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Interview with Lauren K. Williams

Toufik Mansour

Photo by Rose Lincoln/Harvard University

Lauren Williams obtained her B.Sc. in mathematics from Harvard University in 2000, and her Ph.D. in 2005 at the Massachusetts Institute of Technology (MIT) under the supervision of Richard P. Stanley. After postdoctoral positions at the University of California, Berkeley, and Harvard, Williams rejoined the Berkeley mathematics department as an assistant professor in 2009 and was promoted to associate professor in 2013 and then full professor in 2016. Starting in the fall of 2018, she rejoined the Harvard mathematics department as a full professor, making her the second-ever tenured female math professor at Harvard. In 2012, Professor Williams became one of the inaugural fellows of the American Mathematical Society. She is the 2016 winner of the Association for Women in Mathematics and Microsoft Research Prize in Algebra and Number Theory, and an invited speaker at the 2022 ICM.

Mansour: Professor Williams, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Williams: I generally define combinatorics to be the study of finite or discrete structures. Mark Haiman once remarked to me that "combinatorics is not a field, it is an attitude." I like this because it does not "pigeonhole" the field of combinatorics into some list of topics. Instead, we can think of combinatorics as a certain perspective on mathematics, and a combinatorialist as a mathematician who views the various problems and topics in the mathematical world through the lens of combinatorics, to see if her/his combinatorial *attitude* can bring any insight.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Williams: I think it is important for the development of the field that we in combinatorics keep an eye on what is going on in adjacent fields of mathematics, such as representation theory and algebraic geometry, as well as adjacent fields of science, such as computer science and physics. Sometimes techniques from other fields can be used to attack longstanding problems from combinatorics (for example the recent developments using Hodge theory to prove unimodality and log-concavity statements[1]). Conversely, sometimes problems in these fields can be attacked using combinatorial methods, and can moreover lead to new combinatorial tools.

Mansour: What have been some of the main goals of your research?

Williams: My main goal is to understand this corner of mathematics a little better and hopefully find beautiful results. I personally am drawn to topics at the interface of combinatorics and some other fields. For example, I have a series of papers on the asymmetric simple exclusion process[2,3] (ASEP), a model

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[1] J. Huh, *Combinatorial applications of the Hodge-Riemann relations*, Proc. Int. Cong. of Math., Rio de Janeiro, Vol. 3 (2018), 3079–3098.

[2] J. Macdonald, J. Gibbs, and A. Pipkin, *Kinetics of biopolymerization on nucleic acid templates*, Biopolymers 6, 1968.

[3] F. Spitzer, *Interaction of Markov processes*, Adv. in Math. 5 (1970), 246–290.

of hopping particles that comes from *statistical physics*; I also did some work on soliton solutions to the KP equation[4], which comes from *integrable systems*; I have a few papers on *mirror symmetry*[5,6] for Grassmannians and related spaces; and lately, I have been working[7,8] on the amplituhedron, which comes from *scattering amplitudes* in $\mathcal{N} = 4$ *super Yang-Mills theory.* These topics may not sound very combinatorial, but in all instances, our main results can be stated and proved using algebra and combinatorics.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Williams: I was fortunate to go to good public schools which had active math clubs, excellent math teachers, and allowed acceleration in math starting in 4th or 5th grade (for example, students who already knew the grade-level material could take math with students a grade or so above them).

Also in 4th grade, I wound up winning a local math contest, which led to several teachers kind of taking me under their wing and (later) telling me about summer programs I should consider, like the Ross Young Scholars program (a high school number theory program). My experience there and at several other summer programs, including the Research Science Institute at MIT and the Math Olympiad Program, was very influential.

My dad is an engineer and my mom an English teacher, so while they were certainly very supportive of my interest in math, they were equally supportive of my other interests. My three younger sisters also wound up being interested in math; I remember spending quite a lot of time teaching them math and codes. In my senior year of high school, three of us were even on a math team together (the local math team which went to the American Regions Math League in Las Vegas). I coached the team, and I recall that while the boys on the team were super respectful, my sisters would shoot paper airplanes at me from the back of the room.

Mansour: Were there specific problems that made you first interested in combinatorics?

Williams: Apparently, when I was about three, I asked my mom "How old is Alex?" about a baby I encountered. My mom said "she's not quite one" – to which I replied matter-of-factly "Then she's zero. Alex is zero." So I think I had a "discrete" point of view from an early age.

But more seriously, I got exposed to some combinatorics in high school from math competitions, the Ross program, and the first "Art of Problem Solving" book. Then when I was at the Research Science Institute at MIT, I worked on a problem about self-avoiding walks under the guidance of Satomi Okazaki (who was then a graduate student of Richard Stanley). Satomi had me read Wilf's book[9] "generatingfunctionology" as well as a short paper by Zeilberger[10]: I remember being rather dazzled by generating functions! Once I was exposed to abstract algebra and representation theory in college, I also felt an affinity for these fields. I liked the precise answers and statements in algebra and enumerative/algebraic combinatorics.

Mansour: What was the reason you chose MIT for your Ph.D. and your advisor, Richard Stanley?

Williams: After visiting graduate schools, I found myself feeling torn about whether to go to MIT or to UC Berkeley. In the end, I did both – I was a graduate student at MIT and worked with Richard Stanley, but I spent the fall of my third year of graduate school at Berkeley, where I worked with Bernd Sturmfels and learned tropical geom-

[4]B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. 15 (1970), 539–541.

[5]K. Rietsch and L. K. Williams, *Newton-Okounkov bodies, cluster duality and mirror symmetry for Grassmannians*, Duke Math. J. 168:18 (2019), 3437–3527.

[6]C. Pech, K. Rietsch, and L. K. Williams, *On Landau-Ginzburg models for quadrics and flat sections of Dubrovin connections*, Adv. in Math. (Zelevinsky issue) 300 (2016), 275–319.

[7]T. Lukowski, M. Parisi, and L. K. Williams, *The positive tropical Grassmannian, the hypersimplex, and the* $m = 2$ *amplituhedron*, arXiv:2002.06164v2.

[8]M. Parisi, M. Sherman-Bennett, and L. K. Williams, *The* $m = 2$ *amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers*, arxiv:2104.08254

[9]H. Wilf, *generatingfunctionology*, Academic Press, New York, 1990.

[10]D. Zeilberger, *self-Avoiding Walks, the language of science, and Fibonacci numbers*, J. Stat. Planning and Inference 54 (1996), 135–138.

etry. Richard and Bernd have very different perspectives on mathematics, but both influenced me a great deal. I picked up from Richard the aesthetic in mathematics that the answer should be beautiful (meaning simple or elegant); and if it is not, maybe you asked the wrong question. Meanwhile, Bernd has a very broad perspective on the scope of mathematics/algebra/combinatorics, and an appreciation for applications.

Mansour: How was mathematics at MIT at that time?

Williams: I was a graduate student between 2001 and 2005; mathematics at MIT at that time was very exciting, with many excellent mathematicians who were based in Boston or visited Boston. Sara Billey was at MIT during my first year, while Igor Pak and Alex Postnikov were assistant professors throughout my time as a graduate student, so I took classes from both of them. Richard Stanley had perhaps ten students at the time, and I found the older graduate students to be very supportive of the younger ones.

In terms of mathematics, I remember that cluster algebras – which had been discovered by Fomin and Zelevinsky around 2000 – were very much "in the air." I attended many talks on cluster algebras, most of them given by Russians, and most of them rather incomprehensible to me – but the speakers were so excited that I made an effort and eventually came to understand what was going on. Zelevinsky came to MIT regularly, and I was lucky to have a number of conversations with him. One year I was talking so frequently with Russians that I started dropping articles in conversation with them!

Mansour: What was the problem you worked on in your thesis?

Williams: The first problem I worked on came from Postnikov: it was to find a formula for the rank-generating function for the cells in the totally nonnegative Grassmannian[11]. This project ignited my interest in total positivity, which has been a theme in much of my research. Moreover, as it happens, my formula[11] led to a new q-analogue $E_{k,n}(q)$ of the Eulerian numbers, which specializes at $q = 0$ to the Narayana numbers. This q-analogue turned out to arise naturally in the study of the asymmetric simple exclusion process and led to a series of papers on particle processes with Sylvie Corteel (and later others, including Olya Mandelshtam[12]).

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Williams: If I find a conjecture which is consistent with the examples I have done and is sufficiently beautiful, I generally feel rather certain it must be true. I tend to be a very optimistic or "positive" person and it has become a sort of game for me to make conjectures based on very small sets of data. In general, one hopes that the simplest statement that is consistent with the data should be correct, and in my experience, this hope is often realized.

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Williams: I hesitate to answer such a broad and subjective question, but in my own corner of mathematics, I have found ideas from the fields of total positivity, cluster algebras, and tropical geometry to be very influential.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your own case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Williams: To me, it is the *motivation* that separates pure and applied mathematics. In applied math, the motivation comes from real-world problems and it is the *question* that drives the research. On the other hand, pure math is studied for its own intrinsic interest and is judged to a large extent on the "beauty" of the answer (for example, a theorem with a simple, elegant statement). This is not to say that applied math cannot be beautiful, however, or that pure mathematics cannot have applications.

Given the above framework, I would classify myself as a pure mathematician. But I am delighted when I find beautiful problems or results which also have "applications." For example, the ASEP was introduced by biologists to model translation in protein synthesis,

[11] L. K. Williams, *Enumeration of totally positive Grassmann cells*, Adv. in Math. 190:2 (2005), 319–342.

[12] S. Corteel, O. Mandelshtam, and L. K. Williams, *Combinatorics of the two-species ASEP and Koornwinder moments*, Adv. in Math. 321 (2017), 160–204.

and soliton solutions of the KP equation model shallow-water waves, and the "volume" of the amplituhedron compute scattering amplitudes.

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Williams: I would say that it is important early on to figure out what parts of mathematics *you* are most interested in and also what other mathematicians find interesting. The goal is to find something at the intersection. Going to lots of conferences and seminar talks can help with this. I think it would be impossible to find success or happiness in a research career unless you are personally motivated by the problems you are working on.

I would also recommend that you try to resist the temptation to compare yourself to others and to surround yourself as much as possible with people who support you. Conversely, you should be kind and supportive to others. The math world is small, so if you stay in this field you will keep running into the same people for years.

Mansour: While we see that there are more women in science and technology fields today than ever before, bias still affects women in their scientific careers. What do you think about this issue?

Williams: First, I want to point out that underrepresentation of women and minorities is a problem at the top levels of nearly every field I can think of, including STEM, politics, business. I think there are a lot of contributing factors here, of which bias (implicit or explicit) is just one. Part of me feels pessimistic about this problem because change has been so slow and not always in the right direction. (A recent NSF survey reported that in 2010, 29.4% of doctorates in math and statistics in the US were awarded to women; the number was up to 32% in 2014 but back down to 29.1% in 2020. All figures are significantly lower if restricted to US citizens or if restricted to math but not statistics.) Separately, a lot of research has highlighted the devastating effects of the pandemic on the careers of many women with young children. But part of me is optimistic because of the brilliant young women and mathematicians from minority groups that I have met in the last decade; and because of the various senior mathematicians I know (both men and women) who are doing what they can to help.

I think there is a lot that can be done to improve the situation. For example, universities can adopt sensible parental leave policies (the UC system has an excellent "active service modified duty" policy, which I benefited greatly from). And we can all try to recognize and encourage talent even when it presents differently than we might expect. I remember being astonished one year when I realized after grading exams that the best student in my large abstract algebra class was a woman I had never noticed before, someone who had never spoken in class. The same thing happened the following year, except this time the highest score came from a quiet Hispanic man. Even for me, it is so easy to assume that (future) mathematicians will look or behave a certain way.

Mansour: Would you tell us about your interests besides mathematics?

Williams: I enjoy reading, running, playing chamber music, and traveling. Math has been a wonderful excuse to explore the world; although the pandemic has put a rather serious pause on travel for most of us. Until fairly recently, the time I could spend on my hobbies was severely limited because of my young children, but they are now getting old enough that we can do some of these things together.

Mansour: We would like to ask some more specific mathematical questions. Grassmannians, total positivity, and combinatorial questions related to them play an important role in your research. Would you tell us about these mathematical objects and concepts, and the role of combinatorics in their study?

Williams: The Grassmannian $Gr_{k,n}$ is the set of all k-dimensional vector spaces in an n-dimensional vector space. It is intimately connected to the theory of matroids, and its cell decomposition and cohomology can be described using Young tableaux and Schur polynomials[13] . So even if the Grassmannian *per se* is a continuous (rather than discrete) object, in analyzing it one quickly encounters some of the most fundamental objects in combinatorics.

[13] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, Translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs, 6. Cours Spécialisés [Specialized Courses], 3. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001

The classical theory of total positivity concerns square matrices with all minors positive. Lusztig[14] generalized this theory by introducing the totally positive and nonnegative part of a generalized partial flag variety G/P, and Postnikov[15] initiated the combinatorial study of the positive Grassmannian $Gr_{k,n}^{\geq 0}$, the subset where all Plücker coordinates are nonnegative. This object turns out to be incredibly beautiful and rich. For instance, Postnikov showed that the positive Grassmannian is a cell complex, with cells that can be described using various combinatorial objects including Le-diagrams, decorated permutations, and planar bicolored graphs.

Remarkably, the positive Grassmannian has beautiful connections to other parts of mathematics. For example, one can associate a *soliton solution of the KP equation* to each point A in the real Grassmannian, and Kodama[16] and I showed that this solution will be regular for all times t if and only if A came from the positive Grassmannian. Moreover, in this case, the discrete properties of the solution[17] can be read off beautifully from the combinatorial objects labeling the cell containing A.

Mansour: The asymmetric simple exclusion process (ASEP) is a stochastic model of particles on a one-dimensional lattice, which has been extensively studied in combinatorics, probability, and statistical physics. The study of asymptotic fluctuations of the ASEP on the *infinite* one-dimensional integer lattice made a surprising connection with Young tableaux combinatorics. In your joint work with Corteel[18] from 2011, you studied a distinct model of the ASEP, a one-dimensional *finite* system with open boundaries, and developed further tableau combinatorics for this new model by introducing *staircase tableaux*. This was very interesting work and a highly nontrivial generalization of the Matrix Ansatz[19] of B. Derrida et al. Would you tell us about this work and related future developments you expect in this direction?

Williams: The seed of my collaboration with Corteel began when she started studying the probability that in the ASEP on a lattice of n sites, exactly k sites are occupied by a particle. She knew the answer had to interpolate between Eulerian and Narayana numbers (based on the value of a hopping rate q), and eventually found and proved that my q-analogue $E_{k,n}(q)$ of Eulerian numbers fit the bill.

I was very surprised by her result because for me, $E_{k,n}(q)$ came from the positive Grassmannian; there was no reason to expect a probabilistic interpretation. But since I understood how to think of $E_{k,n}(q)$ as a generating function for certain tableaux (Le-diagrams), I started looking for a combinatorial refinement of her result that would express *all* steady-state probabilities. I found such an interpretation, and after struggling through the physics papers on the ASEP, realized that the Matrix Ansatz might be useful. The Matrix Ansatz says that if one can find (typically infinite-dimensional) matrices D and E and vectors W, V satisfying several algebraic relations, then one can compute ASEP probabilities as certain matrix products. After a while, I realized that I should try to write down matrices D, E that encoded the recursive structure of the tableaux I was using, then prove they satisfied the requisite relations.

I subsequently corresponded with Corteel, and she figured out how to introduce two more statistics on my tableaux so as to generalize my theorem to the version of the ASEP where particles can enter the lattice at the left at rate α and exit at the right at rate β (instead of at rate 1). She actually emailed me her idea just a couple of days before giving birth to her first child; so when I wrote back immediately suggesting that I update my paper and add her as an author, there was a very slight delay, before

[14] G. Lusztig, *Total positivity in reductive groups, in: Lie theory and geometry, in honor of Bertram Kostant*, Progress in Math. 123, Birkhauser, 1994.

[15] A. Postnikov, *Total positivity, Grassmannians, and networks*, arXiv:math/0609764.

[16] Y. Kodama and L. K. Williams, *The Deodhar decomposition of the Grassmannian and the regularity of KP solitons*, Adv. Math. 244 (2013), 979–1032.

[17] Y. Kodama and L. K. Williams, *KP solitons, total positivity, and cluster algebras*, Proceedings of the National Academy of Sciences, published online ahead of print May 11, 2011, doi:10.1073/pnas.1102627108.

[18] S. Corteel and L. K. Williams, *Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials*, Duke Math. J. 159:3 (2011), 385–415.

[19] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier, *Exact solution of a 1D asymmetric exclusion model using a matrix formulation*, J. Phys. A 26:7 (1993), 1493–1517.

[20] Some years later we wrote another paper that made its (arXiv) debut at more or less the same time as my son.

I received a "Yes!" and a baby photo.[20]

So that was our first paper together, which appeared in 2007. The natural follow-up question was to generalize our result to the case when particles can enter and exit at both the left and right sides of the lattice. This was a much harder problem. It took more than a year to invent the appropriate combinatorial objects (staircase tableaux) and perhaps another year after that to prove our conjecture. The proof required us to generalize the Matrix Ansatz, as you mentioned above and appeared in our 2011 paper.

Mansour: In the same 2011 work with Corteel[18], you obtained the first combinatorial formula for the moments of the Askey-Wilson polynomials. What are Askey-Wilson polynomials? Why are they important? Would you comment on some future directions?

Williams: The Askey-Wilson[21] polynomials are a family of orthogonal polynomials $P_n(x; a, b, c, d; q)$ in one variable x depending on additional parameters a, b, c, d, q. They are at the top of the hierarchy of classical orthogonal polynomials, meanings that they degenerate or specialize to the other classical orthogonal polynomials, including Laguerre polynomials, Hermite polynomials, etc.

Before our work, Uchiyama-Sasamoto-Wadati[22] had given a solution to the Matrix Ansatz using Askey-Wilson polynomials, which implies that the partition function for the ASEP is closely related to the moments of the Askey-Wilson polynomials. Once we had proved our tableaux formula for the stationary distribution we obtained as a corollary a tableaux formula for the Askey-Wilson moments. This had been a long-standing open problem for classical orthogonal polynomials.

As for future directions, Askey-Wilson polynomials are the one-variable case of the multivariable *Koornwinder polynomials*, also known as Macdonald polynomials of affine type C. So a natural follow-up problem is to generalize the previous results in a way that Askey-Wilson polynomials get replaced by Koornwinder polynomials. We made the first step in this direction with our paper[23] "Macdonald-Koornwinder moments" with Corteel and a follow-up paper with Corteel and Mandelshtam. In short, the relevant particle model is the *multispecies* ASEP with open boundaries.

Mansour: In 2018, you, with Corteel and Mandelshtam[24], used the exclusion process to give a direct combinatorial characterization of the symmetric (and some nonsymmetric) Macdonald polynomials. These polynomials arise in different fields such as in physics as eigenfunctions of the Ruijsenaars–Schneider model, in probability related to random growth models, and in torus knots. What are Macdonald's polynomials? Why do they play an important role in different fields? Would you comment on some future directions related to their combinatorial studies?

Williams: Let me start by commenting that this work with Corteel and Mandelshtam has a lot of parallels to the work I discussed in my previous answer. Just as Macdonald polynomials of affine type C are related to the multispecies exclusion process on a line with open boundaries, Macdonald polynomials of affine type A (the usual Macdonald polynomials) are related to the multispecies exclusion process on a ring. We were motivated to study this topic after seeing a paper of Cantini-deGier-Wheeler[25] which explained that the partition function for the multispecies ASEP on a ring was proportional to the specialization of a Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$ when each $x_1 = x_2 = \cdots = x_n = q = 1$. Meanwhile, James Martin[26] had given a formula for the stationary distribution of this version of the ASEP in terms of *multiline queues*. So we had the idea to try to express arbitrary Macdonald polynomials in terms of multiline queues. This wound up working beautifully and led us (together with Haglund and Mason) to introduce some new quasisymmetric Macdonald polynomials.

[21]R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54 (1985), no. 319.

[22]M. Uchiyama, T. Sasamoto, and M. Wadati, *Asymmetric simple exclusion process with open boundaries and Askey-Wilson polynomials*, J. Phys. A. 37:18 (2004), 4985–5002.

[23]S. Corteel and L. K. Williams, *Macdonald-Koornwinder moments and the two-species exclusion process*, Selecta Math. 24:3 (2018), 2275–2317.

[24]S. Corteel, O. Mandelshtam, and L. K. Williams, *From multiline queues to Macdonald polynomials via the exclusion process*, arXiv:1811.01024.

[25]L. Cantini, J. de Gier, and M. Wheeler, *Matrix product formula for Macdonald polynomials*, J. Phys. A 48(38) (2015), 384001.

[26]J. B. Martin, *Stationary distributions of the multi-type ASEPs*, arXiv:1810.10650.

Regarding the Macdonald polynomials and their importance, I will just say that Macdonald polynomials $P_\lambda(\mathbf{x}; q, t)$ are homogeneous symmetric polynomials in $\Lambda(q, t)$ which are orthogonal with respect to the Macdonald inner product, and they generalize important families of polynomials such as Schur polynomials, Hall-Littlewood polynomials, and Jack polynomials.

Mansour: The *amplituhedron*, introduced in 2013 by theoretical physicists Nima Arkani-Hamed and Jaroslav Trnka[27], is a geometric structure that enables simplified calculation of particle interactions in some quantum field theories. Would you tell us about the combinatorics of amplituhedra and comment on some future research direction?

Williams: Arkani-Hamed and Trnka defined the *tree amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ to be the image of the positive Grassmannian $Gr_{k,n}^{\geq 0}$ under the map $\tilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ which sends a k-plane to its image under a linear map coming from a $n \times (k+m)$ matrix Z with maximal minors positive.

I was very surprised when I first encountered this object and started exploring some of the conjectures about it. I did not have much intuition, but as far as I could ascertain, the conjectures were correct.

The amplituhedron generalizes many nice objects. When $k + m = n$ it recovers $Gr_{k,n}^{\geq 0}$; when $k = 1$ it reduces to a cyclic polytope; and when $m = 1$ (by joint work with Steven Karp[28] it can be identified with the bounded complex of a cyclic hyperplane arrangement). Physicists are interested in *tiling* the amplituhedron $\mathcal{A}_{n,k,m}(Z)$, i.e. in subdividing it using the images of cells of $Gr_{k,n}^{\geq 0}$. With Karp and Yan X. Zhang[29] we conjectured that when m is even, the number of top-dimensional strata comprising a tiling of $\mathcal{A}_{n,k,m}(Z)$ equals the number of plane partitions contained in a $k \times (n-k-m) \times \frac{m}{2}$ box; this conjecture is wide open.

In Fall 2019, during a thematic program on scattering amplitudes and total positivity that I co-organized, one of our visitors Matteo Parisi showed me some computations regarding the number of "good" tilings of $\mathcal{A}_{n,k,2}(Z)$. I recognized these numbers as coinciding with the f-vector of the *positive tropical Grassmannian* $\mathrm{Trop}^+ Gr_{k+1,n}$, an object I had introduced with Speyer in 2003. This was extremely puzzling, but we eventually discovered a conjectural link between these objects via positroid subdivisions of the hypersimplex (joint with Lukowski and Parisi[7]). Very recently with Parisi and Melissa Sherman-Bennett[8], we proved this conjecture: we proved that tilings of the hypersimplex $\Delta_{k+1,n}$ (using the moment map from $Gr_{k+1,n}^{\geq 0}$ to $\Delta_{k+1,n}$) are in bijection with tilings of the amplituhedron $\mathcal{A}_{n,k,2}$. This is very strange, because $\Delta_{k+1,n}$ is an $(n-1)$-dimensional polytope, while $\mathcal{A}_{n,k,2}$ is a $2k$-dimensional non-polytopal subset of $Gr_{k,k+2}$. Even though we have proof, I feel like we do not really understand *why* the statement should be true. It also open problem whether there is a similar phenomenon for other m.

Mansour: Professor Lauren Williams, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

[27] N. Arkani-Hamed and J. Trnka, *The amplituhedron*, J. High Energy Phys. 10 (2014), 33.

[28] S. Karp and L. K. Williams, *The $m = 1$ amplituhedron and cyclic hyperplane arrangements*, International Mathematics Research Notices 5 (2019), 1401–1462.

[29] S. Karp, L. K. Williams, and Y. X. Zhang, *Decompositions of amplituhedra*, Annales de l'Institut Henri Poincare D 7:3 (2020), 303–363.