k = g$, we say that the tuple $(t_1, \ldots, t_k)$ forms a *reflection factorization* of g of length k.

Say that a reflection factorization $t_1 \cdots t_k = g$ of an element $g \in W$ is *full* (relative to W) if the factors generate the full group, i.e., if $W = \langle t_1, \ldots, t_k \rangle$. Observe that, by definition, every reflection factorization of any element g is full *relative to the subgroup generated by the factors*. The *full reflection length* $\ell_W^{\text{full}}(g)$ of g is the minimum length of a full reflection factorization:

$$\ell_W^{\text{full}}(g) := \min \Big\{ k : \exists\ t_1, \ldots, t_k \in \mathcal{R} \text{ such that } t_1 \cdots t_k = g \text{ and } \langle t_1, \ldots, t_k \rangle = W \Big\}.$$

As we wish to enumerate factorizations of arbitrary length N, it is convenient to encode the answers via generating functions. We denote by $\mathcal{F}_W(g; z)$ the exponential generating function for the number of reflection factorizations of g of arbitrary length:

$$\mathcal{F}_W(g; z) := \sum_{N \geq 0} \# \left\{ (t_1, \ldots, t_N) \in \mathcal{R}^N : t_1 \cdots t_N = g \right\} \cdot \frac{z^N}{N!}. \tag{2.5}$$

By definition, the lowest-order term of $\mathcal{F}_W(g; z)$ occurs in degree $\ell_W^{\text{red}}(g)$.

Similarly, we denote by $\mathcal{F}_W^{\text{full}}(g; t)$ the exponential generating function for the number of *full* reflection factorizations of g of arbitrary length:

$$\mathcal{F}_W^{\text{full}}(g; z) := \sum_{N \geq 0} \# \left\{ (t_1, \ldots, t_N) \in \mathcal{R}^N : t_1 \cdots t_N = g \text{ and } \langle t_1, \ldots, t_N \rangle = W \right\} \cdot \frac{z^N}{N!}. \tag{2.6}$$

We further write $F_W^{\text{full}}(g)$ for the number of minimum-length full factorizations (i.e., with $N = \ell_W^{\text{full}}(g)$), so that the lowest-order term of $\mathcal{F}_W(g; t)$ is equal to $F_W^{\text{full}}(g) \cdot z^{\ell_W^{\text{full}}(g)} / \ell_W^{\text{full}}(g)!$.

*See, for example, the proof of [24, Prop. 3.30], which concerns factorizations in a certain "standard form" but does not use this form in any essential way. The key observation is that, after choosing a spanning tree, one may choose a diagonal element that simultaneously conjugates the corresponding reflections to transpositions, generating the symmetric group $\mathfrak{S}_n$ (per Proposition 2.1, below).

2.3 Transitive factorizations

In the symmetric group $\mathfrak{S}_n$, it is of particular interest to study *transitive* factorizations, that is, those for which the factors generate a subgroup that acts transitively on the set $\{1, \ldots, n\}$ (see, e.g., [2, 16, 20] and the survey [18]). We are interested in the case when all factors are transpositions (these are the reflections in $\mathfrak{S}_n$). The next result gives equivalent characterizations of the transitive transposition factorizations; it is well known and we only provide a proof for completeness.

Proposition 2.1 (folklore)**.** *In the symmetric group $\mathfrak{S}_n$, a transposition factorization is transitive if and only if it is connected, if and only if it is full.*

Proof. Let $t_1 \cdots t_N = g$ be a factorization in the symmetric group $\mathfrak{S}_n$ for which all t_k are transpositions. We write $W := \langle t_i \rangle$ for the group generated by the factors and G for the graph formed by the edges (i_k, j_k) corresponding to each transposition $t_k = (i_k j_k)$.

We will first assume that the factorization is *connected*, meaning that G is connected as a graph. For any two indices i and j, G must then contain some path $(i, i_1, i_2, \ldots, i_s, j)$. By iterating the conjugation $(bc) \cdot (ab) \cdot (bc) = (ac)$, we have

$$(i_s j) \cdot (i_{s-1} i_s) \cdots (i_1 i_2) \cdot (i i_1) \cdot (i_1 i_2) \cdots (i_{s-1} i_s) \cdot (i_s j) = (ij).$$

This means that the group W generated by the factors of the connected factorization contains all transpositions (ij) and hence equals the full symmetric group $\mathfrak{S}_n$. In other words, a connected factorization must be *full* and hence trivially must be *transitive* also.

If on the other hand the factorization is not connected, it determines a partition Π of the set $\{1, \ldots, n\}$ corresponding to the connected components of the graph G. By the argument of the previous paragraph, the group W contains at least the group $\mathfrak{S}_\Pi$ (i.e., the permutations that respect the partition Π). However, all the factors t_k are also included in $\mathfrak{S}_\Pi$, so $W = \mathfrak{S}_\Pi \neq \mathfrak{S}_n$. Moreover, it is trivial to see that $\mathfrak{S}_\Pi$ does not act transitively on $\{1, \ldots, n\}$ (since it must respect the parts of Π). That is, a non-connected factorization is never full and never transitive. This completes the proof. $\square$

There have been several attempts to generalize the notion of transitivity to reflection groups other than $\mathfrak{S}_n$. In [1], the authors consider the hyperoctahedral group $G(2, 1, n)$ (the Coxeter group of type B_n) in its natural permutation action on $E_2 := \{\pm e_1, \ldots, \pm e_n\}$, where e_i is a standard basis vector. They enumerate transitive reflection factorizations of arbitrary elements under two notions of transitivity: a strong version (that they call *admissibility*), in which the factors act transitively on E_2, and a weaker version (that they call *near-admissibility*), in which the factors merely act transitively on the coordinate axes (equivalently, on the pairs $\{\{\pm e_i\} : i = 1, \ldots, n\}$).

Both notions of transitivity considered by [1] have been recently studied in the infinite family $G(m, p, n)$ of complex reflection groups. It is easy to see that a reflection factorization in $G(m, p, n)$ is transitive in the weaker sense (i.e., that the group generated by the factors acts transitively on the coordinate axes in $\mathbb{C}^n$) if and only if the factorization is connected. In [27], the authors consider connected reflection factorizations of arbitrary elements in $G(m, p, n)$, and give an analogue of the polynomiality property (see [19, §1.1] and references therein) of the usual Hurwitz numbers (famous due to its connection with the ELSV formula [12]). And in [23], the authors enumerate factorizations of a Coxeter element in $G(m, 1, n)$ or $G(m, m, n)$ into arbitrary factors that are transitive in the stronger sense of the action on $E_m := \{\zeta_m^k e_j : k = 0, \ldots, m-1, j = 1, \ldots, n\}$.

The main drawback of results based on these definitions of transitivity is that it is unclear how to generalize them to reflection groups outside the infinite family (see, e.g., [23, Ques. 8.2]). Indeed, the groups $G(m, p, n)$ are precisely the *imprimitive* irreducible complex reflection groups, i.e., the groups W for which there is a decomposition $V = V_1 \oplus \cdots \oplus V_n$ of the vector space on which W acts that is respected by the action of W; for other (*primitive*) groups, there is no natural set analogous to $\{1, \ldots, n\}$ for $\mathfrak{S}_n$. Thus, one thesis of the present work and its sequels is that fullness is the "correct" generalization of transitivity from $\mathfrak{S}_n$ to reflection groups because it allows one to pose questions uniformly.† And, moreover, the main result of [10] will establish that such questions may have uniform answers. Before we move on to the main result of the present paper, we offer one final point in support of this thesis: it is not only in $\mathfrak{S}_n$ but also in $G(m, m, n)$ that there is a coincidence between factorizations that are full and those that are transitive (in an appropriate sense).

Proposition 2.2. *A set of reflections in $W = G(m, m, n)$ acts transitively on the set $E_m = \{\zeta_m^k e_j : k = 0, \ldots, m-1, j = 1, \ldots, n\}$ (where e_j is a standard basis vector of $\mathbb{C}^n$ and ζ_m is the primitive mth root of unity) if and only if it generates the full group W.*

†For factorizations whose factors are not necessarily reflections, the notion of fullness would require that there is no *proper* reflection subgroup containing all the factors.

Proof. Any set of reflections of $W = G(m, m, n)$ generates a reflection subgroup W' of W. If the set is not connected, then W' does not act transitively on the coordinate axes of $\mathbb{C}^n$, let alone on the set E_m. Therefore we restrict our attention to connected sets of reflections. It follows from the observations in the last paragraph of Section 2.1 that every connected subgroup of $G(m, m, n)$ is conjugate by an element of $G(m, 1, n)$ to a subgroup of the form $G(p, p, n)$ for some $p \mid m$. Conjugation by elements of $G(m, 1, n)$ preserves both the properties of transitivity and fullness, so it is enough to observe that $G(p, p, n)$ acts transitively on E_m if and only if $p = m$. $\square$

Of course, this equivalence does not extend to $G(m, p, n)$ for $p < m$, since each such group contains the proper reflection subgroup $G(m, m, n)$.

3. A general approach to counting via the Frobenius lemma

In the previous section we defined, in Equations (2.5) and (2.6), exponential generating functions $\mathcal{F}_W(g; z)$ and $\mathcal{F}_W^{\text{full}}(g; z)$ that enumerate factorizations whose factors are taken from the set of reflections $\mathcal{R}$ of the complex reflection group W. Because $\mathcal{R}$ is closed under conjugation, a traditional approach via representation theory, due originally to Frobenius [13], allows us to express both series as a finite sum of exponentials in z. We start with the case of arbitrary (i.e., not necessarily full) reflection factorizations. Following [4, (4.3)], we have the formula

$$\mathcal{F}_W(g; z) = \frac{1}{\#W} \cdot \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(g^{-1}) \cdot \exp\left(\frac{\chi(\mathcal{R})}{\chi(1)} \cdot z\right), \tag{3.1}$$

where $\widehat{W}$ denotes the set of irreducible complex characters of W and $\chi(\mathcal{R}) := \sum_{t \in \mathcal{R}} \chi(t)$. For any reflection group W, the normalized character values $\chi(\mathcal{R})/\chi(1)$ are integers (see [3, Def. 3.14 and Prop. 3.15]) and thus $\mathcal{F}_W(g; z)$ can be written as a Laurent polynomial in the variable $X := e^z$.

3.1 Full factorizations after an inclusion-exclusion argument

Any factorization $(t_1, \ldots, t_N)$ is full for the subgroup $W' := \langle t_1, \ldots, t_N \rangle$ generated by its elements. Therefore, we may view the series $\mathcal{F}_W(g; z)$ in (2.5) as the sum, over all reflection subgroups W' of W that contain g, of the generating functions $\mathcal{F}_{W'}^{\text{full}}(g; z)$ of full factorizations of g with respect to W'. By applying the principle of inclusion-exclusion, we conclude that

$$\mathcal{F}_W^{\text{full}}(g; z) = \sum_{g \in W' \leq W} \mu(W, W') \cdot \mathcal{F}_{W'}(g; z), \tag{3.2}$$

where the Möbius function is computed in the poset of all reflection subgroups W' of W, ordered by reverse inclusion. This is analogous to the usual construction [36, §5.2] of taking the logarithm of an exponential generating function to count "connected components", which was also what led Goulden and Jackson to rediscover the Hurwitz formula of Theorem 1.1, see [16, §4]. Indeed, in the case of the symmetric group $\mathfrak{S}_n$, the poset of reflection subgroups $W' \leq W$ is isomorphic to the set partition lattice of $\{1, \ldots, n\}$, and Möbius inversion corresponds precisely to taking the logarithm of generating functions.

The irreducible characters for complex reflection groups W can be constructed via the computer software SageMath and CHEVIE [15, 30], which realize reflection groups via their permutation action on roots. This allows for explicit calculation of the functions $\mathcal{F}_W(g; z)$ and $\mathcal{F}_W^{\text{full}}(g; z)$ for the exceptional groups G_4 to G_{36}. For the interested reader, the series for the case $g = \text{id}$ for all exceptional groups are attached to the arXiv version [8] of this paper as an auxiliary file. (For the reflection group $G_{37} = E_8$, the lattice of reflection subgroups and its Möbius function are very complicated, so computation of $\mathcal{F}_{E_8}^{\text{full}}(g; z)$ for arbitrary g requires significant computational resources.)

Example 3.1. *For example, consider the case of an arbitrary element g in the cyclic group $W = G(m, 1, 1)$ with $m > 1$, in which all non-identity elements are reflections. It is easy to show that the number a_k of ways to write an element g of a finite group G as a product of k non-identity elements is*

$$a_k = \begin{cases} \dfrac{(\#G - 1)^k - (-1)^k}{\#G}, & g \neq \text{id} \\[2ex] \dfrac{(\#G - 1)^k + (\#G - 1)(-1)^k}{\#G}, & g = \text{id} \end{cases}$$

(see [14, Example 13, p. 10] for the abelian case). Consequently, we have

$$\mathcal{F}_W(g;\log X) = \begin{cases} \dfrac{X^m-1}{mX}, & g \neq \mathrm{id} \\ \dfrac{X^m+(m-1)}{mX}, & g = \mathrm{id} \end{cases}$$

(in fact one can show this also by applying (3.1)*).*

The subgroups of W are the cyclic groups $G(m,r,1) \cong \mathbb{Z}/(m/r)\mathbb{Z}$ for $r \mid m$, and their poset of inclusions is isomorphic to the poset of positive integer divisors of m. Furthermore, an element g of $G(m,1,1)$ belongs to $G(m,r,1)$ if and only if its order $\mathrm{ord}(g)$ divides r. Therefore, doing the Möbius inversion over all subgroups, we have (after some simplification) that

$$\mathcal{F}_W^{\mathrm{full}}(g;\log X) = \frac{1}{X} \sum_{\substack{r\,:\, r\mid m \\ and\, \mathrm{ord}(g)\mid r}} \mu(m/r) \frac{X^r-1}{r} = \frac{X-1}{X} \sum_{\substack{r\,:\, r\mid m \\ and\, \mathrm{ord}(g)\mid r}} \mu(m/r) \frac{[r]_X}{r}$$

for all g in W, where $[r]_X := 1+X+\ldots+X^{r-1}$.

Another example, in the case of the real reflection group $H_3 = G_{23}$, is given below as Example 3.2.

Remark 3.1. *The same representation-theoretic approach can easily be used to obtain the reflection length and full reflection length of any element $g \in W$. Indeed, as mentioned below Equations* (2.5) *and* (2.6)*, one need only read off the lowest-order terms of the generating functions $\mathcal{F}_W(g;z)$ and $\mathcal{F}_W^{\mathrm{full}}(g;z)$, respectively.*

Remark 3.2. *The two formulas* (3.1) *and* (3.2) *imply that the generating functions $\mathcal{F}_W^{\mathrm{full}}(g;z)$ are finite sums of exponentials e^{kz}. In fact, there are much stronger constraints on their structure. As we will see in Proposition 3.1 in the next section, the numbers k appearing in the exponents are integers and belong to an interval of length at most $h \cdot n$, where h is the Coxeter number and n the rank of the irreducible group W.*

We warn the reader that several relevant results in factorization enumeration (e.g., [16, 17, 19]) concern generating functions that vary the index n of the symmetric group $\mathfrak{S}_n$ but fix the genus, whereas we fix the group but vary the genus (equivalently, the number of factors). These generating functions, though similar to ours at first glance, have much more complicated structures and asymptotics.

3.2 Additional structure for the generating functions

We give now some further properties of the functions $\mathcal{F}_W^{\mathrm{full}}(g;z)$ that are not necessary for what follows, but that might shed some light on the structural characteristics of our main theorems.

Proposition 3.1. *Let W be any complex reflection group and g any element of W. The exponential generating function $\mathcal{F}_W^{\mathrm{full}}(g;z)$ of full reflection factorizations of g can always be expressed in terms of the variable $X := e^z$ as a Laurent polynomial of the form*

$$\mathcal{F}_W^{\mathrm{full}}(g;\log X) = \frac{1}{\#W} \cdot \Phi_W(g;X) \cdot (X-1)^{\ell_W^{\mathrm{full}}(g)} \cdot \frac{1}{X^{\#\mathcal{A}_W}},$$

with $\Phi_W(g;X)$ a monic polynomial in X of degree $h \cdot n - \ell_W^{\mathrm{full}}(g)$ and $\mathcal{A}_W$ the reflection arrangement of W.

Proof. As mentioned in the first paragraph of Section 3, the generating function $\mathcal{F}_W^{\mathrm{full}}(g;z)$ is always a Laurent polynomial in terms of the variable $X := e^z$. Since $e^z = 1 + z + \ldots$, the multiplicity of the root $X = 1$ in this polynomial agrees with the degree $\ell_W^{\mathrm{full}}(g)$ of the lowest-order term in the z-expansion of the series. The highest and lowest degree of this Laurent polynomial is determined by the inequalities of [6, Prop. 3.2] regarding the normalized traces $\chi(\mathcal{R})/\chi(1)$; it is monic because only the trivial representation χ_{triv} has normalized trace equal to $\#\mathcal{R}$ (as in the proof of [6, Thm. 3.6]). ☐

Example 3.2 (Counting full reflection factorizations of the identity element in H_3)**.** *We give here in some detail a snapshot of the calculation of the generating function $\mathcal{F}_{H_3}^{\mathrm{full}}(\mathrm{id};z)$ counting full factorizations of the identity element in the real reflection group $H_3 = G_{23}$. Möbius inversion on the poset of reflection subgroups of a reflection group W can be computed recursively, as follows: assume that we have calculated in all reflection subgroups $W' \lneq W$ the corresponding generating functions $\mathcal{F}_{W'}^{\mathrm{full}}(g;z)$ of reflection factorizations of g that are full in W'; then $\mathcal{F}_W^{\mathrm{full}}(g;z)$ is the result of subtracting all of these from the generating function $\mathcal{F}_W(g;z)$ of* not necessarily full *reflection factorizations in W.*

In Table 1 below we see this information for the reflection group H_3. Up to conjugacy, it has six classes of proper reflection subgroups (see also [5, §6]), including the trivial subgroup $\{\mathrm{id}\}$*. The first column records the*

Coxeter type of the classes, the second column records the number of different subgroups in each class, and the last column displays the already-calculated generating functions. Notice that we have given them in terms of the variable $X := e^z$ so that one can immediately see the structural properties of Proposition 3.1. In particular, the exponent of the factor $X - 1$ is the full reflection length of the identity in each subgroup W'.

Type of W'	$\#[W']$	$\mathcal{F}^{\text{full}}_{W'}(\text{id}; \log X)$
$\{\text{id}\}$	1	1
A_1	15	$\frac{1}{2} \cdot (X-1)^2 \cdot \frac{1}{X}$
A_1^2	15	$\frac{1}{4} \cdot (X-1)^4 \cdot \frac{1}{X^2}$
A_2	10	$\frac{1}{6} \cdot (X^2+4X+1) \cdot (X-1)^4 \cdot \frac{1}{X^3}$
$I_2(5)$	6	$\frac{1}{10} \cdot (X^6+4X^5+10X^4+20X^3+10X^2+4X+1) \cdot (X-1)^4 \cdot \frac{1}{X^5}$
A_1^3	5	$\frac{1}{8} \cdot (X-1)^6 \cdot \frac{1}{X^3}$

Table 1: The generating functions $\mathcal{F}^{\text{full}}_{W'}(\text{id}; z)$ for all proper reflection subgroups W' of H_3, expressed as Laurent polynomials in $X = e^z$.

Now, a direct computer calculation as in (3.1) *gives that the generating function for not-necessarily-full reflection factorizations of the identity is*

$$\mathcal{F}_{H_3}(\text{id}; \log X) = \frac{1}{120} \cdot (X^{30} + 18X^{20} + 25X^{18} + 32X^{15} + 25X^{12} + 18X^{10} + 1) \cdot \frac{1}{X^{15}}.$$

We can finally write

$$\begin{aligned}
\mathcal{F}^{\text{full}}_{H_3}(\text{id}; \log X) &= \mathcal{F}_{H_3}(\text{id}; \log X) - \sum_{W' \lneq W} \mathcal{F}^{\text{full}}_{W'}(\text{id}; \log X) \\
&= \frac{1}{120} \cdot \Big(X^{24} + 6X^{23} + 21X^{22} + 56X^{21} + 126X^{20} + 252X^{19} + 462X^{18} \\
&\quad + 792X^{17} + 1287X^{16} + 2002X^{15} + 2949X^{14} + 4044X^{13} \\
&\quad + 4804X^{12} + 4044X^{11} + 2949X^{10} + 2002X^{9} + 1287X^{8} + 792X^{7} \\
&\quad + 462X^{6} + 252X^{5} + 126X^{4} + 56X^{3} + 21X^{2} + 6X + 1 \Big) \cdot (X-1)^6 \cdot \frac{1}{X^{15}}.
\end{aligned}$$

Remark 3.3. *The polynomials $\Phi_W(g; X)$ that appear in Proposition 3.1 are very interesting: in real reflection groups, it is easy to show that they are palindromic, but furthermore, it seems (based on empirical data) that they have positive integer coefficients; in most cases (but not always: $G_2 = I_2(6) = G(6,6,2)$ is an exception) their coefficients are unimodal, and their roots have an interesting structure (see Figure 1 and Appendix A). Neither palindromicity nor positivity holds for all complex types; for instance, in the cyclic group $G(6,1,1) \cong \mathbb{Z}/6\mathbb{Z}$ we have*

$$\Phi_{G(6,1,1)}(\text{id}; X) = X^4 + 2X^3 + 3X^2 + 2X - 2.$$

Remark 3.4. *While the machinery of the Frobenius lemma gives a powerful tool to study factorizations in any particular group, the answers it produces for an infinite family of groups may not be usable. For example, in the symmetric group, one can produce explicit formulas for the number of genus-*0 *and genus-*1 *factorizations of an arbitrary element, but such formulas are unknown for the whole generating series $\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}(g; z)$. However, there is one case where some structure is indeed known, namely the case of the identity element $g = \text{id}$. In the seminal work [25], Okounkov showed that an analog of the series $\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}(\text{id}; z)$ satisfies a Toda equation, proving and extending a conjecture of Pandharipande [26]. This implied a recursion for the series which later on Dubrovin–Yang–Zagier [11, Thm. 1, eq. (17)] managed to replace by the following simpler quadratic recursion; as usual, we phrase it in terms of the parameter $X = e^z$ (but notice that [11] count factorizations modulo simultaneous*

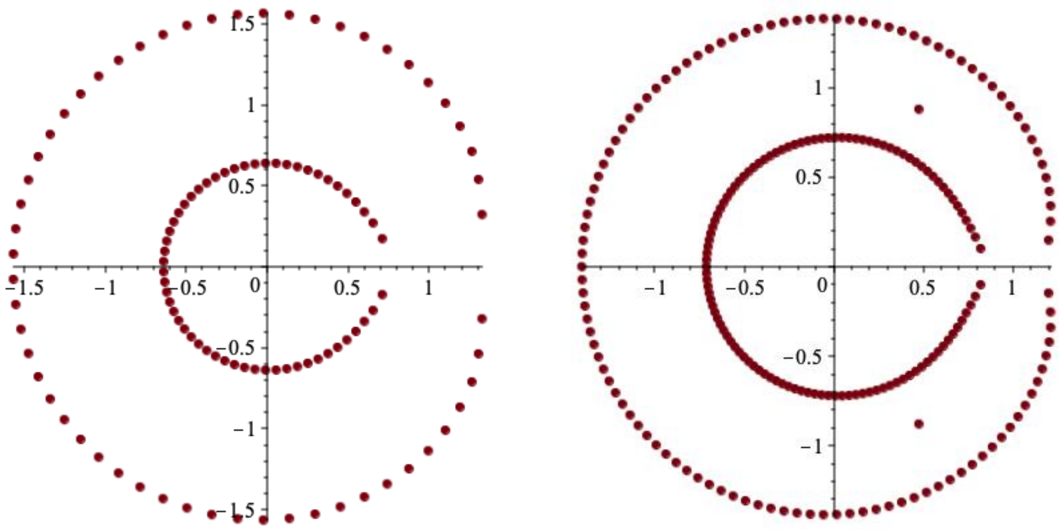


Figure 1: Roots of the polynomials $\Phi_{E_7}(\mathrm{id}; X)$ and $\Phi_{E_8}(\mathrm{id}; X)$. The roots come in pairs $(a, 1/a)$ because the polynomials are palindromic, and this explains the two "components" in the figures, but we cannot a priori justify the rest of the structure (i.e., the orderly arrangement of the roots).

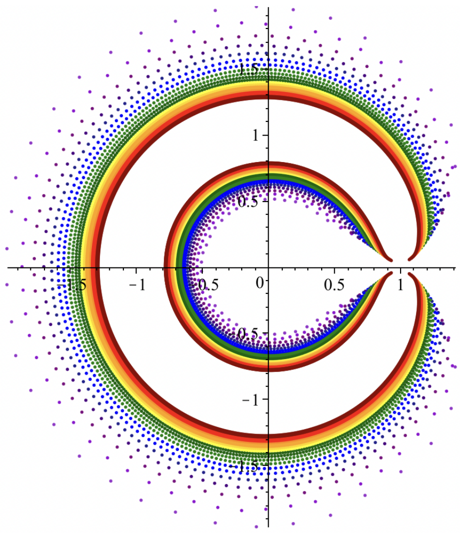


Figure 2: Plot of the roots of $\Phi_{\mathfrak{S}_{2n}}(\mathrm{id}, X)$ for $n = 5, \ldots, 25$. (We skip the roots for odd n to make the plot clearer.)

conjugation by a group element, so that their generating function $C_n(X)$ is related to ours via the equation $\mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}(\mathrm{id}; \log X) = n! \cdot C_n(X)$):

$$n^2(n-1) \cdot \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}(\mathrm{id}; \log X) = \sum_{k=1}^{n-1} k(n-k)^2 \cdot \binom{n}{k} \cdot (X^k - 2 + X^{-k}) \cdot \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_k}(\mathrm{id}; \log X) \cdot \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_{n-k}}(\mathrm{id}; \log X). \quad (3.3)$$

This allows one to compute the function $\mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}(\mathrm{id}; z)$ efficiently.

Remark 3.5. *From the recurrence (3.3), the polynomials $\Phi_{\mathfrak{S}_n}(\mathrm{id}, X)$ have integer coefficients and are palindromic. This forces their roots to come in pairs $(a, 1/a)$ and $(a, \overline{a})$ and thus the plots of the roots have two components corresponding to the $(a, 1/a)$ pairs. Figure 2 shows plots of the roots of $\Phi_{\mathfrak{S}_{2n}}(\mathrm{id}, X)$ for $n = 5, \ldots, 25$. Are these points converging to a limit shape as $n \to \infty$ (for example, to the unit circle)?*

4. Counting full factorizations in the combinatorial family

In this section, we give a formula for the generating function that counts full reflection factorizations of arbitrary length (not just minimum length) of an element g in a group $G(m,p,n)$ from the infinite family. Our formula is valid for arbitrary g and expresses the result in terms of similar generating functions in the symmetric group $\mathfrak{S}_n$ and in the cyclic group $G(m,p,1) \cong \mathbb{Z}/(m/p)\mathbb{Z}$. To avoid the cumbersome notation $\mathcal{F}^{\text{full}}_{G(m,p,n)}(g;z)$ we write the generating functions for the infinite family simply as $\mathcal{F}^{\text{full}}_{m,p,n}(g;z)$.

Theorem 4.1. *For an element $g \in G(m,p,n)$ with k cycles, of colors $a_1,\ldots,a_k$, let $d = \gcd(a_1,\ldots,a_k,p)$. We have*

$$\mathcal{F}^{\text{full}}_{m,p,n}(g;z) = \frac{1}{m^{n-1}} \cdot \mathcal{F}^{\text{full}}_{m,p,1}\big(\zeta_m^{\text{col}(g)}; n\cdot z\big) \cdot \sum_{r\colon r|d} \big(\mu(r)\cdot r^{n+k-2} \cdot \mathcal{F}^{\text{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g);(m/r)\cdot z\big)\big),$$

where μ is the number-theoretic Möbius function.

(Since $p \mid m$ and any two integer representatives of $a_i \in \mathbb{Z}/m\mathbb{Z}$ differ by a multiple of m, the number d is well defined as an integer, and is always a divisor of p.)

Remark 4.1. *In the case that $p = 1$, then always $d = 1$ and so the summation simplifies to a single term. In the case that $p = m$, the group $G(m,m,1)$ is the trivial group, which is still a reflection group but with an empty set of reflections $\mathcal{R} = \emptyset$; then by construction (see* (3.1) *and* (3.2)*) we will have that $\mathcal{F}^{\text{full}}_{m,m,1}(\text{id};z) = 1$.*

Our approach is broadly similar to that of [23] and Polak–Ross [27]; we use the maps π to project factorizations from $G(m,p,n)$ to a simpler subgroup (either $G(p,p,n)$ or $\mathfrak{S}_n$) and then compute the size of each fiber of the projection. Our main tool for counting preimages (Lemma 4.2) is adapted directly from [27]; however, because we consider full (rather than connected) factorizations, we require an additional inclusion-exclusion argument over a certain subgroup lattice.

The structure of the proof is as follows: first, we characterize generating sets of reflections in $G(m,p,n)$ in terms of their projections into $G(m,p,1)$ and into $G(p,p,n)$. Second, we use this characterization to express $\mathcal{F}^{\text{full}}_{m,p,n}(g;z)$ in terms of related series in the cyclic group $G(m,p,1)$ and the color-0 subgroup $G(p,p,n)$. Finally, we show how to express $\mathcal{F}^{\text{full}}_{p,p,n}(g;z)$ in terms of related series in the symmetric group $\mathfrak{S}_n$.

4.1 Characterization of generating sets

We begin by giving a characterization for generating sets of reflections in $G(m,p,n)$ in terms of their projections.

Lemma 4.1. *A set S of reflections in $G(m,p,n)$ generates the whole group if and only if $\text{col}(S)$ generates $p\mathbb{Z}/m\mathbb{Z}$ and $\pi_{m/p}(S)$ generates $G(p,p,n)$.*

Proof. If S generates $G(m,p,n)$ then $\pi_{m/p}(S)$ generates $\pi_{m/p}(G(m,p,n)) = G(p,p,n)$ and $\text{col}(S)$ generates $\text{col}(G(m,p,n)) = p\mathbb{Z}/m\mathbb{Z}$ because both projections are surjective.

Conversely, suppose S is a set of reflections in $G(m,p,n)$ such that $\text{col}(S)$ generates $p\mathbb{Z}/m\mathbb{Z}$ and $\pi_{m/p}(S)$ generates $G(p,p,n)$. Write S as a disjoint union $S = S_{\text{tr}} \cup S_{\text{diag}}$, where S_{tr} consists of transposition-like reflections and S_{diag} consists of diagonal reflections. Consider any element $w = [u;(a_1,\ldots,a_n)]$ in $G(m,p,n)$ (so $p \mid a_1+\ldots+a_n$). By definition, we have $\pi_{m/p}(g) = [u;(m/p\cdot a_1,\ldots,m/p\cdot a_n)] \in G(p,p,n)$. By hypothesis, we can write

$$\pi_{m/p}(g) = \pi_{m/p}(s_1)\cdots\pi_{m/p}(s_k)$$

for some reflections $s_1,\ldots,s_k$ in S. Then

$$g\cdot(s_1\cdots s_k)^{-1} = [\text{id};(b_1,\ldots,b_k)]$$

belongs to the kernel of $\pi_{m/p}$, that is, p divides b_i for $i = 1,\ldots,n$. To finish, it suffices to show that this element $[\text{id};(b_1,\ldots,b_k)]$ belongs to the subgroup $\langle S\rangle$ generated by S. Since $\pi_{m/p}(S)$ generates $G(p,p,n) \supseteq \mathfrak{S}_n$, it must be that S_{tr} is connected, and so products of elements of S_{tr} yield arbitrary underlying permutations. Conjugating a diagonal reflection by such elements produces the diagonal reflections with the same color and arbitrary position for the nontrivial entry, and hence the subgroup

$$\Big\langle\ \langle S\rangle\text{-conjugates of } S_{\text{diag}}\ \Big\rangle$$

generated by S_{diag} and all of its conjugates by products of elements of S are equal to the diagonal subgroup

$$\Big\{\ [\text{id};(a_1,\ldots,a_n)]\colon p \text{ divides } a_1+\ldots+a_n\ \Big\}.$$

This subgroup manifestly contains $\ker(\pi_{m/p})$. Thus $g \in \langle S\rangle$. Since g was arbitrary, $\langle S\rangle = G(m,p,n)$, as claimed. $\square$

Lemma 4.2. *Let $(t_1, \ldots, t_\ell)$ be a sequence of permutations in $\mathfrak{S}_n$ with the property that its subsequence of transpositions is connected, and let $g = t_1 \cdots t_\ell$. There exists a subsequence $(t_{i_1}, \ldots, t_{i_{n-1}})$ of $n-1$ transpositions with the following properties: for every choice of $\{\widetilde{t_j} : j \neq i_1, \ldots, i_{n-1}\}$ and $\widetilde{g} \in G(m,1,n)$ such that*

- $\pi_{m/1}(\widetilde{t_j}) = t_j$ *for* $j \neq i_1, \ldots, i_{n-1}$,
- $\pi_{m/1}(\widetilde{g}) = g$, *and*
- $\operatorname{col}(\widetilde{g}) = \displaystyle\sum_{j \neq i_1, \ldots, i_{n-1}} \operatorname{col}(\widetilde{t_j})$,

there exists a unique choice of reflections $\widetilde{t}_{i_1}, \ldots, \widetilde{t}_{i_{n-1}}$ in $G(m,1,n)$ such that $\pi_{m/1}(\widetilde{t_{i_j}}) = t_{i_j}$ and $\widetilde{g} = \widetilde{t}_1 \cdots \widetilde{t}_\ell$.

Proof. This is essentially Lemma 3.2 and the argument immediately following it in [27]. $\square$

4.2 From $G(m,p,n)$ to $G(p,p,n)$

Next, we show how to express the generating function for full factorizations of an element $g \in G(m,p,n)$ in terms of the series of $G(p,p,n)$-full factorizations of its projection $\pi_{m/p}(g)$.

Proposition 4.1. *Suppose $p < m$. Then for any element $g \in G(m,p,n)$, we have*

$$\mathcal{F}^{\text{full}}_{m,p,n}(g;z) = \frac{1}{(m/p)^{n-1}} \cdot \mathcal{F}^{\text{full}}_{p,p,n}(\pi_{m/p}(g);(m/p)\cdot z) \cdot \mathcal{F}^{\text{full}}_{m,p,1}(\zeta_m^{\operatorname{col}(g)}; n \cdot z).$$

Proof. Let $W = G(m,p,n)$ with $p < m$ and let g be an arbitrary element of W. For each reflection factorization f of g, let f_1 be the result of deleting all copies of the identity from $\pi_{m/p}(f)$ (equivalently, the result of applying $\pi_{m/p}$ only to the transposition-like factors in f) and let f_2 be the result of deleting all copies of the identity from $\zeta_m^{\operatorname{col}(f)}$ (equivalently, of applying $\zeta_m^{\operatorname{col}}$ only to the diagonal factors in f). Then f_1 is a $G(p,p,n)$-reflection factorization of $\pi_{m/p}(g)$ and f_2 is a $G(m,p,1)$-reflection factorization of $\zeta_m^{\operatorname{col}(g)}$, and their lengths sum to the length of f. By Lemma 4.1, f generates W if and only if f_1 generates $G(p,p,n)$ and f_2 generates $G(m,p,1)$. We now fix such a pair (f_1,f_2) and consider how many preimages f it has among the reflection factorizations of g.

Let $k = \#f_1$ and $\ell - k = \#f_2$. Then the number of preimages of (f_1,f_2) among factorizations of *all* elements in W is $\binom{\ell}{k} \cdot (m/p)^k \cdot n^{\ell-k}$: the binomial coefficient counts the ways to assign k positions to transposition-like factors from among ℓ positions, the factor $(m/p)^k$ counts the ways to choose for each of k elements of f_1 a reflection preimage under $\pi_{m/p}$, and the factor $n^{\ell-k}$ counts the ways to choose for each of $\ell-k$ elements of f_2 a reflection preimage under $\zeta_m^{\operatorname{col}}$.

We claim that among these preimages, precisely $\binom{\ell}{k} \cdot (m/p)^{k-(n-1)} \cdot n^{\ell-k}$ are factorizations of g. Indeed, following the first step of the construction in the previous paragraph, choose a length-ℓ factorization f^* of $\pi_{m/p}(g)$ by shuffling f_1 with $\ell-k$ copies of the identity. Now project f^* to the symmetric group, giving us an $\mathfrak{S}_n$-factorization $\pi_{m/1}(f^*)$ of $\pi_{m/1}(g)$. By construction, $\pi_{m/1}(f_1)$ is equal to the subsequence of transpositions in $\pi_{m/1}(f^*)$. Since f_1 generates $G(p,p,n)$, it is connected, and so $\pi_{m/1}(f_1)$ is connected as well. Therefore, Lemma 4.2 provides a special subsequence $t_{i_1}, \ldots, t_{i_{n-1}}$ of $\pi_{m/1}(f^*)$. As in the previous paragraph, there are $n^{\ell-k} \cdot (m/p)^{k-(n-1)}$ ways to choose reflection preimages $\widetilde{t_j}$ in $G(m,p,n)$ of the factors t_j in f^* for $j \neq i_1, \ldots, i_{n-1}$, consistent with the restrictions that the projections under $\pi_{m/p}$ and $\zeta_m^{\operatorname{col}}$ should give f_1 and f_2. By Lemma 4.2, there is a unique way to lift $\pi_{m/1}(t_{i_1}), \ldots, \pi_{m/1}(t_{i_{n-1}})$ to reflections $\widetilde{t}_{i_1}, \ldots, \widetilde{t}_{i_{n-1}}$ in $G(m,p,n)$ so that $g = \widetilde{t}_1 \cdots \widetilde{t}_\ell$. To complete the counting argument, it remains to show that this unique choice is compatible with f_1, i.e., that $\pi_{m/p}(\widetilde{t}_{i_j}) = t_{i_j}$ for $j = 1, \ldots, n-1$.

Let $\widetilde{f} = (\widetilde{t}_1, \ldots, \widetilde{t}_\ell)$. By construction, $\pi_{m/p}(\widetilde{f})$ and f^* are both $G(p,p,n)$-factorizations of $\pi_{m/p}(g)$, with corresponding factors sharing the same underlying permutation, that agree at all positions except possibly $i_1, \ldots, i_{n-1}$. But by Lemma 4.2, there is a unique $G(p,p,n)$-factorization of $\pi_{m/p}(g)$ that agrees with f^* except at the positions $i_1, \ldots, i_{n-1}$. Thus $\pi_{m/p}(\widetilde{f}) = f^*$, so that the constructed factorization $\widetilde{f}$ really does map to the pair (f_1,f_2).

In summary, so far we have shown that there is a map from full length-ℓ $G(m,p,n)$-reflection factorizations of g to pairs (f_1,f_2) such that, for some integer k, f_1 is a full length-k $G(p,p,n)$-reflection factorization of $\pi_{m/p}(g)$ and f_2 is a full length-$(\ell-k)$ $G(m,p,1)$-reflection factorization of $\zeta_m^{\operatorname{col}(g)}$, and that each such pair (f_1,f_2) has

$\binom{\ell}{k}(m/p)^{k-(n-1)}n^{\ell-k}$ preimages under this map. Converting the preceding statement to a generating-function calculation, we have

$$\begin{aligned}
[z^\ell]\mathcal{F}^{\text{full}}_{m,p,n}(g;z) &= \frac{1}{\ell!}\cdot\#\left\{\begin{array}{c}\text{length-}\ell\ G(m,p,n)\text{-refn. facts.}\\ \text{of } g \text{ that generate } G(m,p,n)\end{array}\right\}\\
&= \sum_k \frac{(m/p)^{k-(n-1)}n^{\ell-k}}{k!(\ell-k)!}\cdot\#\left\{\begin{array}{c}\text{length-}k\\ G(p,p,n)\text{-refn.}\\ \text{facts. of } \pi_{m/p}(g) \text{ that}\\ \text{generate } G(p,p,n)\end{array}\right\}\cdot\#\left\{\begin{array}{c}\text{length-}(\ell-k)\\ G(m,p,1)\text{-refn.}\\ \text{facts. of } \zeta_m^{\mathrm{col}(g)} \text{ that}\\ \text{generate } G(m,p,1)\end{array}\right\}\\
&= (p/m)^{n-1}\cdot\sum_k (m/p)^k n^{\ell-k}\cdot[z^k]\mathcal{F}^{\text{full}}_{p,p,n}(\pi_{m/p}(g);z)\cdot[z^{\ell-k}]\mathcal{F}^{\text{full}}_{m,p,1}(\zeta_m^{\mathrm{col}(g)};z)\\
&= (p/m)^{n-1}\cdot[z^\ell]\left(\mathcal{F}^{\text{full}}_{p,p,n}(\pi_{m/p}(g);(m/p)\cdot z)\cdot\mathcal{F}^{\text{full}}_{m,p,1}(\zeta_m^{\mathrm{col}(g)};n\cdot z)\right),
\end{aligned}$$

and the proposition follows immediately. □

4.3 From $G(p,p,n)$ to $\mathfrak{S}_n$

Next, we show how to express the generating function for full factorizations of an element $g \in G(p,p,n)$ in terms of the series of $\mathfrak{S}_n$-full factorizations of its underlying permutation $\pi_{p/1}(g)$.

Proposition 4.2. *Fix an element $g \in G(p,p,n)$ with k cycles, of colors $a_1,\ldots,a_k$, and let $d = \gcd(a_1,\ldots,a_k,p)$. Then*

$$\mathcal{F}^{\text{full}}_{p,p,n}(g;z) = \frac{1}{p^{n-1}}\sum_{r|d}\left(\mu(r)\cdot r^{n+k-2}\cdot\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}(\pi_{p/1}(g);(p/r)\cdot z)\right),$$

where μ is the number-theoretic Möbius function.

(Since any two integer representatives of a_i differ by a multiple of p, the number d is well defined as an integer; it could alternatively be defined as the unique positive integer such that $\{a_1,\ldots,a_k\}$ generates the cyclic subgroup $d\mathbb{Z}/p\mathbb{Z}$ of $\mathbb{Z}/p\mathbb{Z}$.)

Proof. By hypothesis, the colors of all cycles of g are multiples of d. Therefore, recalling the characterization of conjugacy classes in $G(p,1,n)$ from Section 2, we have that g is conjugate by an element of $G(p,1,n)$ to an element of $G(p/d,p/d,n)$. (Here, and in the rest of this proof, we write $G(p/r,p/r,n)$ for the particular subgroup of $G(p,p,n)$ defined in (2.1), rather than the isomorphism class of such groups.) Since $W = G(p,p,n)$ is a normal subgroup of $G(p,1,n)$, this conjugation extends to a bijection between W-full reflection factorizations, and so for convenience, we replace g with its conjugate in $G(p/d,p/d,n)$.

Applying $\pi_{p/1}$ to any reflection factorization of g that is full with respect to W produces a connected $\mathfrak{S}_n$-factorization of $\pi_{p/1}(g)$. The main idea of the proof is to count preimages of each such $\mathfrak{S}_n$-factorization. The remainder of the argument has three main components: first, we enumerate the reflection subgroups of each type that may be generated by such a preimage; then we show that the number of preimages which generate a subgroup of a given type only depends on its isomorphism type; finally, we count the preimages according to the type of subgroup they generate. The final answer then follows from an inclusion-exclusion argument.

Fix a connected transposition factorization $f = (t_1,\ldots,t_\ell)$ of $\pi_{p/1}(g)$. Since f is connected, every subgroup of $W = G(p,p,n)$ that is generated by a lift of f is conjugate in $G(p,1,n)$ to $G(p/r,p/r,n)$ for some $r \mid p$. Furthermore, since the cycle colors of g generate $d\mathbb{Z}/p\mathbb{Z}$, when we restrict to lifts of f that are factorizations of g, we have by Lemma 4.1 that each one generates a subgroup conjugate to $G(p/r,p/r,n)$ for some $r \mid d$ (not just $r \mid p$). Moreover, as observed in Section 2, it is enough to allow conjugation only by *diagonal* elements of $G(p,1,n)$.

We next consider how many distinct $G(p,1,n)$-conjugates of $H = G(p/r,p/r,n)$ in $W = G(p,p,n)$ contain the element $g = [u; d\cdot a]$. Let $\delta = [\mathrm{id}; (d_1,\ldots,d_n)] \in G(p,1,1)^n \subset G(p,1,n)$. By (2.4), we have $\delta^{-1}[v;(b_1,\ldots,b_n)]\delta = [v;(b_1+d_1-d_{v(1)},\ldots,b_n+d_n-d_{v(n)})]$ for any $[v;b] \in W$. Thus if $g = \delta^{-1}g'\delta$ then $g' = [u;b]$ for some n-tuple $b = (b_1,\ldots,b_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ of colors. Consequently, $g \in \delta^{-1}H\delta$ if and only if

$$[u; d\cdot a] = [u;(rb_1+d_1-d_{u(1)},\ldots,rb_n+d_n-d_{u(n)})]$$

for some b. Since $r \mid d$, such a tuple b exists if and only if $r \mid d_i - d_{u(i)}$ for $i = 1,\ldots,n$. There are $p^k\cdot(p/r)^{n-k}$ choices δ such that these equations hold: the color of one element from each of the k cycles of u may be chosen arbitrarily, and the remaining colors in the cycle may be chosen to be any of the p/r colors that differ from the first choice by a multiple of r. Among these choices for δ, there are $p\cdot(p/r)^{n-1}$ that normalize H (every two entries of δ must differ by a multiple of r), so by the orbit-stabilizer theorem there are r^{k-1} distinct copies of H that contain g.

Furthermore, suppose that $\delta^{-1}H\delta$ is an isomorphic copy of H containing g, so that

$$g = \delta^{-1}[u; r \cdot b]\delta = [u; (rb_1 + d_1 - d_{u(1)}, \ldots, rb_n + d_n - d_{u(n)})].$$

Choose a new diagonal element $\delta' = [\mathrm{id}; (d'_1, \ldots, d'_n)]$ as follows: select one entry j in each cycle of u and set $d'_j = 0$, and determine the other values of d'_i by the relation $d'_i - d'_{u(i)} = d_i - d_{u(i)}$ for $i = 1, \ldots, n$. By construction, we have (first) that $(\delta')^{-1}[u; r \cdot b]\delta' = g$, and (second) that $d_i \in r\mathbb{Z}/p\mathbb{Z}$ for each i and so $\delta' \in H$. Consequently $\delta^{-1}\delta'$ commutes with g and $(\delta^{-1}\delta')H(\delta^{-1}\delta')^{-1} = \delta^{-1}H\delta$. Thus, conjugation by $\delta^{-1}\delta'$ extends to a bijection between lifts of f that are H-reflection factorizations of g and lifts of f that are $\delta^{-1}H\delta$-reflection factorizations of g – or in other words, all the r^{k-1} distinct copies of H that contain g also contain the same number of factorizations of g that are lifts of f.

Now let us count lifts of f according to what subgroup they generate. Let $a_{p/r}$ be the number of lifts of f that factor g and generate the group $G(p/r, p/r, n)$, and let $b_{p/r}$ be the (generally larger) number of lifts of f that factor g and generate any subgroup of $G(p/r, p/r, n)$. We next establish a relationship between the as and the bs. It follows from the arguments of the two last paragraphs that every lift of f that factors g and generates a subgroup of $G(p/r, p/r, n)$ in particular generates, for some integer r' such that $r \mid r' \mid d$, one of the $(r'/r)^{k-1}$ distinct subgroups of $G(p/r, p/r, n)$ that are isomorphic to $G(p/r', p/r', n)$ and contain g; and that the number of factorizations that generate each of these subgroups is $a_{p/r'}$. Therefore for any $r \mid d$ we have

$$b_{p/r} = \sum_{r' \colon r|r'|d} (r'/r)^{k-1} \cdot a_{p/r'},$$

or equivalently

$$r^{k-1} \cdot b_{p/r} = \sum_{r' \colon r|r'|d} (r')^{k-1} \cdot a_{p/r'}.$$

By Möbius inversion, it follows that the number $a_p = a_{p/1}$ of lifts of f that factor g and generate the full group $W = G(p, p, n)$ is

$$a_p = \sum_{r|d} \mu(r) \cdot r^{k-1} \cdot b_{p/r}. \tag{4.1}$$

Next, we compute the number $b_{p/r}$.

Since f is connected, Lemma 4.2 promises a special subsequence $t_{i_1}, \ldots, t_{i_{n-1}}$ with the following property: each of the $p^{\ell-(n-1)}$ ways of lifting the t_j for $j \neq i_1, \ldots, i_{n-1}$ into W determines a unique lift of f to a W-factorization of g. Moreover, if each of the $\ell-(n-1)$ non-special factors is lifted to a reflection in $G(p/r, p/r, n)$, then they and the product g all live inside $G(p/r, 1, n)$; in this case, Lemma 4.2 promises that the remaining special factors will also lift to transposition-like reflections inside $G(p/r, 1, n)$. Thus, $b_{p/r} = (p/r)^{\ell-(n-1)}$ of the lifts generate a subgroup of $G(p/r, p/r, n)$. Substituting this into (4.1), we conclude that the number $a_p = a_{p/1}$ of lifts of f that factor g and generate the full group $W = G(p, p, n)$ is

$$a_p = p^{k-1} \cdot \sum_{r|d} \mu(r) \cdot (p/r)^{\ell-n-k+2}.$$

Now taking into account all choices of f, we have

$$[z^\ell]\mathcal{F}^{\mathrm{full}}_{p,p,n}(g; z) = p^{k-1} \cdot \sum_{r|d} \mu(r) \cdot (p/r)^{\ell-n-k+2} \cdot [z^\ell]\mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}(g; z).$$

The desired result follows immediately. □

4.4 Completing the proof of Theorem 4.1

Let g be an arbitrary element of $W = G(m, p, n)$, and suppose that g has k cycles, of colors $a_1, \ldots, a_k \in \mathbb{Z}/m\mathbb{Z}$. By Proposition 4.1, we have

$$\mathcal{F}^{\mathrm{full}}_{m,p,n}(g; z) = \frac{1}{(m/p)^{n-1}} \cdot \mathcal{F}^{\mathrm{full}}_{p,p,n}(\pi_{m/p}(g); (m/p) \cdot z) \cdot \mathcal{F}^{\mathrm{full}}_{m,p,1}(\zeta_m^{\mathrm{col}(g)}; n \cdot z). \tag{4.2}$$

Viewed as an element of $G(m, p, n)$, $\pi_{m/p}(g)$ has cycles of colors $\frac{m}{p}a_i \in \mathbb{Z}/m\mathbb{Z}$; therefore, when viewed as an element of $G(p, p, n)$, its cycles have colors $a_i \in \mathbb{Z}/p\mathbb{Z}$. Setting $d = \gcd(a_1, \ldots, a_k, p)$, we have by Proposition 4.2 that

$$\mathcal{F}^{\mathrm{full}}_{p,p,n}(\pi_{m/p}(g); (m/p) \cdot z) = \frac{1}{p^{n-1}} \sum_{r|d} \left(\mu(r) \cdot r^{n+k-2} \cdot \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}\big(\pi_{p/1}(\pi_{m/p}(g)); (p/r) \cdot (m/p) \cdot z\big) \right)$$

$$= \frac{1}{p^{n-1}} \sum_{r|d} \left(\mu(r) \cdot r^{n+k-2} \cdot \mathcal{F}^{\text{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g); (m/r) \cdot z\big) \right).$$

Plugging this into (4.2) immediately gives the result. □

5. Recovering leading terms

In this section we extract the leading term of the generating series $\mathcal{F}^{\text{full}}_{m,p,n}(g; z)$ in Theorem 4.1 to obtain the number of *minimum-length* full factorizations of an arbitrary element in $G(m, p, n)$. The answer will involve the Euler totient function $\varphi(m)$, the *Jordan totient function*

$$J_2(m) := \sum_{d|m} \mu(m/d) \cdot d^2, \tag{5.1}$$

which counts elements of order m in the group $(\mathbb{Z}/m\mathbb{Z})^2$ [34, `https://oeis.org/A007434`], and Hurwitz numbers of the symmetric group of genus 0 (given by Theorem 1.1) and genus 1. The latter also have an explicit formula.

Theorem 5.1 (Goulden–Jackson [17]; Vakil [38])**.** *For $\lambda = (\lambda_1, \ldots, \lambda_k)$, the number of genus-1 transitive transposition factorizations in $\mathfrak{S}_n$ of a permutation of cycle type λ is*

$$H_1(\lambda) = \frac{1}{24}(n+k)! \left(\prod_{i=1}^{k} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!} \right) \left(n^k - n^{k-1} - \sum_{i=2}^{k} (i-2)! \cdot e_i(\lambda) \cdot n^{k-i} \right),$$

where e_i denotes the ith elementary symmetric function.

In order to extract the leading coefficients of $\mathcal{F}^{\text{full}}_{m,p,n}(g; z)$, we need first to determine the full reflection length $\ell^{\text{full}}_W(g)$ for g in $W = G(m, p, n)$. We do this by calculating from Theorem 4.1 the degree of the leading term in the generating function $\mathcal{F}^{\text{full}}_{m,p,n}(g; z)$. Although the relation between generating functions in Theorem 4.1 is stated uniformly for all m, p, n, it is most convenient to formulate our corollary separately for $G(m, m, n)$ and for $G(m, p, n)$ with $p < m$. This is because of the appearance of the cyclic group $p\mathbb{Z}/m\mathbb{Z}$ in Theorem 4.1 – when $p = m$, this group is trivial and does not contribute to the (full) reflection length (see Remark 4.1).

Corollary 5.1. *Let $W = G(m, p, n)$. For an element $g \in W$ with k cycles, of colors $a_1, \ldots, a_k$, let $d = \gcd(a_1, \ldots, a_k, p)$ and $a = \gcd(\operatorname{col}(g), m)/p$. If $m = p$, we have*

$$\ell^{\text{full}}_W(g) = \begin{cases} n + k - 2, & \text{if } d = 1 \\ n + k, & \text{if } d \neq 1, \end{cases}$$

while if $m \neq p$, we have

$$\ell^{\text{full}}_W(g) = \begin{cases} n + k - 1, & \text{if } a = 1 \text{ and } d = 1 \\ n + k, & \text{if } a \neq 1 \text{ and } d = 1 \\ n + k + 1, & \text{if } a = 1 \text{ and } d \neq 1 \\ n + k + 2, & \text{if } a \neq 1 \text{ and } d \neq 1. \end{cases}$$

Proof. We need only compute the degree (in z) of the lowest-order term of the generating function $\mathcal{F}^{\text{full}}_{m,p,n}(g; z)$. Looking at the right side of the equation in Theorem 4.1, we make the following observations.

(1) When $m = p$, the group $G(m, m, 1)$ is the trivial group and the generating function $\mathcal{F}^{\text{full}}_{m,m,1}(\zeta_m^{\operatorname{col}(g)}; n \cdot z)$ equals 1 as we explain in Remark 4.1, contributing the factor z^0 for the degree of the lowest-order term of $\mathcal{F}^{\text{full}}_{m,p,n}(g; z)$.

(2) When $m \neq p$, the group $G(m, p, 1)$ is the cyclic group of order m/p and therefore the generating function $\mathcal{F}^{\text{full}}_{m,p,1}(\zeta_m^{\operatorname{col}(g)}; n \cdot z)$ will either contribute a factor of z^1 (if $\zeta_m^{\operatorname{col}(g)}$ generates $G(m, p, 1)$) or z^2 (if not) to the degree of the lowest-order monomial. The condition $\zeta_m^{\operatorname{col}(g)}$ generates $G(m, p, 1)$ is equivalent to $\gcd(\operatorname{col}(g), m) = p$, i.e., to $a = 1$.

(3) When $d = 1$, the sum factor has a unique term $\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}(\pi_{m/1}(g); m \cdot z)$. By Theorem 1.1, since $\pi_{m/1}(g)$ has k cycles, this generating function will contribute the factor z^{n+k-2}.

(4) When $d \neq 1$, the coefficient of z^{n+k-2} in $\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}(\pi_{m/1}(g); (m/r)\cdot z)$ is a multiple of

$$\sum_{r\colon r|d} \mu(r)\cdot r^{n+k-2}\cdot (m/r)^{n+k-2} = m^{n+k-2}\cdot \sum_{r\colon r|d}\mu(r) = 0,$$

leaving z^{n+k} as the contribution to the lowest-order monomial. Indeed, its coefficient

$$\sum_{r\colon r|d} \mu(r)\cdot r^{n+k-2}\cdot (m/r)^{n+k} = \frac{m^{n+k}}{d^2}\sum_{b\colon b|d}\mu(d/b)\cdot b^2$$

equals $\dfrac{m^{n+k}}{d^2}J_2(d)$ by (5.1) and thus is nonzero.

The statement of the corollary is immediate after the previous points. □

Remark 5.1. *It is interesting to observe that the formulas for full reflection length in Corollary 5.1 are efficiently computable (indeed, the computation is completely straightforward). By contrast, although an explicit combinatorial formula exists for reflection length in $G(m,p,n)$ [33, Thm. 4.4], it is computationally intractable in general – see [24, Rem. 2.4].*

Theorem 5.2. *For an element $g \in G(m,p,n)$ with k cycles, of colors $a_1,\ldots,a_k$, let $d = \gcd(a_1,\ldots,a_k,p)$ and $a = \gcd(\mathrm{col}(g),m)/p$. If $m = p$, we have*

$$F^{\text{full}}_{m,m,n}(g) = \begin{cases} m^{k-1}\cdot H_0(\lambda), & \text{if } d = 1\\ m^{k+1}\cdot \frac{J_2(d)}{d^2}\cdot H_1(\lambda), & \text{if } d\neq 1,\end{cases}$$

while if $m \neq p$, we have

$$F^{\text{full}}_{m,p,n}(g) = \begin{cases} n(n+k-1)\cdot m^{k-1}\cdot H_0(\lambda), & \text{if } a = 1 \text{ and } d = 1\\ \frac{n^2(n+k)(n+k-1)m^k}{2}\cdot\frac{\varphi(a)}{pa}\cdot H_0(\lambda), & \text{if } a\neq 1 \text{ and } d = 1\\ n(n+k+1)m^{k+1}\cdot\frac{J_2(d)}{d^2}\cdot H_1(\lambda), & \text{if } a = 1 \text{ and } d\neq 1\\ \frac{n^2(n+k+2)(n+k+1)m^{k+2}}{2}\cdot\frac{\varphi(a)}{pa}\cdot\frac{J_2(d)}{d^2}\cdot H_1(\lambda), & \text{if } a\neq 1 \text{ and } d\neq 1,\end{cases}$$

where λ is the cycle type of the underlying permutation $\pi_{m/1}(g)$.

Proof. Since g has k cycles, the full reflection length of the permutation $\pi_{m/1}(g)$ is $n+k-2$. We consider the same cases as of Corollary 5.1.

If $m = p$, then in all cases $\mathcal{F}^{\text{full}}_{m,m,1}\big(\zeta_m^{\mathrm{col}(g)}; n\cdot z\big) = 1$. If $d = 1$ then $\ell^{\text{full}}_W(g) = n+k-2$ by Corollary 5.1. Thus by Theorem 4.1 we have

$$\begin{aligned} F^{\text{full}}_{m,m,n}(g) &= \frac{(n+k-2)!}{m^{n-1}}\cdot [z^{n+k-2}]\,\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g); m\cdot z\big)\\ &= \frac{(n+k-2)!}{m^{n-1}}\cdot\frac{m^{n+k-2}H_0(\lambda)}{(n+k-2)!}\\ &= m^{k-1}\cdot H_0(\lambda).\end{aligned}$$

If instead $d\neq 1$, then $\ell^{\text{full}}_W(g) = n+k$ by Corollary 5.1. Thus by Theorem 4.1 we have

$$\begin{aligned} F^{\text{full}}_{m,m,n}(g) &= \frac{(n+k)!}{m^{n-1}}\cdot [z^{n+k}]\sum_{r:r|d}\big(\mu(r) r^{n+k-2}\mathcal{F}^{\text{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g);(m/r)\cdot z\big)\big)\\ &= \frac{1}{m^{n-1}}\cdot H_1(\lambda)\cdot\sum_{r:r|d}\mu(r)\cdot r^{n+k-2}\cdot (m/r)^{n+k}.\end{aligned}$$

By (5.1), this simplifies to

$$\frac{1}{m^{n-1}}\cdot H_1(\lambda)\cdot\frac{m^{n+k}}{d^2}\cdot J_2(d) = \frac{m^{k+1}}{d^2}\cdot J_2(d)\cdot H_1(\lambda),$$

as claimed.

Now suppose $m \neq p$. Observe that $\frac{m}{\gcd(\mathrm{col}(g),m)} = \frac{m}{pa}$ is precisely the order of $\mathrm{col}(g)$ in the cyclic group $\mathbb{Z}/m\mathbb{Z}$ (or any subgroup thereof that contains it). In particular, $\zeta_m^{\mathrm{col}(g)}$ generates $G(m,p,1) \cong \mathbb{Z}/(m/p)\mathbb{Z}$ if and only if $a = 1$.

If $d = 1$, then, as in the case $p = m$, the summation in Theorem 4.1 consists of a single term, whose lowest-degree term is $H_0(\lambda) \cdot \frac{m^{n+k-2} z^{n+k-2}}{(n+k-2)!}$. If $a = 1$, so that $\zeta_m^{\mathrm{col}(g)}$ generates $G(m,p,1)$, then the lowest-degree term of $\mathcal{F}^{\mathrm{full}}_{m,p,1}(\zeta_m^{\mathrm{col}(g)}; n \cdot z)$ is $n \cdot z$. It follows that the term of degree $\ell_W^{\mathrm{full}}(g) = n + k - 1$ in this case is

$$F^{\mathrm{full}}_{m,p,n}(g) = \frac{(n+k-1)!}{m^{n-1}} \cdot H_0(\lambda) \cdot \frac{m^{n+k-2} z^{n+k-2}}{(n+k-2)!} \cdot n = n(n+k-1) \cdot m^{k-1} \cdot H_0(\lambda),$$

as claimed.

Still considering the case $p \neq m$ and $d = 1$, let us suppose instead that $a \neq 1$. In this case, the contribution from $\mathcal{F}^{\mathrm{full}}_{m,p,1}(\zeta_m^{\mathrm{col}(g)}; n \cdot z)$ is $c \cdot n^2 \cdot z^2/2$ where c is the number of full factorizations of length 2 of $\zeta_m^{\mathrm{col}(g)}$ in the cyclic group $G(m,p,1)$. More generally, let $a(R,N)$ be the number of length-2 (not necessarily reflection) factorizations of an element of order N/R in the cyclic group of order N that do not lie in a proper subgroup. The number of all length-2 factorizations of such an element is $b(R,N) = N$, and consequently

$$N = \sum_{r\colon r|R} a(R/r, N/r)$$

for all N such that $R \mid N$. By Möbius inversion, it follows that

$$a(R,N) = \sum_{r:r|R} \mu(r) \cdot b(R/r, N/r) = \frac{N}{R} \sum_{r:r|R} \mu(r) \cdot R/r = \frac{N}{R} \varphi(R).$$

Now specializing to our particular case $N = \frac{m}{p}$ and $R = a$, since the element of order N/R is not a generator, the factorizations counted by $a(R,N)$ are actually full reflection factorizations. Thus we have $c = \frac{m}{pa}\varphi(a)$ and we obtain

$$\begin{aligned} F^{\mathrm{full}}_{m,p,n}(g) &= \frac{n^2 \cdot (n+k)!}{2 \cdot m^{n-1}} \cdot \frac{m}{pa} \cdot \varphi(a) \cdot \frac{m^{n+k+2} \cdot H_0(\lambda)}{(n+k-2)!} \\ &= \frac{n^2 (n+k)(n+k-1) m^k}{2pa} \cdot \varphi(a) \cdot H_0(\lambda), \end{aligned}$$

as claimed.

On the other hand, if $d \neq 1$, then, again as in the case $p = m$, the lowest-degree term from the summation factor is

$$[z^{n+k}] \sum_{r,r|d} \big(\mu(r) r^{n+k-2} \mathcal{F}^{\mathrm{full}}_{\mathfrak{S}_n}\big(\pi_{m/1}(g); (m/r) \cdot z\big)\big) = \frac{m^{n+k} \cdot J_2(d) \cdot H_1(\lambda)}{(n+k)! \cdot d^2}.$$

Consequently, when $a = 1$ we have

$$\begin{aligned} F^{\mathrm{full}}_{m,p,n}(g) &= \frac{(n+k+1)!}{m^{n-1}} \cdot n \cdot \frac{m^{n+k} \cdot J_2(d) \cdot H_1(\lambda)}{(n+k)! \cdot d^2} \\ &= \frac{n(n+k+1) m^{k+1}}{d^2} \cdot J_2(d) \cdot H_1(\lambda), \end{aligned}$$

while when $a \neq 1$ we instead have

$$\begin{aligned} F^{\mathrm{full}}_{m,p,n}(g) &= \frac{n^2 \cdot (n+k+2)!}{2 \cdot m^{n-1}} \cdot \frac{m}{pa} \cdot \varphi(a) \cdot \frac{m^{n+k} \cdot J_2(d) \cdot H_1(\lambda)}{(n+k)! \cdot d^2} \\ &= \frac{n^2 (n+k+1)(n+k+2) m^{k+2}}{2d^2 pa} \cdot \varphi(a) \cdot J_2(d) \cdot H_1(\lambda), \end{aligned}$$

as claimed. □

Remark 5.2. *If W is a complex reflection group of exceptional type, we can recover the number $F_W^{\mathrm{full}}(g)$ of minimum-length full reflection factorizations of an element g by computing the series $\mathcal{F}_W^{\mathrm{full}}(g; z)$ as in Section 3 and extracting the lowest-order term of the series. When we express the generating function as a Laurent polynomial in $X = e^z$ as in Proposition 3.1, this gives*

$$F_W^{\mathrm{full}}(g) = \frac{1}{\#W} \cdot \Phi_W(g; 1) \cdot \ell_W^{\mathrm{full}}(g)!.$$

For example, continuing where Example 3.2 left off, we have that $\Phi_{H_3}(\mathrm{id};1) = 28800$ *and so the number of minimum-length full reflection factorizations of the identity in* H_3 *is given by*

$$\begin{aligned} F^{\mathrm{full}}_{H_3}(\mathrm{id}) &= \frac{1}{\#H_3} \cdot \Phi_{H_3}(\mathrm{id};1) \cdot \ell^{\mathrm{full}}_{H_3}(\mathrm{id})! \\ &= \frac{1}{120} \cdot 28800 \cdot 6! \\ &= 172800. \end{aligned}$$

Acknowledgements

We thank David Jackson and Jiayuan Wang for their helpful comments and suggestions. This work was facilitated by computer experiments using Sage [30], its algebraic combinatorics features developed by the Sage-Combinat community [29], and CHEVIE [15]. We thank the anonymous referee for helpful comments and suggestions.

The second-named author was partially supported by an ORAU Powe award and by a grant from the Simons Foundation (634530). The third-named author was partially supported by NSF grant DMS-1855536.

An extended abstract of this work appeared as [7].

References

[1] G. Bini, I. P. Goulden, and D. M. Jackson, *Transitive factorizations in the hyperoctahedral group*, Canad. J. Math., 60 (2008), 297–312.

[2] M. Bousquet-Mélou and G. Schaeffer, *Enumeration of planar constellations*, Adv. in Appl. Math. 24 (2000), 337–368.

[3] G. Chapuy and T. Douvropoulos, *Coxeter factorizations with generalized Jucys-Murphy weights and matrix tree theorems for reflection groups*, Arxiv: 2012.04519, 2020.

[4] G. Chapuy and C. Stump, *Counting factorizations of Coxeter elements into products of reflections*, J. Lond. Math. Soc. (2) 90 (2014), 919–939.

[5] J. M. Douglass, G. Pfeiffer, and G. Röhrle, *On reflection subgroups of finite Coxeter groups*, Comm. Algebra 41 (2013), 2574–2592.

[6] T. Douvropoulos, *On enumerating factorizations in reflection groups*, Arxiv: 1811.06566, 2018.

[7] T. Douvropoulos, J. B. Lewis, and A. H. Morales, *Hurwitz numbers for reflection groups*, Sém. Lothar. Combin. 85B (2021), Article 66.

[8] T. Douvropoulos, J. B. Lewis, and A. H. Morales, *Hurwitz numbers for reflection groups I: Generatingfunctionology*, Arxiv: 2112.03427, 2021.

[9] T. Douvropoulos, J. B. Lewis, and A. H. Morales, *Hurwitz numbers for reflection groups II: Parabolic quasi-Coxeter elements*, In preparation.

[10] T. Douvropoulos, J. B. Lewis, and A. H. Morales, *Hurwitz numbers for reflection groups III: Uniform formulas*, In preparation.

[11] B. Dubrovin, D. Yang, and D. Zagier, *Classical Hurwitz numbers and related combinatorics*, Mosc. Math. J. 17 (2017), 601–633.

[12] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. 146 (2001), 297–327.

[13] F. G. Frobenius, *Uber gruppencharacktere*, In *Gesammelte Abhandlungen. Bände III*, Springer-Verlag, Berlin-New York, 1968. Herausgegeben von J.-P. Serre.

[14] Z. Gao, A. MacFie, and Q. Wang, *Counting compositions over finite abelian groups*, Electron. J. Combin. 25 (2018), Paper No. 2.19.

[15] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, *CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras*, Appl. Algebra Engrg. Comm. Comput. 7 (1996), 175–210.

[16] I. P. Goulden and D. M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. 125 (1997), 51–60.

[17] I. P. Goulden and D. M. Jackson, *A proof of a conjecture for the number of ramified coverings of the sphere by the torus*, J. Combin. Theory Ser. A 88 (1999), 246–258.

[18] I. P. Goulden and D. M. Jackson, *Transitive factorizations of permutations and geometry*, In P. Hersh, T. Lam, P. Pylyavskyy, and V. Reiner, editors, *The mathematical legacy of Richard P. Stanley*, pages 189–201. Amer. Math. Soc., Providence, RI, USA (2016).

[19] I. P. Goulden, D. M. Jackson, and R. Vakil, *Towards the geometry of double Hurwitz numbers*, Adv. Math. 198 (2005), 43–92.

[20] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 39 (1891), 1–60.

[21] D. M. Jackson, *Some combinatorial problems associated with products of conjugacy classes of the symmetric group*, J. Combin. Theory Ser. A 49 (1988), 363–369.

[22] S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications*, volume 141 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, Germany (2004).

[23] J. B. Lewis and A. H. Morales, *Factorization problems in complex reflection groups*, Canad. J. Math. 73 (2021), 899–946.

[24] J. B. Lewis and J. Wang, *The Hurwitz action in complex reflection groups*, Combinatorial Theory 2 (2022), #12.

[25] A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. 7 (2000), 447–453.

[26] R. Pandharipande, *The Toda equations and the Gromov-Witten theory of the Riemann sphere*, Lett. Math. Phys. 53 (2000), 59–74.

[27] E. Polak and D. Ross, *Polynomiality of factorizations in reflection groups*, 2020, to appear Canadian J. of Math. Available at `https://doi.org/10.4153/S0008414X21000663`. Arxiv: 2004.13213.

[28] E. W. Read, *On the finite imprimitive unitary reflection groups*, J. Algebra 45 (1977), 439–452.

[29] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, Available at `https://wiki.sagemath.org/combinat`.

[30] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.0)*, Available at `https://www.sagemath.org`.

[31] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. 6 (1954), 274–304.

[32] J. Shi, *Congruence classes of presentations for the complex reflection groups $G(m,1,n)$ and $G(m,m,n)$*, Indag. Math. (N.S.) 16 (2005), 267–288.

[33] J. Shi, *Formula for the reflection length of elements in the group $G(m,p,n)$*, J. Algebra 316 (2007), 284–296.

[34] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2019. Available at `https://oeis.org`.

[35] R. P. Stanley, *Factorization of permutations into n-cycles*, Discrete Math. 37 (1981), 255–262.

[36] R. P. Stanley, *Enumerative combinatorics*, Volume 2, Cambridge University Press, Cambridge, UK (1999).

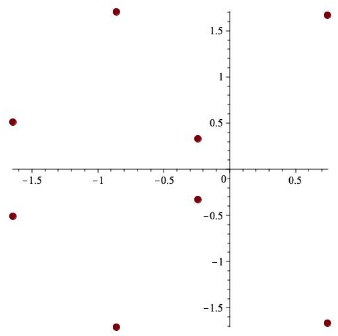
[37] V. Strehl, *Minimal transitive products of transpositions—the reconstruction of a proof of A. Hurwitz*, Sém. Lothar. Combin. 37 (1996), Article S37c.

[38] R. Vakil, *Genus 0 and 1 Hurwitz numbers: recursions, formulas, and graph-theoretic interpretations*, Trans. Amer. Math. Soc. 353 (2001), 4025–4038.

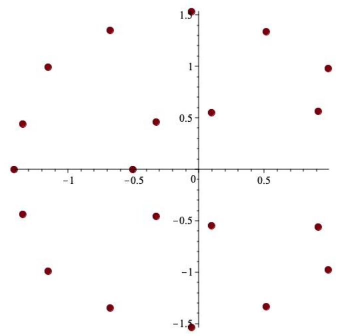
A. Root plots of the polynomials $\Phi_W(\mathrm{id}; X)$

We give below the plots of the roots of the polynomials $\Phi_W(\mathrm{id}; X)$ of Proposition 3.1 in the complex plane for all exceptional complex reflection groups. For the polynomials themselves, see the data file attached as a supplementary file to the arXiv version of this paper [8].

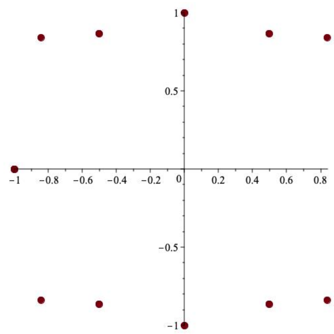
Rank 2



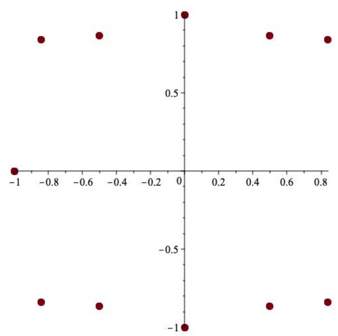
G_4



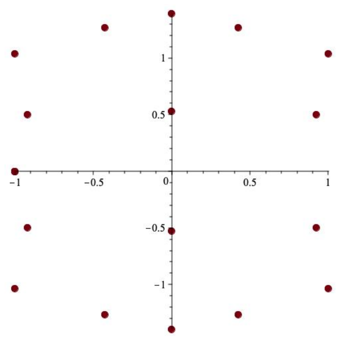
G_5



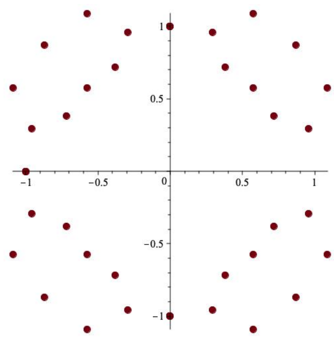
G_6



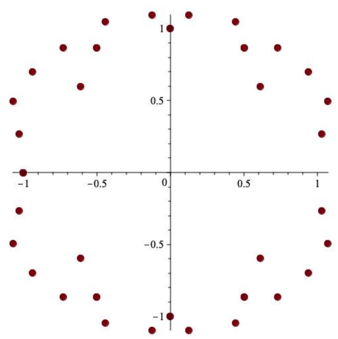
G_7



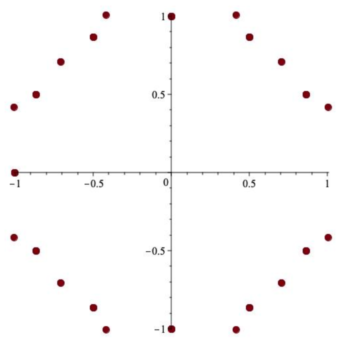
G_8



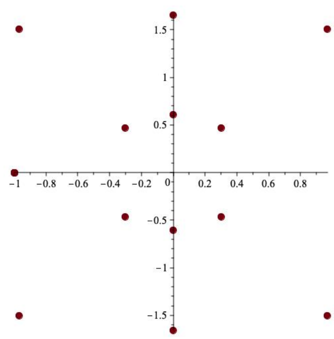
G_9



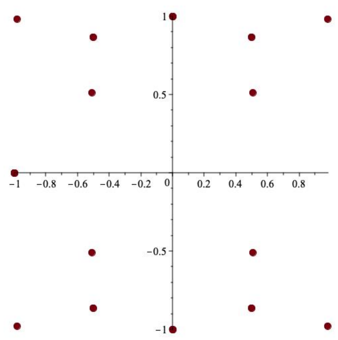
G_{10}



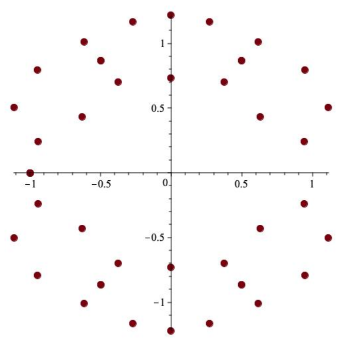
G_{11}



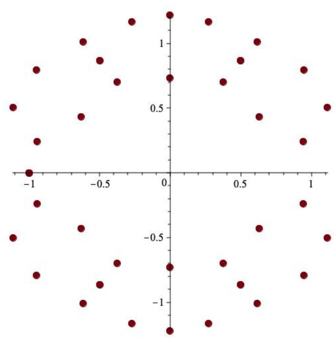
G_{12}



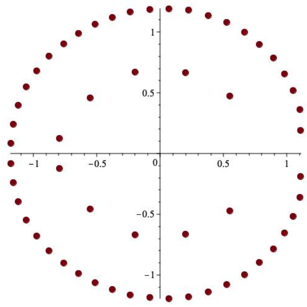
G_{13}



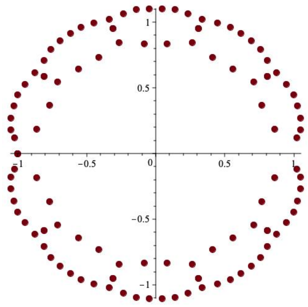
G_{14}



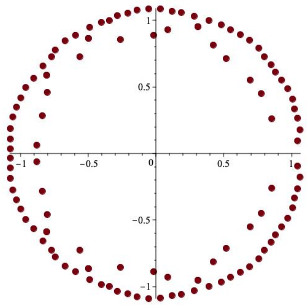
G_{15}



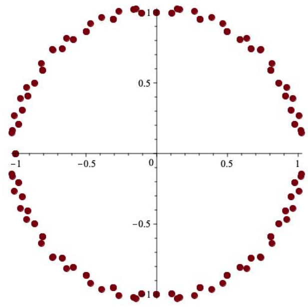
G_{16}



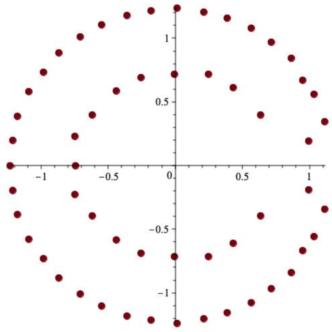
G_{17}



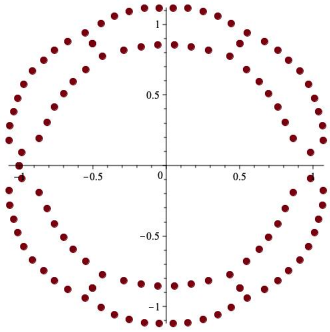
G_{18}



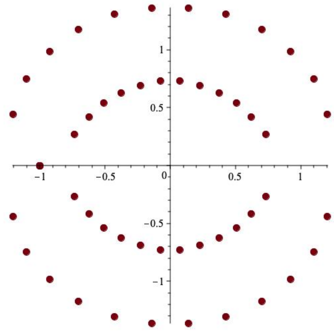
G_{19}



G_{20}

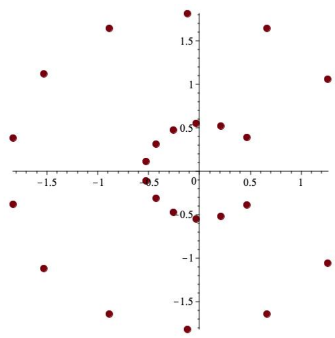


G_{21}

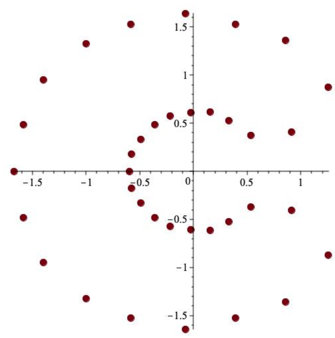


G_{22}

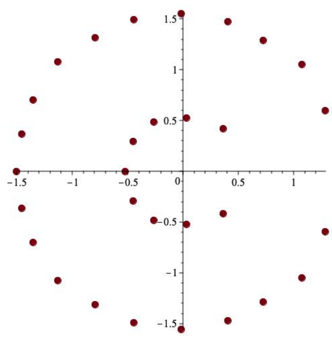
Rank 3



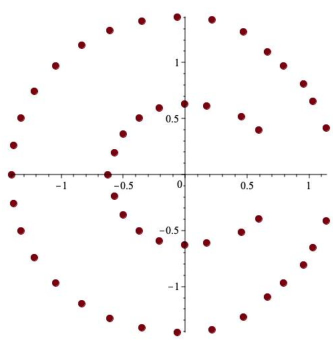
$G_{23} = H_3$



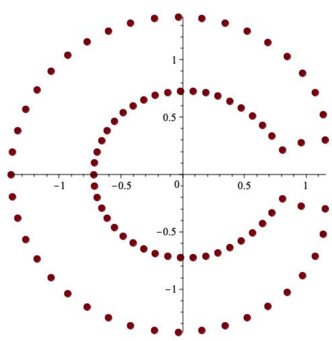
G_{24}



G_{25}

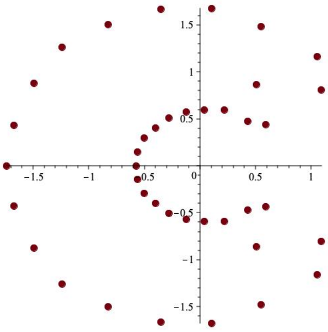


G_{26}

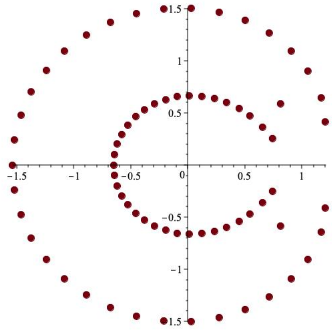


G_{27}

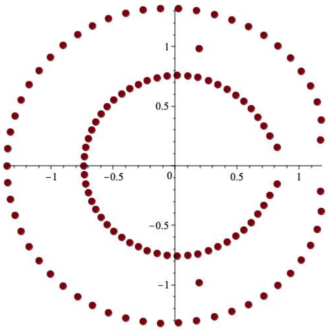
Rank 4



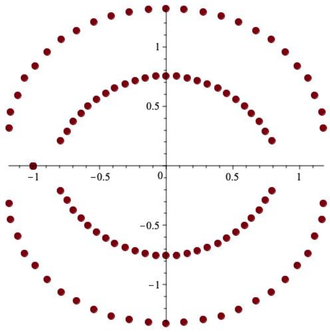
$G_{28} = F_4$



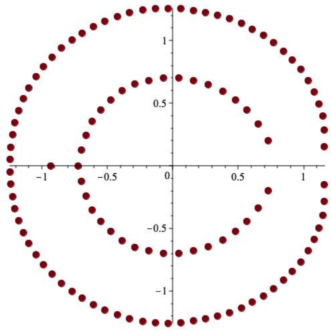
G_{29}



$G_{30} = H_4$

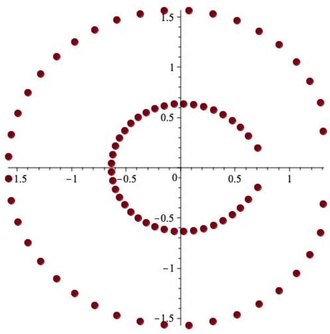


G_{31}

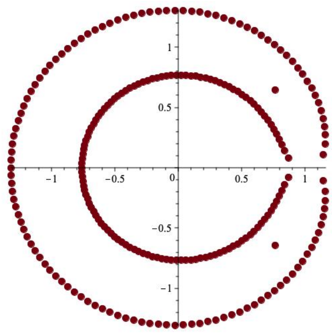


G_{32}

Ranks 5 and 6

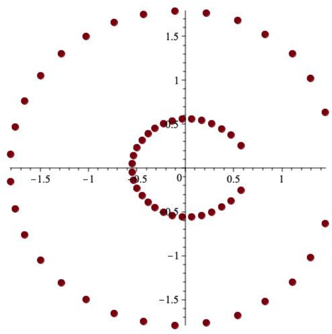


G_{33}

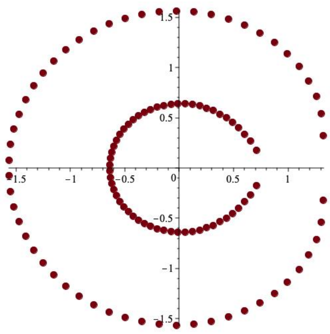


G_{34}

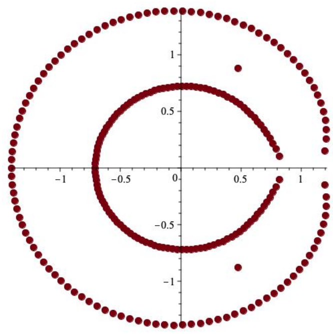
E-series



$G_{35} = E_6$



$G_{36} = E_7$



$G_{37} = E_8$