Boolean Intersection Ideals of Permutations in the Bruhat Order

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ABSTRACT: Motivated by recent work with Mazorchuk, we characterize the conditions under which the intersection of two principal order ideals in the Bruhat order is boolean. That characterization is presented in three versions: in terms of reduced words, in terms of permutation patterns, and in terms of permutation support. The equivalence of these properties follows from an analysis of what it means to have a specific letter repeated in a permutation's reduced words; namely, that a specific 321-pattern appears.

1. Introduction

In recent work with Mazorchuk, we studied intersections of a boolean principal order ideal with an arbitrary principal order ideal in the Bruhat order of the symmetric group [5]. That boolean requirement was enough to guarantee that the grade of the simple module indexed by a boolean element is given by Lusztig's **a**-function (see [4]). This revealed, among other things, an important relevance to understanding intersections of principal order ideals in the Bruhat order. It is somewhat surprising that, despite the quantity of literature about Coxeter groups and the Bruhat order, not much has been written about the intersections of principal order ideals in these posets, and the results of [5] highlight the need to address this lack. That work also reinforced the value of boolean intervals and order ideals in this poset, which have been studied previously by the author and others, but in a somewhat narrow setting usually restricted to principal order ideals taken independently [3, 5–7, 9].

In the present work, we acknowledge these two themes by studying a related question; namely, given two arbitrary principal order ideals in the Bruhat order, when is their intersection boolean? Not only does this take an important step towards understanding intersections more generally, but it also gives key insight into how two principal order ideals (and the permutations that define them) can interact, subject to this order relation. Certainly, if either permutation itself is boolean then their intersection will also be boolean, but that is merely a special case. We answer the general question by first characterizing a more general property relating reduced words and permutation patterns, which is related, in some ways, to previous work [8, 11].

We begin this note with definitions and a presentation of the problem. In Section 3, we use previous work to give our first characterization of boolean intersection ideals in terms of an "interlacing" property of reduced words (Theorem 3.1). Section 4 explores the interlacing property more deeply in its own right, and shows that interlacing in w is equivalent to a particular 321-pattern appearing in a permutation $w^{[k]}$ (Theorem 4.1). We conclude with Section 5, giving a pattern characterization in Corollary 5.1 for permutations v and w that is equivalent to $B(v) \cap B(w)$ being boolean, as well as a characterization in terms of support (Corollary 5.2). In Sections 3 and 5, we also address features that one might hope to have in these boolean intersections, but which do not always hold.

2.   Definitions and notation

This work is concerned with permutations in the symmetric group $\mathfrak{S}_n$, under the Bruhat order. For any $w \in \mathfrak{S}_n$, we will write $B(w)$ for the principal order ideal of w. Our interest is in intersections of the form

$$B(v) \cap B(w),$$

which have not previously received much attention. By the subword property (see [2]), this intersection is an order ideal.

The Coxeter group $\mathfrak{S}_n$ is generated by the adjacent transpositions $\{\sigma_i : i \in [1, n-1]\}$. As in [8, 11], we will be interested in both the one-line representation of a permutation $w \in \mathfrak{S}_n$ and in the reduced words $R(w)$ for w. To indicate that a string s of values represents a reduced word and not a permutation in one-line notation, we will write $[s]$. Permutations are composed from right to left, and so

$$R(2431) = \{[1232], [1323], [3123]\},$$

and we can write $2431 = [1232] = [1323] = [3123]$. The example of $R(2431)$ includes an important feature that we highlight with a definition.

Definition 2.1. *Consider a permutation $w \in \mathfrak{S}_n$ whose reduced words contain both k and $k + 1$ for some $k \in [1, n-2]$. If w has a reduced word with one (or both) of the forms*

$$[\cdots k \cdots (k+1) \cdots k \cdots] \qquad or \qquad [\cdots (k+1) \cdots k \cdots (k+1) \cdots], \tag{1}$$

then k and $k + 1$ are interlaced *in w.*

Thus 2431 interlaces 2 and 3, and it does not interlace 1 and 2.

Note that the hypothesis on "a reduced word" in Definition 2.1 can equivalently be replaced by the same hypothesis on *all* reduced words of w. This is because any two reduced words for w are related to each other by a sequence of commutation and braid moves.

As studied previously, a permutation is *boolean* if its principal order ideal in the Bruhat order is isomorphic to a boolean algebra [3, 5–7, 9]. Here we extend that definition to order ideals.

Definition 2.2. *An order ideal in the Bruhat order is* boolean *if all of its elements are boolean elements.*

The principal order ideal of any boolean element is certainly boolean. Perhaps more interestingly, the 5-element order ideal consisting of the permutations $B(2143) \cup B(1324) = \{1234, 2134, 1324, 1243, 2143\} \subset \mathfrak{S}_4$ is also boolean. This order ideal is depicted in Figure 1.
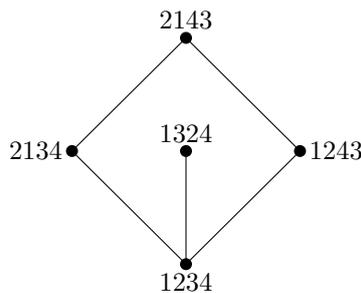
Figure 1: The boolean order ideal $B(2143) \cup B(1324) \subset \mathfrak{S}_4$.

The purpose of this note is to answer the following question.

Question 2.1. *Under what circumstances is $B(v) \cap B(w)$ a boolean order ideal?*

3.   Characterization in terms of interlacing

Two previous results will be key to answering Question 2.1. The first of these is a characterization of boolean permutations. Although we state this in terms of permutations and the symmetric group, a version of this characterization exists for any Coxeter group.

Proposition 3.1 ([9]). *A permutation is boolean if and only if its reduced words contain no repeated letters. This is equivalent to the permutation avoiding the patterns 321 and 3412.*

Fix permutations $v, w \in \mathfrak{S}_n$. Determining whether $B(v) \cap B(w)$ is boolean amounts to checking each $u \in B(v) \cap B(w)$ against the equivalent conditions of Proposition 3.1: the intersection $B(v) \cap B(w)$ fails to be boolean if and only if it contains some u whose reduced words contain repeated letters. That is, the question amounts to determining whether such a u has a reduced word containing two copies of some $k \in [1, n-1]$. By the subword property, then, we can say the following.

Theorem 3.1. *The intersection $B(v) \cap B(w)$ is not boolean if and only if $[k(k+1)k] \in B(v) \cap B(w)$ for some $k \in [1, n-2]$. Equivalently, the intersection ideal $B(v) \cap B(w)$ fails to be boolean if and only if there exists $k \in [1, n-2]$ for which k and $k+1$ are interlaced in both v and w.*

While this does, in a sense, answer Question 2.1, we can clarify that answer further. In particular, previous work has shown connections between reduced words and permutation patterns, and we can make use of those relationships here. To do so, we will call upon the following result.

Proposition 3.2 ([5])**.** *Fix a permutation $w \in \mathfrak{S}_n$. Then k and $k+1$ are interlaced in w if and only if there exists $i \in [1, k]$ and $j \in [k+2, n]$ such that $w(i) > k+1$ and $w(j) < k+1$.*

We find it illuminating here to also highlight a red herring. In [12], we introduced the language of straddling patterns to describe particular occurrences of 321- and 3412-patterns, and their relationship to the number of times each letter $k \in [1, n-1]$ can appear in reduced words of a permutation $w \in \mathfrak{S}_n$. A particular result from that work, namely [12, Theorem 3.3], looks at first glance like it might be helpful for characterizing permutations for which $B(v) \cap B(w)$ is boolean. However, a subtlety of 3412-pattern containment dashes those hopes. For example, there is no k for which the permutations $34125, 14523 \in \mathfrak{S}_5$ both straddle k in both position and value. However, $B(34125) \cap B(14523)$ is not boolean because it contains $B([232])$. The hiccup here is that $R(34125) = \{[2132], [2312]\}$ and $R(14523) = \{[3243], [3423]\}$, and none of these contains both 232 and 323 as subwords.

4. The meaning of interlaced letters

Our goal now is to understand what it means for letters to be interlaced in a permutation. Because interlacing is defined by the configurations in (1) and because $R(321) = \{[121], [212]\}$, it is tempting to expect a relationship between interlacing and 321-patterns. That intuition is correct, but the relationship is not straightforward. For an indication of why not, recall the interlacing in the 321-avoiding permutation 3412 discussed above.

As we will show, interlaced letters in w will imply the presence of a 321-pattern in a permutation $w^{[k]}$. This $w^{[k]}$ will be related to w, but will fix $k+1$ and will change the rest of w as little as possible while being careful about which inversions are created by the changes that are necessary. Moreover, we will show that interlacing in w is equivalent to a specific and describable 321-pattern in the permutation $w^{[k]}$.

Definition 4.1. *Fix a permutation $w \in \mathfrak{S}_n$ and a value $k \in [1, n-2]$. Set $m := w^{-1}(k+1)$ and define $w^{[k]}$ as follows.*

- *If $m = k+1$, then $w^{[k]} := w$.*

- *If $m > k+1$ and $w(k+1) > k+1$, or if $m < k+1$ and $w(k+1) < k+1$, then let $w^{[k]} \in \mathfrak{S}_n$ be the permutation defined by*

$$w^{[k]}(i) := \begin{cases} w(i) & \text{for } i \notin \{m, k+1\}, \\ w(m) = k+1 & \text{for } i = k+1, \text{ and} \\ w(k+1) & \text{for } i = m. \end{cases}$$

- *If $m > k+1$ and $w(k+1) < k+1$, then there is necessarily some $t < k+1$ with $w(t) > k+1$, and we (arbitrarily) pick the maximal such t. If $m < k+1$ and $w(k+1) > k+1$, then there is necessarily some $t > k+1$ with $w(t) < k+1$, and we (arbitrarily) pick the minimal such t. In either case, let $w^{[k]} \in \mathfrak{S}_n$ be the permutation defined by*

$$w^{[k]}(i) := \begin{cases} w(i) & \text{for } i \notin \{m, k+1, t\}, \\ w(m) = k+1 & \text{for } i = k+1, \\ w(k+1) & \text{for } i = t, \text{ and} \\ w(t) & \text{for } i = m. \end{cases}$$

Note what changes between w and $w^{[k]}$ in each case of Definition 4.1. In the first case, nothing changes. In the second, the values $k+1$ and $w(k+1)$ create an inversion in w and are then swapped in the one-line

notation to form $w^{[k]}$. The third scenario is when the values $k + 1$ and $w(k + 1)$ do not form an inversion, and so we find a value $w(t)$ that forms an inversion with both of them. In the first subcase, the permutation w has a 312-pattern in positions $t < k + 1 < m$, and the values in those positions get permuted to form a 123-pattern in $w^{[k]}$. In the second subcase, the permutation w has a 231-pattern in positions $m < k + 1 < t$, and the values in those positions get permuted to form a 123-pattern in $w^{[k]}$. In all situations, $k + 1$ is a fixed point of the permutation $w^{[k]}$.

This permutation $w^{[k]}$ is what will contain the designated 321-pattern when k and $k + 1$ are interlaced in w. The particularity of that pattern is defined by how it occurs in the permutation.

Definition 4.2. *Let w be a permutation fixing a value h. If w contains a 321-pattern with middle value equal to (and appearing in position) h, then we will say that w has a 321-pattern* centered *at h.*

Having defined $w^{[k]}$ and centering, we can now describe interlaced letters in terms of 321-patterns.

Theorem 4.1. *A permutation w interlaces k and $k+1$ if and only if the permutation $w^{[k]}$ contains a 321-pattern centered at $k + 1$.*

Proof. By Proposition 3.2, we have that w interlaces k and $k + 1$ if and only if there exist i and j such that

$$i < k + 1 < j \qquad \text{and} \qquad w(i) > k + 1 > w(j). \tag{2}$$

These inequalities bear a notable resemblance to the definition of a 321-pattern, but the middle term in the latter set is not quite what one would need unless w were to fix $k + 1$.

As described in Definition 4.1, the permutation $w^{[k]}$ is constructed from w by, among other things, moving the value $k + 1$ into position $k + 1$.

If $w(k + 1) = k + 1$, then $w = w^{[k]}$ and the inequalities in (2) would describe the desired occurrence of 321 in $w^{[k]}$.

Now assume that $w(k+1) \neq k + 1$. Thus $w \neq w^{[k]}$, and we recall the definitions of m and t from Definition 4.1. In particular, there are no inversions among the positions $\{k+1, m, t\}$ (or just $\{k + 1, m\}$ in the second category of the definition) in $w^{[k]}$. Therefore there is a 321-pattern in $w^{[k]}$ in positions $i < k + 1 < j$ and having values $w^{[k]}(i) > w^{[k]}(k + 1) = k + 1 > w^{[k]}(j)$ if and only if neither i nor j is equal to m or t (if t is defined). That is, there is such a 321-pattern if and only if $w^{[k]}(i) = w(i)$ and $w^{[k]}(j) = w(j)$. Thus i and j satisfy the inequalities of (2), and so there is such a 321-pattern in $w^{[k]}$ if and only if k and $k + 1$ are interlaced in w. $\square$

It is helpful to demonstrate Definition 4.1 and Theorem 4.1 with examples.

Example 4.1. *Consider $w = 462135 \in \mathfrak{S}_6$.*

(a) *If $k = 1$, then we are in the second category of Definition 4.1, meaning that $w^{[1]} = 426135$. There is a 321-pattern in positions $1 < 2 < 4$ of this permutation, centered at $2 = k + 1$. This confirms the fact that w interlaces 1 and 2, as we see, for example, in the reduced word $[53412312] \in R(w)$.*

(b) *For $k = 2$, we are in the third category of the definition and $t = 2$. Then $w^{[2]} = 423165$, which has the desired 321-pattern in positions $1 < 3 < 4$, centered at $3 = k + 1$. This confirms the fact that w interlaces 2 and 3, as we see in $[53412312] \in R(w)$.*

(c) *With $k = 4$, we are again in the third category with $t = 2$. Then $w^{[4]} = 432156$, which has no 321-pattern centered at $5 = 4 + 1$. This confirms the fact that w does not interlace 4 and 5, as we see from its reduced words $[53412312]$, and so on.*

5. Characterizations in terms of patterns and support

Theorem 4.1 builds off of results like Proposition 3.1 and [1, Theorem 2.1] from the literature. The last of these shows that being 321-avoiding is equivalent to having no consecutive substring $k(k + 1)k$ in any reduced words. In Proposition 3.1, having all distinct letters prevents both 321- and 3412-patterns. In the result of this paper, on the other hand, repeating the letter k and having, without loss of generality, a $k + 1$ between the repeated letters, forces a 321-pattern in $w^{[k]}$ centered at $k + 1$. We can use this language to expand upon the result of Theorem 3.1, characterizing boolean intersection ideals $B(v) \cap B(w)$ in terms of patterns and giving a more complete answer to Question 2.1.

Corollary 5.1. *Fix permutations $v, w \in \mathfrak{S}_n$. The intersection $B(v) \cap B(w)$ is boolean if and only if, for all $k \in [1, n - 2]$, at most one of the permutations $v^{[k]}$ and $w^{[k]}$ has a 321-pattern centered at $k + 1$.*

We demonstrate this characterization using boolean and non-boolean examples.

k	$u^{[k]}$, permuted letters in red	$w^{[k]}$, permuted letters in red	321-pattern in $u^{[k]}$ centered at $k+1$?	321-pattern in $w^{[k]}$ centered at $k+1$?
1	5213674	1237546	yes	no
2	1235674	5137246	no	yes
3	3214675	5134276	no	yes
4	4213576	2137546	no	yes
5	5213467	5134267	no	no

Table 1: Data for $u = 5213674$ and $w = 5137246$.

Example 5.1. *Consider the principal order ideal of $w = 5137246$, intersected with the principal order ideals of two different permutations: $u = 5213674$ and $v = 3512674$.*

(a) *To determine the structure of $B(u) \cap B(w)$, we compute the values shown in Table 1. Each row of the table has at least one "no" in its last two columns, so the intersection $B(u) \cap B(w)$ is boolean. We can confirm this by computing reduced words, such as $u = [4321256]$ and $w = [64323154]$, and using Theorem 3.1. In fact, $B(u) \cap B(w)$ is the union of two boolean principal order ideals, $B([43215]) \cup B([64321])$.*

(b) *To determine the structure of $B(v) \cap B(w)$, we compute the values shown in Table 2. The row for*

k	$v^{[k]}$, permuted letters in red	$w^{[k]}$, permuted letters in red	321-pattern in $v^{[k]}$ centered at $k+1$?	321-pattern in $w^{[k]}$ centered at $k+1$?
1	3215674	1237546	yes	no
2	1532674	5137246	yes	yes
3	3214675	5134276	no	yes
4	3412576	2137546	no	yes
5	3512467	5134267	no	no

Table 2: Data for $v = 3512674$ and $w = 5137246$.

$k = 2$ has "yes" in both of the last two columns, meaning that $B(v) \cap B(w)$ is not boolean. Indeed, $1432567 = [232] \in B(v) \cap B(w)$, and so the non-boolean poset $B([232])$ is a subset of this intersection ideal.

Because $k+1$ is a fixed point of $w^{[k]}$, we can use [10, Lemma 2.8] to frame boolean intersection ideals in one more light, now in terms of "support."

Definition 5.1. *The* support *of a permutation $w \in \mathfrak{S}_n$ is the set $\mathsf{supp}(w) \subseteq [1, n-1]$ consisting of all letters that appear in reduced words for w.*

We can use [10, Lemma 2.8] and Corollary 5.1 to characterize boolean intersection ideals by support. As a starting point, note that if u fixes $k+1$, then the lemma implies that $k \in \mathsf{supp}(u)$ if and only if $k+1 \in \mathsf{supp}(u)$.

Corollary 5.2. *For permutations $v, w \in \mathfrak{S}_n$, the following statements are equivalent:*

- *the intersection $B(v) \cap B(w)$ is boolean;*

- *k is in the support of at most one of $v^{[k]}$ and $w^{[k]}$, for all $k \in [1, n-2]$;*

- *$k+1$ is in the support of at most one of $v^{[k]}$ and $w^{[k]}$, for all $k \in [1, n-2]$; and*

- *$\{k, k+1\} \cap \mathsf{supp}(u) = \emptyset$ for at least one $u \in \{v^{[k]}, w^{[k]}\}$, for all $k \in [1, n-2]$.*

We conclude this note with another red herring. From Theorem 3.1 and [5, Lemma 5.10], one might hope that when an intersection $B(v) \cap B(w)$ is known to be boolean, perhaps that intersection is equal to $B(v') \cap B(w')$ for some permutations v' and w' that are themselves boolean. Sadly this is not always the case, as we can see with $v = [123] = 2341$ and $w = [2132] = 3412$. The intersection of their principal order ideals is equal to $B(v) \setminus \{v\}$, and yet any boolean w' whose principal order ideal contains both $[12]$ and $[23]$ will necessarily also contain $[123] = v$, in which case the intersection ideal would be all of $B(v)$. This example is depicted in Figure 2.
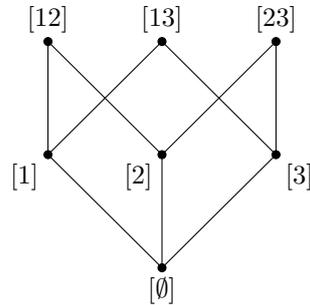
Figure 2: The boolean intersection ideal $B([123]) \cap B([2132])$.

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