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Interview with Xavier Viennot

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Photo by Nicolas Viennot

Xavier Viennot completed his studies at Ecole Normale Supérieure Ulm in 1969. He obtained a PhD at the University of Paris in 1971, under the direction of Marcel-Paul Schüzenberger. Since 1969 he has been working as Researcher at CNRS (National Centre for Scientific Research, France). His present position is *Emeritus Research Director* at CNRS, member of the combinatorial group at Laboratoire Bordelais de Recherche en Informatique (LaBRI), Université de Bordeaux. Professor Viennot held visiting positions at 20 different universities or research centres in Europe, North and South America, India, China, and Australia. He has received several awards, including A. Châtelet medal (for his work in algebra) in 1974 and Silver Medal at CNRS (for his work in combinatorics and control theory) in 1992. He has given numerous communications and colloquia and has been invited speaker in some major conferences in pure mathematics, combinatorics, computer science and physics. The main research domains of Xavier Viennot is enumerative, algebraic and bijective combinatorics.

Mansour:[a] Professor Viennot, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Viennot: First, I want to thank you for your invitation to this interview with no limit in the length of my answers and anecdotes. There are many possible definitions, and many different kinds of combinatorics, such as enumerative, algebraic, bijective, analytic, "existentialist", extremal, geometric, probabilistic, "integrable" (i.e. combinatorial physics), and even magic. Let us say that combinatorics is the study of finite mathematical objects having a poor underlying structure. In other words, some objects where a kid can give examples once you explain to him the definition (permutations, Young tableaux, movements of a tower on a chessboard, tilings the same chessboard with dimers, etc). Enumerating finite structures is not necessarily combinatorics (just think of the number of simple finite groups!). Nowadays combinatorics, especially *bijective* combinatorics, is rather an *attitude* that is transversal to all mathematics (and also physics, computer science, theoretical biology, etc) where classical (and non-classical) theories are studied with a "combinatorial" point of view. This denomination *"attitude"* is due to Volker Strehl, one of the creators in 1980 of the *"Séminaire Lotharingien de Combinatoire"*[1].

I believe that the world, at a very small scale, such as Planck length is discrete. Michel Mendès-France, a number theorist in Bordeaux, was talking about the fractal character of time and that we are becoming older by infinitesimal little jumps of time. We can think of space-time particles. In this sense, our entire world is combinatorics and the continuum is just an illusion in the same way when you look at a computer screen with a high resolu-

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[1]See http://www.mat.univie.ac.at/~slc/.

tion or a movie. G.C. Rota said that mathematicians were first studying continuum as a first step before going to the study of the finite. With this (broad) point of view, the beauty of Nature is a reflection of the underlying beauty of combinatorics. Even if it is a belief, I love this point of view.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Viennot: As I just said, I believe that combinatorics is an *attitude* transversal to all mathematics. The relation with the rest of mathematics is deep, fruitful, amazing, spectacular. Many parts of mathematics, especially algebra, analysis, etc, can be rewritten with a combinatorial point of view. Classical theories have a new birth among the garden of combinatorial objects (for example, the theory of symmetric functions, representation of groups and Lie algebra, or in analysis, continued fractions and orthogonal polynomials). Even some very difficult problems can be solved using combinatorics therapy.

I have been very lucky to belong to the generation of students where this relation was starting in all directions. Like the birth of a star from different pieces of matter scattered in the space, the birth of modern combinatorics is developing from many individual facts and formulae. We are in a golden age of this part of mathematics. Everything was waiting to be developed, from the foundations of the house to the bathroom of the seventh floor until some sophisticated details of the balcony on the tenth floor.

Mansour:[b] What have been some of the main goals of your research?

Viennot: Some people like to solve some specific problems. My preferences are to try to unify different facts or formulae, try to understand them, to develop a theory or a methodology from them, in the spirit of bijective combinatorics, trying to put some order in the jungle of various bijections, to develop some aspects of these new theories or methodologies, or reproved in a better way some known fact or formulae. I like to find the ultimate elegant bijection which will explain at the same time different formulae, even if bijective proofs already exist for these formulae. Of course, I have also been happy to work on specific open problems, find a formula or a bijection for a specific enumerative problem, such as the enumeration of polyominoes and directed animals. An example was to find a bijection to prove the amazing formula[2] 3^n for the number of compact source directed animals of size $(n+1)^m$.

In particular, I have been developing the following methodologies: commutations and heaps of pieces, combinatorial theory of orthogonal polynomials, and continued fractions, combinatorial theory of differential equations (with Pierre Leroux) and what I call the *"cellular ansatz"* (i.e. the relation between bijective combinatorics on a grid and quadratic algebra). The starting point is bijections such are RSK (the Robinson-Schensted-Knuth correspondence[3,4,5]) viewed with Fomin's *"local rules"*[6] or bijections coming from the PASEP model in physics[d] relating permutations and some tableaux[7].

In the last ten years I concentrated my efforts to write (euh sorry ... to speak), a book (in fact a *"video book"*) on bijective combinatorics. This video-book is in four "Volumes" (called "Parts"), each Part corresponds to a course given at the IMSc (The Institute of Mathematical Science, Chennai, India). This video-book called *"The Art of Bijective Combinatorics"*[8] (ABjC for short) is the achievement of the 1985 seminal paper on the theory of heaps of pieces[9] (Part II), the 1983 monograph on orthogonal polynomials[10] (Part IV) and the development of the methodology I called the *"cellular ansatz"*[11] (Part III). I am

[2]D. Gouyou-Beauchamps and X. Viennot, *Equivalence of the two-dimensional directed animals problem to a one-dimensional path problem*, Adv. in Appl. Math. 9 (1988), 334–357.

[3]D.E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific Journal of Mathematics 34 (1970), 709–727.

[4]G. de B. Robinson, *On the representations of the symmetric group*, American Journal of Mathematics, 60:3 (1938), 745–760.

[5]C. Schensted, *Longest increasing and decreasing subsequences*, Canadian Journal of Mathematics 13 (1961), 179–191.

[6]S. Fomin, *Generalised Robinson-Schensted-Knuth correspondence*, Journal of Soviet Mathematics, 41 (1988), 979–991. Translation from Zapiski Nauchn Sem. LOMI 155 (1986), 156–175.

[7]E. Steingrımsson and L. Williams, *Permutation tableaux and permutation patterns*, J. Combin. Theory Ser. A 114:2 (2007), 211–234.

[8]See `http://www.viennot.org/abjc.html`.

[9]See `http://www.viennot.org/abjc2.html`.

[10]See `http://www.viennot.org/abjc4.html`.

[11]See `http://www.viennot.org/abjc3-abstract.html`.

very extremely grateful to IMSc and my colleague and friend Amritanshu Prasad (Amri) to have made possible this video book.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Viennot: At school, I loved physics and mathematics and was a good student, nothing more. My father was an engineer and he wanted me to be the same and pushed me to prepare for the exams of the "great schools" as they are called in France, such as the prestigious Ecole Polytechnique. I succeeded at the "Ecole Normale Supérieure", rue d'Ulm, where instead of an engineering school and Ecole Polytechnique, I was attracted by the perfume of freedom with the passion for science and research.

Mansour: Were there specific problems that made you first interested in combinatorics?

Viennot: There were no specific problems that made my interest in combinatorics. In general, I just loved discrete and concrete mathematics. After my "thèse d'Etat", I learnt about the Robinson-Schensted correspondence between permutations and pairs of Young tableaux and wrote my first combinatorial paper[j]. Then I read Foata-Schützenberger[12] Lecture Note in Mathematics on Eulerian polynomials and was surprised by the interpretations of secant and tangent numbers with alternating permutations and Désiré André's permutations[13]. Since then, I started my bijective exploration in the fascinating world with the combinatorial interpretations of the Genocchi numbers[14].

Mansour: What was the reason you chose the University of Paris for your Ph.D. and your advisor Marcel-Paul Schützenberger?

Viennot: As I mentioned above, I succeeded for Ecole Normale Supérieure. In France, after secondary schools, there are two options: university or the so-called "great schools". But paradoxically for Ecole Normale Supérieure, we follow courses at university (mainly the University of Paris, very close to "rue d'Ulm" in the Latin quarter) and have only a few courses at the Ecole given by famous professors. Bruhat was the director for mathematics studies, after 25 years of direction by Henri Cartan and his famous seminar. The studies are four years long. The ambiance was very stimulating, science and letters students are mixed and living together in the hot ambiance of the Latin quarter in the middle of Paris. Freedom was the rule. I became friend with Alain Connes, who entered the school one year after me. He get the Fields medal, respecting the tradition that the 10 Fields medalists in France (as today), all of them come from ENS Ulm (!). There was also the famous Physics Laboratory of ENS. I remember the special events when Alfred Kastler received the Nobel Price in Physics. Former students became Prime Minister (such as Laurent Fabius, promotion 1966) or President of France such as Georges Pompidou. I remember a visit with some friends from ENS at Hotel Matignon when he was Prime Minister of President De Gaulle. It was the time when candid students were joking with a Prime Minister, few months before the May 1968 "revolution". The school was created during the French revolution of 1789 and was initially intended for the formation of college professors. At the historical entrance one can read "Décret de la Convention, 9 Brumaire, An III".

We also had the privilege to have some special lectures. This was the way I met Marco Schützenberger for the first time. He gave a lecture on a topic in theoretical computer science. When you meet such an original person, it is like falling in love. This one-hour lecture decided for me the rest of my scientific life.

Mansour:[c] What was the problem you worked on in your thesis?

Viennot: In France, at that time, the process was the following: a "thèse de 3ème cycle", a kind of "light" Ph.D., where you are supposed to spend 2-3 years, now replaced by "thesis" (= Ph.D.) and several years later, the "big thesis" called "Thèse d'Etat", now equivalent to what is called "Habilitation". I am not going to describe my "thèse de 3ème cycle" which contains some notions related to automata theory and

[12] D. Foata and M.-P. Schützenberger, *Théorie géométrique des polynômes eulériens*, Lecture Notes in Mathematics, 138, Berlin, Springer-Verlag, 1970. Electronic reedition in the section "book" of SLC, see [1].

[13] D. Foata and M.-P. Schützenberger, *Nombres d'Euler et permutations alternantes*, A survey of Combinatorial Theory, J.N. Srivastava et al. eds., 173–187, Amsterdam, North-Holland, 1973. Available `http://irma.math.unistra.fr/~foata/paper/pub18.pdf`.

[14] D. Dumont, *Interprétations combinatoires des nombres de Genocchi*, Duke Math. J. 41 (1974), 305–318.

the premise of my "Thèse d'Etat", but will go directly to this last one.

Schützenbeger[15] introduced the notion of "factorizations" of free monoids. A particular example is the so-called "Lyndon words" which are in bijection with a basis of the free Lie algebra. The subject of the "Thèse d'Etat" was to make a general theory of the relation between various classes of "complete" factorizations and the constructions of the various basis of free Lie algebra, including all known basis. For general factorizations, the relation is with the construction of a decomposition of the free Lie algebra into a direct sum of (free) Lie subalgebra, together with the construction of a "basic family" freely generating each subalgebra.

Such topic has a flavor of theoretical computer science with the study and constructions on words and free monoids, together with combinatorial algebra (groups and Lie algebra defined by relations and generators, Michel Lazard's commutators calculus in free groups[16], etc).

Pierre Cartier was the referee of my "Thèse d'Etat" and for publication as a Lecture Notes in Maths he asked me to restrict myself to "complete" factorizations, equivalent to the basis of free Lie algebra, and thus to rewrite completely the manuscript (!). Luckily, at that time, when you enter at CNRS as a trainee, not only you have an adviser but also a "Godfather", in the person of Dominique Foata (in French "parrain", there is no mention of God). I am extremely grateful to him for the help to write these Lecture Notes. With his many corrections, I learned how to write.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Viennot: In the first step, I like to start from a specific identity, find a combinatorial interpretation of both sides, and then find a bijection that will give proof of the identity. This can be very difficult, maybe impossible. You need some imagination. Then I will try to relate this bijection with other bijections and interpretations, and in a second step lift all these constructions in a more general framework of general theory. Maybe in a third step use such methodology for the construction of new bijections.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Viennot: When I work hard on a specific research problem or in the construction of a more general bijective methodology, I have sometimes the feeling to be connected to a wonderful word underlying the so-called "real" word that we can see. It can be a brief flash or a delicious sensation of floating on a cloud (a few hours or even several days). I have a strong impression that something is true before having the proof. In the beginning, it cannot be put in terms of words, but I feel that it is true. Maybe it will take weeks or months to give the proof and write it. Sometimes it happens to me that long after, I found a counter-example, but with a slight modification of the definitions involved, the initial intuitive idea became true.

Mansour:[d] What three results do you consider the most influential in combinatorics during the last thirty years?

Viennot: Of course I will talk only about my favorite and familiar world of enumerative with my bijective combinatorics glasses. Instead of selecting three particular results, I think it is better to quote three hot and influential areas of research where there is a package of major results.

First of all, I will quote the long-standing conjecture (Mills, Robins, and Rumsey[17]) for the enumeration of *alternating sign matrices* (ASM). Such objects are very simple to define and can be viewed as an extension of permutations. It has been first solved by Zeilberger[18] in 1992, in association with the proof by Andrews[19] for the number of the so-called *totally symmetric self-complementary plane partitions* (TSSCPP), which are enumerated by the same number. A second shorter proof, using the six-vertex model in statistical physics was given

[15] M.-P. Schützenbeger, *On a factorization of free monoids*, Proc. Amer. Math. Soc. 16 (1965), 21–24.
[16] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Annales Sci. ENS, 3, 71 (1954), 101–190.
[17] W. H. Mills, D. P. Robbins, and H. C. Rumsey Jr. *Proof of the Macdonald conjecture*, Invent. Math. 66:1 (1982), 73–87.
[18] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. 3:2 (1996), Article R13.
[19] G. E. Andrews, *Plane partitions V: The TSSCPP conjecture*, J. Combin. Theory Ser. A 66:1 (1994), 28–39.
[20] G. Kuperberg, *Another proof of the alternating sign matrix conjecture*, Int. Math. Res. Not. 3 (1996), 139–150.

by Kuperberg[20] in 1996.

Other combinatorial objects are enumerated by the same numbers as ASM: *descending plane partitions* (DPP) introduced by Andrews[21] in 1979, *alternating sign triangles* (AST) introduced by Ayyer, Behrend and Fischer[22] in 2016. At the end of the 90's a book on these topics has been written by Bressoud[23], just before another conjecture was discovered by Razumov and Stroganov[24]: these numbers appear in Physics with quantum spins chains, conjecture solved by physicists Cantini and Sportiello[25] in 2010 with bijective techniques. Finding bijections between these classes is a challenge for more than 30 years. Very recently a bijective construction has been given by Fischer and Konvalinka[26]. But many mysteries are still open. I do not see an interest to select which one is the most influential between these remarkable results that solve hard conjectures.

A second fascinating result (to me) is the first combinatorial interpretation of the moments of the Askey-Wilson polynomials by Corteel and Williams[27] in 2009 with the so-called *staircase tableaux*. These polynomials are at the top level classification of orthogonal polynomials (the famous Askey tableau[28]). When I start my life as a combinatorist, it was time for interpretation of Hermite polynomials, at the bottom level of the Askey tableau. At the end of the 70's, I remember a talk at Oberwolfach given by Foata[29] about his proof of Mehler's formula for Hermite polynomials. It was the time of blackboards and chalks (but with colors chalks, a must for combinatorics!). Nobody could surpass Richard Askey, who was called "the fastest chalk in West"! Then in the 80's appear transparencies, overhead projectors, and came slowly the combinatorial climbing of the Askey tableau with interpretations of the coefficients or generating function for Laguerre, Jacobi, Hahn, ... polynomials (by Bergeron, Leroux, Foata, Strehl, Kreweras, Labelle, Yeh[30], ...). The proof of Mehler's formula is now an exercise in a standard course on bijective combinatorics.

The interpretation of the moments of Askey-Wilson polynomials is in fact the culmination of a series of works in physics and in combinatorics. In physics, it is the famous PASEP (also called ASEP) model (partially asymmetric exclusion process), a toy model in the physics of dynamic systems far from equilibrium and the computation of the stationary probabilities. The most general case (with five parameters q, α, β, γ, and δ) has been solved in 2003 by Uchiyama, Sasamoto, and Wadati[31]. The Askey-Wilson polynomials are central in this resolution. In combinatorics, it is the interpretation of these probabilities in terms of *alternative tableaux*[32] (in bijection with permutations in the case of 3 parameters q, α, and β) coming naturally from a quadratic algebra related to this PASEP, in the same way, the famous RSK correspondence between Young tableaux and permutations is related to the Heisenberg quadratic algebra[f].

A third fascinating subject is diagonal harmonics and the resolution of the so-called $(n+1)^{n-1}$ conjecture[33], $n!$ conjecture[34], delta

[21] G. E. Andrews, *Plane partitions (III): The weak Macdonald conjecture*, Invent. Math. 53 (1979), 193–225.

[22] A. Ayyer, R. E. Behrend, and I. Fischer, *Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order*, Adv. Math. 367 (2020), Article 107125.

[23] D. Bressoud, *Proofs and confirmations. The story of the alternating sign matrix conjecture*, MAA Spectrum, Mathematical Association of America and Cambridge University Press, Washington, DC and Cambridge, 1999.

[24] A. V. Razumov and Yu. G. Stroganov, *Spin chains and combinatorics*, J.Phys. A34 (2001), Article 3185.

[25] L. Cantini and A. Sportiello, *Proof of the Razumov–Stroganov conjecture*, J. Combin. Theory, Ser. A 118:5 (2011), 1549–1574.

[26] I. Fischer and M. Konvalinka, *The first bijective proof of the refined ASM*, Sém. Lothar. Combin. 84B (2020), Article #18.

[27] S. Corteel and L. K. Williams, *Staircase tableaux, the asymmetric exclusion process, and Askey-Wilson polynomials*, PNAS 107:15 (2010), 6726–6730.

[28] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54:319 (1985), 1–55.

[29] D. Foata, *A combinatorial proof of the Mehler formula*, J. Combin. Theory, Ser. A 24:3 (1978), 367–376.

[30] X. Viennot, *The Art of Bijective Combinatorics, Part IV, A combinatorial theory of orthogonal polynomials and continued fractions*, Chapter 5a and 5b, IMSc, Chennai, 2019. See `http://www.viennot.org/abjc4-ch5.html`.

[31] M. Uchiyama, T. Sasamoto, and M. Wadati, *Asymmetric simple exclusion process with open boundaries and Askey–Wilson polynomials*, J. Phys. A: Math. Gen. 37:18 (2004), Article 4985.

[32] X. Viennot, *Alternative tableaux, permutations and partially asymmetric exclusion process*, talk in the workshop "Statistical Mechanics and Quantum Field Theory Methods in Combinatorial Enumeration", Isaac Newton Institute for Mathematical Sciences, 23 April 2008. See `https://sms.cam.ac.uk/media/1004`.

[33] A. M. Garsia and M. Haiman; *A graded representation model for the Macdonald polynomials*, Proc. Nat. Acad. Sci. 90:8 (1993), Article 36073610.

[34] M. Haiman, *Hilberts schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14:4 (2001), 941–1006.

conjecture[35], shuffle conjecture[36,37], (q,t)-Catalan[38], etc. We are in deep algebraic combinatorics. Many people are working hard for thirty years, in particular Garsia the pioneer in this story, Bergeron, Haiman, Haglund, Loehr, Armstrong, and many others. But many mysteries are waiting to be explained. I am not qualified to talk on that subject, I just follow from far away this subject, talking with my friends Adriano Garsia and François Bergeron, except that I have just been involved in some related work with Louis-François Préville-Ratelle[39], see my Answer[f] about Tamari and diagonal harmonics.

Mansour:[e] What are the top three open questions in your list?

Viennot: In the previous question, I mentioned 3 fascinating areas of research with remarkable achievements. In each of these areas, the story is far from the end. I will mention only open questions which bother me for a long time. First, coming back to the interpretation of the Askey-Wilson polynomials with some tableaux. The interpretation of the moments of Askey-Wilson polynomials has been improved with Corteel, Stanley, Stanton, and Williams[40]. These polynomials are q-polynomials with 4 parameters α, β, γ, and δ having natural symmetries. One cannot see these symmetries on the tableaux. An ultimate interpretation explaining these symmetries is missing.

Maybe this symmetry problem is related to another analog problem. Dumont and Foata[41] discovered a remarkable ternary symmetry on Genochi numbers. To my knowledge, today, there is no combinatorial interpretation of Genocchi numbers where the ternary symmetry appears "naturally". A discussion is given in [42]. Jousuat-Vergès[43] gave an interpretation with the alternating tableaux of the PASEP[11]. You may ask what is the relation with the symmetry problem for Askey-Wilson polynomials? Gessel and Zeng showed that the 3-variables polynomials expressing the ternary symmetry of Genocchi numbers are moments of some orthogonal polynomials known as continuous dual Hahn polynomials, an important sequence in the Askey-Wilson hierarchy. I suspect the existence of a super combinatorial object above the staircase tableaux interpreting the moments of Askey-Wilson polynomials where the 4-parameters symmetry appear naturally, together with the 3-variables related to Hahn polynomials and Genocchi polynomials, and why not going through the interpretation of Koorwinder polynomials discussed in [f] and going down to the famous problem of finding a natural explanation to the symmetry of the (q,t)-Catalan numbers discussed just below. It is possible to dream.

In the same philosophy about combinatorial interpretations where you can "see" in a "natural" way the symmetry, I have always been puzzled by the different interpretations of the (q,t)-Catalan polynomials[38] (Dyck paths, *area*, *bounce* parameter, ...) mentioned above, but where the symmetry of the two parameters q and t is hidden. I remember Adriano Garsia introducing me to this problem 30 years ago when we were in the magnificent environment at Mittag-Leffler Institute.

I would have many other top open questions about bijective proofs and combinatorial interpretations. Let's mention a question that remains in a corner of my mind for 40 years. In statistical physics, there is the so-called hard hexagon gas model. This model was solved

[35] J. Haglund, J. B. Remmel, and A. T. Wilson, *The delta conjecture*, Discrete Math. Theor. Comput. Sci. proc. BC (2016), 611–622.

[36] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. 126 (2005), 195–232.

[37] E. Carlsson and A. Mellit, *A proof of the shuffle conjecture*, J. Amer. Math. Soc. 31 (2018), 661–697.

[38] A. Garsia and J. Haglund, *A proof of the q,t-Catalan positivity conjecture*, Adv. Math. 175 (2003), 319–334.

[39] L.-F. Préville-Ratelle and X. Viennot, *The enumeration of generalized Tamari intervals*, Trans. Amer. Math. Soc. 369 (2017), 5219–5239. The Journal gave a wrong title, it should be "An extension of Tamari lattice".

[40] S. Corteel, R. Stanley, D. Stanton, and L. Williams, *Formulae for Askey-Wilson moments and enumeration of staircase tableaux*, Tran. Amer. Math. Soc. 364:11 (2012), 6009–6037.

[41] D. Dumont and D. Foata, *Une propriété de symétrie des nombres de Genocchi*, Bull. Soc. Math. France 104 (1976), 433–451.

[42] X. Viennot, *Interprétation combinatoire des nombres d'Euler et de Genocchi*, Séminaire de Théorie des nombres de Bordeaux, Publi de l'Université de Bordeaux I, 1982–83, §5;4, 54–60. See `http://www.xavierviennot.org/xavier/articles_files/Euler_Genocchi81.pdf`.

[43] M. Jousat-Vergès, *Generalized Dumont-Foata polynomials and alternative tableaux*, Sém. Lothar. Combin. 64 (2010), Article B64b.

[44] R. J. Baxter, *Hard hexagons: exact solution*, J. Phys. A: Mathematical and General 13:3 (1980), L61–L70.

[45] G. E. Andrews, *The hard-hexagon model and the Rogers-Ramanujan type identities*, Proc. Natl. Acad. Sci. 78 (1981), 5290–5292.

by Baxter[44] using Roger-Ramanujan identities and 14 analogous identities[45]. "Solved" means giving an expression (in fact a system of equations) for the partition function, or equivalently the so-called density of the gas. G.S. Joyce proved that this density is a formal power series that satisfies an algebraic equation (of degree 12). The open question is to *"explain"* combinatorially this algebricity.

When I was at Tata Institute in Bombay (Mumbai), Deepak Dhar gave me a system of algebraic equations (given by some colleagues working on modular forms). I was surprised. This system involves some simple algebraic equations having a flavor of Catalan numbers and some similarity with the system of equations for the number of planar maps given a long time ago by Tutte, where various combinatorial explanations have been given starting with the first bijection of Cori and Vauquelin[46] (as a difference of two algebraic languages), then the "direct" bijection of Schaeffer[47], and also many others, in particular in relation with physics[48]. Using the theory of heaps of pieces, the density of the gas becomes the generating function of *pyramids of hexagons* on a triangular lattice. From my education in Paris with Schützenberger and being since 40 years at LaBRI, a computer science laboratory, I have a background in theoretical computer science. Algebricity is sometimes related to algebraic languages which can be recognized by automata with one stack. With two stacks you lose the algebricity. From a naive point of view, trying to recognize pyramids of hexagons would need an automaton with many (infinite!) stacks.

Mansour:[f] What kind of mathematics would you like to see in the next ten-to-twenty years as the continuation of your work?

Viennot: From my previous answers you can see that I am more interested in developing general bijective methodologies (starting from particular facts, formulae, or bijections) than solving particular problems. Three main methodologies emerge from my work: heaps of pieces, combinatorial theory of orthogonal polynomials with weighted paths, and a third methodology I propose to call "cellular ansatz". These 3 methodologies are Parts II, III, and IV of ABjC[b].

Heaps of pieces (Part II of ABjC)[49] are related to various topics such as orthogonal polynomials, algebraic graph theory (chromatic, spanning trees, zeros of matching polynomials), representation theory of Lie algebras, fully commutative elements in Coxeter groups, Petri nets in computer science, Ising model and gas model in statistical mechanics, Lorentzian triangulations in quantum gravity. Much beautiful works on the relation between heaps and representation of Lie algebra has been done by Richard Green[50] and its students at Boulder University. I wish the continuation of such fruitful developments.

I am somewhat surprised to see that recently several papers have appeared in the last six months on arXiv where heaps methodology plays a key role, while the basic paper on heaps appears in 1985 and the Lecture Notes (in French) on orthogonal polynomials in 1983. Here are some: Garsia and Ganzberger (heaps and orthogonal polynomials)[51]; Tamm, Pospelov, and Nechaev (heaps in statistical mechanics)[52]; Giscard (heaps, paths and self-avoiding polygons)[53]; Bagno, Biagioli, Jouhet, and Roichman (heaps and fully commutative elements in Coxeter groups)[54]; Cigler and Krattenthaler (heaps and orthogonal polynomials)[55]; Fredes and Marckert (heaps and spanning tree in prob-

[46] R. Cori and B. Vauquelin, *Planar maps are well labeled trees*, Canad. J. Math. 33 (1981), 1023–1042.

[47] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aléatoires*, Ph.D. thesis, Université Bordeaux I, 1998, (1999 SPECIF price). See `http://www.lix.polytechnique.fr/~schaeffe/Biblio/PhD-Schaeffer.pdf`.

[48] J. Bouttier, P. Di Francesco, and E. Guitter, *Planar maps as labeled mobiles*, Electron. J. Combin. 11:1 (2004), Article #R69.

[49] See `http://www.viennot.org/abjc2.html`.

[50] R. M. Green, *Combinatorics of minuscule representations*, Cambridge University Press, 2013, Cambridge Tracts in Maths 199.

[51] A. M. Garsia and G. Ganzberger, *Fibonacci polynomials*, `https://arxiv.org/pdf/2009.10213.pdf`.

[52] M. Tamm and N. Pospelov and S. Nechaev, *Growth rate of 3D heaps of pieces*, arXiv: 2009.12540v2 [cond-math.stat-mech], 2020.

[53] P.-L. Giscard, *Counting walks by their last erased self-avoiding polygons using sieves*, Discrete Math. 344:4 (2021), Article 112305.

[54] E. Bagno, R. Biagioli, F. Jouhet, and Y. Roichman, *Block number, descents and Schur positivity of fully commutative elements in B_n*, `https://arxiv.org/abs/2012.06412`.

[55] J. Cigler and C. Krattenthaler, *Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity*, `https://arxiv.org/abs/2012.03878`.

abilities)[56]; Rani and Arunkumar (heaps and Borcherds-Kac-Moody Lie superalgebras, with some flavor of Lalonde's Lyndon heaps)[57]. With Lyndon heaps, we are in the flavor of Lyndon basis of free Lie algebra[c]. I have no doubts that heaps will continue to have a fruitful and exciting life.

About orthogonal polynomials (part IV of ABjC)[10], my first dream would be to see an ultimate, complete, and unified combinatorial "understanding" of the classical orthogonal polynomials, following two paths in the Askey-Wilson tableau. An ascending path starting from the known interpretations of the coefficients (or generating function) of the polynomials (Hermite, Laguerre, Jacobi, Hahn, ...) and reaching the top with the Askey-Wilson polynomials. A descending path starting from an interpretation of the moments of the Askey-Wilson polynomials where the symmetries of the four parameters can be seen[e], and then going down through Hahn and Jacobi polynomials and reaching the known interpretations of moments for Laguerre, Tchebycheff, Hermite polynomials. Inspiration maybe can come from the "bridge" between Hermite and Askey-Wilson with our (semi)-bijective proof (with Ismail and Stanton[58]) for the famous Askey-Wilson integral expressed as an integral of a product of four q-Hermite polynomials. An exciting fact is a relation between orthogonal polynomials and the combinatorics of ASM (alternating sign matrices) discussed in [d]. Colomo and Pronko[59] showed that some enumeration problems of ASM can be solved using some Hankel determinants related to Continuous Hahn, Meixner-Pollaczek, and continuous dual Hahn polynomials.

Finally, the "cellular ansatz" (Part III of ABjC)[11] contains some general tools for constructing new bijections. Let me explain briefly the philosophy of this *"cellular ansatz"*, which is not written anywhere. We start from RSK defined from a representation of the Heisenberg algebra, as explained by Fomin[6] with "local rules". These local rules are usually defined by attaching 3 Ferrers diagrams to 3 adjacent vertices of an elementary cell on the square lattice. This process can also be applied for the PASEP algebra and leads to the known bijections between permutations and alternating tableaux, corresponding to the PASEP with 3 parameters. Another quadratic algebra related to the PASEP with two types of particle leads, applying the philosophy of the "cellular ansatz" to the notion of *rhombic alternating tableaux* (work with Mandelshtam[60], FPSAC 2016) and bijections with *assemblées of permutations*.

The process of extending alternative tableaux (PASEP with 3 parameters) to staircase tableaux (PASEP with 5 parameters) has an analog with this rhombic alternative tableaux and leads to an interpretation of the Koorwinder polynomials with rhombic staircase tableaux (work of Corteel, Mandelshtam, and Williams[61]). Now I can see as a spectator the development of this new active field where the tower going from Hermite polynomials to Koorwinder polynomials via Askey-Wilson is joining the tower going from Young tableaux, symmetric function, MacDonald polynomials. Maybe an extension of the cellular ansatz philosophy is possible and will be useful?

Going back to the beginning of the motivation of the cellular ansatz with RSK, representation of the Heisenberg algebra and Fomins's local rules, I think it is better to write these local rules by attaching labels on the edges, as shown in the paper[62] 11th GASCom, Athens, 2018. Then RSK can also be defined as the result of another process I call *"demultiplication of equations"* in a quadratic algebra. In general, we can define the dual of this quadratic algebra. In the case of RSK and the Heisenberg algebra is self-dual and no new bijection

[56] L. Fredes and J.-F. Marckert, *Aldous–Broder theorem: extension to the non reversible case and new combinatorial proof*, `https://arxiv.org/pdf/2102.08639.pdf`.

[57] S. Rani and G. Arunkumar, *A study on free roots of Borcherds-Kac-Moody Lie Superalgebras*, `https://arxiv.org/abs/2103.12332`.

[58] M. E. H. Ismail, D. Stanton, and X. Viennot, *The Combinatorics of q-Hermite polynomials and the Askey—Wilson Integral*, Europ. J. Combin. 8:4 (1987), 379–392.

[59] F. Colomo and A.G. Pronko, *Square ice, alternating sign matrices and classical orthogonal polynomials*, arXiv:math-ph/0411076v2, 2004.

[60] O. Mandelshtam and X. Viennot, *Rhombic alternative tableaux and assemblées of permutations*, Europ. J. Combin. 73 (2018), 1–19.

[61] S. Corteel, O. Mandelshtam, and L. Williams, *Combinatorics of the two-species ASEP and Koornwinder moments*, Adv. Math. 321 (2017), 160–204.

[62] See `http://ceur-ws.org/Vol-2113/paper22.pdf`.

is deduced. For the PASEP algebra, the dual algebra is different. The tableaux become the well-known *tree-like tableaux*[63] associated with the PASEP (with 3 parameters), and the analog principle gives rise to new bijections (such as the "Tamil bijection" for binary trees and the "Adela bijection" for permutations).

These are powerful tools that should be developed. In particular for alternating sign matrices (ASM) and *fully packed loops* (FPL), which can also be defined as some tableaux associated with a quadratic algebra with 4 generators.

Another topic I like and which should continue its development is around Tamari lattices. On the set of binary trees enumerated by Catalan numbers, inspired by the associativity property for well parenthesis expressions, Dov Tamari has defined a lattice in his thesis, Université de Paris, 1951. A classical geometric representation is a so-called associahedron.

Let us tell another story. In November 2013, I was invited to give a talk at the workshop in Madrid "Recent trends in Algebraic and Geometric Combinatorics" where many "Tamarists" would be there. I should present something related to Tamari but had nothing in my combinatorial suitcase. In the past I used the notion of *"canopy"* of a binary tree in relation with some bijections solving the enumeration of *convex polyominoes*[l]. The canopy of a binary tree is a word in two letters which has been defined (without giving a name) by Loday and Ronco[64] in the context of Hopf algebras, analog to *up-down sequences* for permutations. For the Madrid workshop, I study the behavior of the canopy in the Tamari lattice and made a modest contribution showing that binary trees with a given canopy form an interval of the Tamari lattice.

Then at the invitation of Luc Lapointe, we move to Talca, Chile, for a few months, and I start to repeat the talk given in Madrid. At the end of the talk, a young québecois post-doc, Louis-François Préville-Ratelle wanted to talk to me. Around a *pisco sour*, he explained to me his work[65] in the context of diagonals harmonic, the problem of extending the Tamari lattice for the so-called rational Catalan combinatorics, corresponding to define an analog of Tamari lattice on paths (with North and East steps) located below a line of rational slope. His adventurous idea was to search for a much more general Tamari lattice for the set of paths that are below any arbitrary path ν with North and East steps. He had a possible definition with 3 beautiful conjectures. He has a strong feeling that this was related to the talk I just gave. The next day we realized[39] that the ν-Tamari he was looking for was in bijection with the interval of binary trees with a given canopy ν.

People were looking for an extension of Tamari lattice to rational Tamari, Louis-François for a much more general extension, and this extension was there, inside the ordinary Tamari! We presented this work at FPSAC 2015 in South Korea and published it in the Transactions of the AMS[39] amazingly the title given in the Transactions is wrong (!), the correct title is "An extension of Tamari lattice", see `https://arxiv.org/pdf/1406.3787.pdf`.

Immediately, this was the starting point of many beautiful works by Fang and Préville-Ratelle[66], Caballos, Padrol and Sarmiento[67,68] and many others, in particular by Defant[69]. I see that we are going from ν-Tamari to the combinatorics of PASEP described above.

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Viennot: Mainstream areas or important subjects will vary from time to time and is subject to one personal's taste. One quality for a domain is to be active, with plenty of facts, formulae, problems that are all connected to each other by a network of links, correspondence. Moreover, another important criterium is when the domain has many interactions with

[63] J.-C. Aval, A. Boussicault, and P. Nadeau, *Tree-like tableaux*, Elec. J. Comb. 20 (2013), Article P34.

[64] J.-L. Loday and M. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. 139:2 (1998), 293–309.

[65] F. Bergeron and L.-F. Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets*, J. Combin. 3:3 (2012), 317–341.

[66] W. Fang and L.-F. Préville-Ratelle, *The enumeration of generalized Tamari intervals*, European J. Combin. 61 (2017), 69–84.

[67] C. Ceballos, A. Padrol, and C. Sarmiento, *Geometry of ν-Tamari lattices in types A and B*, Trans. Amer. Math. Soc. 371 (2019), 2575–2622.

[68] C. Ceballos, A. Padrol, and C. Sarmiento, *The ν-Tamari lattice ν-trees, ν-bracket vectors, and subword complexes*, Electron. J. combin. 27:1 (2020), Article #P1.141.

[69] C. Defant, *Meeting covered elements in ν-Tamari lattices*, arXiv: 2104.03890v1 [math.CO], 2021.

other domains and other fields. Nowadays everything goes so fast. New domains are just born and a few years later appear to be "important", such as the beautiful new notion of *cluster algebra* of Fomin and Zelevinski[70]. Even in a few years, it became a new domain classified in Maths Review 13F60. Although it is classified as a subdomain of commutative algebra, it contains a lot of combinatorics.

In 1977, in "Panomara des mathématiques pures, le choix bourbachique", Dieudonné[71] gave a classification of mathematical theories into six classes. The best seems to be in class IV *"Les problèmes qui s'ordonnent autour d'une théorie générale, féconde et vivante, avec l'apport ininterrompu de problèmes nouveaux"*. In this classification Combinatorics is in class II *"les problèmes sans postérités"*. In more than 40 years what a change! I think "Combinatorics" can be classified in class IV and that the classification of Maths Review is somewhat obsolete. But how to classify the subdomains of a domain which is transverse to all mathematics, with incursion in physics, computer science and theoretical bsiology?

Maybe the best criterium is "beauty". Mathematics is a form of art, beauty is fundamental. The paradox is that at first, one can believe that he or she is creating a piece of this art, a nice bijection, a nice space, but after some more exploration, one will realize that all this was already there, no piece of art has been created, we are just walking in a museum, but not a closed museum with past dirty collections, but a living world, like exploring mountains.

Some mountains like the mountains near Chamonix, Mont-Blanc have been explored in every corner of each face and peaks and it is impossible to inaugurate a new route somewhere. Probably all main peaks in Himalaya and Andes have been climbed (except an important one for Tibetan, Kailash Mount, and should never be climbed). Comparing exploration of maths and mountains, which mountains are the more important? Himalaya or Andes? Alpes or Pyrénées? I went to the four of them and my preference is for Pyrénées, even if the highest mountain is Pic d'Aneto with only 3404 meters high.

Mansour:[g] What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Viennot: You can go by a continuous and fruitful path from the purest mathematics to the most applied mathematics. It is not a linear path, it a network of fruitful connections in all directions. In my case I have been naturally connected via "pure" notions of combinatorics to various topics such as the shape of rivers in Hydrogeology, mRNA (now well known with the vaccine for Covid-19), radiology of lungs, control theory, etc. I spent maybe one year of my life applying pure beautiful concepts related to trees and Catalan numbers to computer graphics[72]. Let me talk a little about this adventure.

The idea comes from a parameter on binary trees, called Strahler number, introduced by hydrogeologists Horton and Strahler in the study of rivers networks, which also appears in computer science as the minimum number of registers needed to compute an arithmetical expression. The generating function has very beautiful properties. Flajolet and coauthors[73] gave asymptotic properties related to Delange function in number theory and also Riemann zeta function. Françon[74] gave a highly recursive bijective proof with Dyck paths. Surprisingly another parameter (complexity of the secondary structure of RNA) totally different, appears to have the same distribution (and it is really difficult to prove it). This is the Ph.D. thesis of Mireille Vauchaussade de Chaumont[75]. With Didier Arquès (from Besançon), and our students Georges Eyrolles and Nicolas Janey[72], we start from

[70]S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15:2 (2002), 497–529.

[71]J. Dieudonné, *Panorama des mathématiques pures. Le choix bourbachique*, Gauthier-Villars, Paris, France, 1977.

[72]D. Arquès, G. Eyrolles, N. Janey, and X. Viennot, *Combinatorial analysis of ramified patterns and computer imagery of trees*, Proc. SIGGRAPH'89, Computer Graphics 23 (1989), 31–40.

[73]P. Flajolet, J. C. Raoult, and J. Vuillemin, *The number of registers required for evaluating arithmetic expressions*, Theoret. Comput. Sci. 9 (1979), 99–105.

[74]J. Françon, *Sur le nombre de registres nécessaires à l'évaluation d'une expression arithmétique*, RAIRO Inform. Théor. 18 (1984), 355–364.

[75]M. V. de Chaumont and X. Viennot, *Enumeration of RNAs by complexity*, Proc. Intern. Conf. of Medicine and Biology, Bari, Italie, 1983. Lecture Notes in Biomathematics, 1985.

this Strahler analysis for binary trees to do some synthetic images of natural trees, we work very hard to get some beautiful images, submit a paper to the big annual meeting SIGGRAPH in USA (20 thousands participants, 5000 for the scientific part) and we were accepted. We get a glimpse of another world.

You have to give a talk in front of thousands of people (Boston 1989) in a huge room used for meetings for presidential elections, with 3 big screens (two for your slides, one for video or for your face which is lighted with very strong projectors, so that you do not see anybody! I cannot use my usual transparencies with an overhead projector and do the usual so-called "viennotique". Here, there are no transparencies, just slides in the old way (on a photographic plate). I followed the advice of my colleague Claude Puech, and after 3 or 4 slides, without seeing anybody, I hear the crowd laughing. I know it was going to go well. After that, we have been invited speakers at IMAGINA'90, the International forum of new images of Monte-Carlo, inauguration by the Prince Albert de Monaco, luxury hotel and helicopter to go to Nice airport, an invitation to give a course at EUROGRAPHICS, etc. Then, having a glimpse of such world, having pushed the Strahler idea at the maximum, it was good to stop and go back to our world of bijective combinatorics.

Let's tell another personal story. Several years after this incursion into the world of Computer Graphics, I have the chance of spending one month at Mittag-Leffler Institute with Donal Knuth. He loves the Strahler parameter with the 3 interpretations on binary trees, Dyck paths, and planar trees (coming from the mathematics of mRNA), and the 3 bijections between them: between binary trees and Dyck paths (Françon[74]), between binary trees and planar trees (Zeilberger[76] who won the price of 10 bottles of "Domaine des Mattes" 1981) and a bijection between planar trees and Dyck paths[77]. Don had worked hard to improve these bijections. The night just before the end of our stay, he found a very nice "direct" bijection[78] replacing the one of Françon, a style of bijection not familiar in combinatorics where the reverse bijection is defined in the same way. But there was a big problem.

His bijection was using an intermediate structure between paths and binary trees, a very classical data structure in computer science, called "heaps". Nothing to do with heaps "à la Xavier". Don wanted me to change the name "heap" throughout my works. I told him that this was not possible, there were already many papers in various domains using the name "heaps". Even worse: in our conversations during this Mittag-Leffler month, I was inspired to introduce a fourth combinatorial interpretation of Strahler numbers, with a new interpretation of Catalan numbers as sequences of heaps of dimers on cylinders of size 2, 4, 8, 16, ... and Don was going to put it in his report with his new bijection using "computer science heaps". Finally, I proposed to call this interpretation "Kepler towers"[79] for different reasons: this is a reminder of Kepler model for solar system and heaps can also be used in quantum gravity, see ^{k}. Don prefered to make a projection on a plane of these sequences of cylinder heaps, looking like a CD Rom, see the cover of Stanley's book on Catalan numbers.

But back home, another problem rises. Don wanted to use heaps (à la Xavier) in his Volume 4 of TAOCP, Pre-fascicle 6A, section 7.2.2.2 about satisfiability, starting with Lovász's local lemma, and of course, he cannot use the term "heaps" in Computer Science books. What to do? After long exchanges of emails, Don agreed to use in his book the French name for heaps, that is *"empilements"*.

Mansour: Do you think that research and the field of teaching mathematics can benefit from the notion of video lectures?

Viennot: About video/lecture, there is a long discussion in Igor Pak's blog[80] "You should watch combinatorics videos!". (May 2, 2015), and a collection of 400 videos (prior to 2015) on his website. There are more and more videos

[76]D. Zeilberger, *A bijection from ordered trees to binary trees that sends the pruning order to the Strahler number*, Discrete Math. 82 (1990), 89–92.

[77]X. Viennot, *A Strahler bijection between Dyck paths and planar trees*, Discrete Math. 246:1-3 (2002), 317–329.

[78]D. Knuth, Program to read: Zeilberger, Françon, and Viennot. See `https://www-cs-faculty.stanford.edu/~knuth/programs/francon.w`.

[79]X. Viennot, *Kepler towers, Catalan numbers and Strahler distribution*, FPSAC'05, special session dedicated to Adriano Garsia, June 2005, Taormina, Italy. `http://www.xavierviennot.org/xavier/articles_files/Kepler-Towers2.pdf`.

[80]See `https://igorpak.wordpress.com/page/3/`.

related to combinatorics. It can be a series of lectures for a full course, or a lecture in a seminar, workshop, or congress. You will find pieces of advice on how to watch videos and also how to give a talk which is going to be recorded and available for the public to watch.

Probably in the future, with the improvement of this kind of communication, which sometimes can be an “art”, there will be a library of videos (satisfying a certain standard for publication) freely available, analogue to ArXiv for papers. I have practiced this kind of activity with many videos recorded and can give some pieces of advice from my own experience of giving and watching videos.

If it is a video recorded in real-life, in front of an audience, it happens very often, that to make a good video, you need a good professional recording the video. I have seen some videos of some conferences where the camera insists on the speaker and not on the slides, or sometimes you even do not see some slides. In that case, you should watch the videos with two screens, one for the video, the other for the slides. An important detail is that when you put your laser pointer on the screen, you will not see it on the video. It is better to have a long stick. Another way, which is now widely used since Covid-19 is to record the speaker with a fixed camera, the slides related to the talk appear on a full screen, the speaker appears in a little window on one of the corners of the screen and the arrow of the computer replaces the laser pointer.

Usually, people do not look at videos in totality, just a few portions or even a few minutes. Looking at a video instead of reading a paper, is like reading the paper where you can see only one line at the same time. The main problem is to be able to go inside a video in the same way you can go back and forth inside a paper. It seems that some tools are or will be created. For example, YouTube is creating chapters in videos where clicking somewhere enables you to go to another chapter.

One possible solution for the videos related to some conferences of courses is given on the website “The Art of Bijective Combinatorics” (ABjC[8]). Videos fit very well to explain bijections. The beauty of bijective combinatorics can be spoiled when bijections are described in a written way. For each conference or course, there are downloadable slides, a full page devoted to a detailed map of the talk, with sections, subsections, key facts, definitions, or propositions. In front of each line, there is the page number of the slides with a link giving the time. If you click on this link, the reader goes directly into the video at the corresponding place up to one second. Thus you can navigate better in the video, in the same way, you navigate in a paper, with 3 screens: the video, the slides, and the website.

In the field of teaching mathematics, there is the idea of “video book”. There are several possible definitions. A possibility is to start from a written book and then every chapter and subchapter are put into videos as if you were giving a course from the written book. A nice example is with the book of Philippe Flajolet and Robert Sedgewick “Analytic combinatorics”[81]. On the left part of the screen, one can navigate inside the contents of the book, for each portion selected, you get the corresponding video. The speaker (here Sedgewick) explains with slides as if he was in a course for students. On the right part of the screen you can see in written words what the speaker says, this text is following exactly the video, with some marks giving the time. This kind of video book is for students who follow a course. There are many other such courses, mainly for computer science and engineering. It is available on the platform CUvids[82] (nothing to do with Covid!) from Princeton University.

Another kind of video book is based on the reverse process. It may be something trying to propose a set of videos replacing a book, where you can navigate in the same way you turn the pages of a book. I try to develop this idea with the “video book” ABjC. It has 3 components, a website, a set of videos (more than 100 hours), and a set of slides. It is divided into 4 volumes (equivalent to four books), each volume (Part) with some chapters and subchapters, each corresponding to a video (about 1h30). In the same way as a book, there are (or will be) prefaces, introductions, references, complementary conferences related to each Part. An advantage compared to a written book is that one can

[81] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.
[82] See `https://cuvids.io/`.

add some complements, links to related talks or papers. But the basic material (slides and videos) should be kept unchanged with a fixed web address.

This video book ABjC is based on a series of courses given at IMSc during 4 years at two different levels at the same time (for students and for researchers). IMSc has a special video room for recording conferences or classes. One camera is for the speaker, another for the public or the students. Slides appear with two small rectangles in some corners of the screen. The only problem is that sometimes there are no micro for the audience and in the video, you do not hear the question, except if the speaker repeats the question. This can be done for a conference but is not natural inside a course. It happens that for some not recorded conferences, I repeat the same conference in the video room at IMSc. See for example the (high speed) talk I gave for the 60th birthday of Christian Krattenthaler in Strobl (in 50') and the same (with exactly the same slides) (Epilogue of ABjC[83]) but in almost 100'. The next step should be to write some papers giving an overview of each of the 24 chapters, and maybe one day to finish with written books. This would be exactly the reverse process than the one of Flajolet and Sedgewick. But, I know (and also my co-authors) that I am bad at writing papers or books.

Most combinatorics videos that are not on a specific site of a University institution are on YouTube. A problem with videos, or a website for a video book, is that the URL address and the link may change with time. Another problem is for our Chinese colleagues. They have no access to YouTube. A solution is to put your video on bilibili, a popular chain for young people who accept videos given by scientists. It is better to go there with your harddisk and give it to the colleague who invited you. Many thanks to Bill Chen for his kind invitation to Tianjin University, September 2019!

Another possibility is with Vimeo which seems to be less commercial than YouTube. Thus I will finish the answer to your question by giving some publicity for the series of videos organized by SMF (the French Mathematical Society) and BNF (Bibliothèque Nationale de France) in Paris. Each year there are 4 conferences (in French) for a wide audience given in the big amphitéâtre of the BNF in Paris, starting in 2005 with Don Zagier about the series of letters between Hardy and Ramanujan. These conferences are recorded by professional technicians with several cameras and mixed in realtime. The title of the series is called *"Un texte, un mathématicien"*[84]. The speaker starts from a historical paper, book or letter, gave the context, and shows how it is related to modern science. In particular, the reader can listen a video of a conference given by Pierre Cartier[85], 17 January 2007, about the "return of figures" in mathematics. There have been some conferences related to combinatorics and I encourage the reader to go there, in particular to a conference[86] given in 2007 (300th anniversary of birthday) about Euler and combinatorics, conference given with two violinists and a storyteller.

Mansour: It seems you have started planting your own vegetable garden. What was the reason you decided to do this? I have been experiencing a complex problem at home recently. One of my daughters, Atil, has decided to follow a vegetarian diet based on some ethical issues. Whenever she wants to discuss it, I try to escape by saying that "Ohh, this is a challenging real-life problem; I am a mathematician, let me finish this paper, later we can talk about it!" But I have no satisfactory answer! What do you think about eating meat not as a nutritional issue but as a philosophical issue?

Viennot: Planting trees, making and eating your own vegetables (of course organic), observing Nature is really a rewarding experience. I hope that every researcher in science can do the same even a tiny garden on a balcony. For example, the below picture (selfie) with two walnut trees is instructive.

The two trees are born naturally 20 years ago from two nuts coming from a big walnut on our piece of land. The nuts were very close and gave two different stems, but which fi-

[83] See `http://www.viennot.org/abjc-epilogue.html`.

[84] See `https://smf.emath.fr/la-smf/cycle-un-texte-un-mathematicien`.

[85] See the conference of Pierre Cartier at BNF, 17 January, 2007. The link to the video is available at `https://smf.emath.fr/smf-dossiers-et-ressources/cartier-pierre-le-symbolisme-mathematique-des-figures-aux-nombres-et`.

[86] See `http://www.viennot.org/abjc-euler.html`.

nally joined together at the basis of forming a unique trunk. The two trees have exactly the same height, avoid each other, all their branches form half circles on the left or the right side of the picture, and between them, not a single branch is going in direction of the other. I even observed a branch on the top starting by going straight towards the other trunk and turning immediately before hitting the other trunk. How he "knows" he was going to annoy his brother? There are some scientific researches in this sense about communications between trees.

The question related to your daughter Atil is delicate, maybe going outside such an interview. I do not want to give pieces of advice or tell you what you should do in your relationship with your daughter Atil. I can only share an experience with my daughter. When she was almost 18, after finishing lycée she was going to study abroad. Before the departure, we went for a trekking in the Pyrénées and for each stop near a lake, she told me the complaints she had to tell me, in particular, she was still remembering that when she was a kid, she wanted to play with me on my keens. My answer was "I have no time, I have to finish some mathematics". Every researcher work at home, even if he or she works at university. We have two offices, one at university and one at home. It is difficult to organize time at home, one for maths, one for children.

About eating meat, in fact, I think that nutritional issues, philosophical, and ethical issues are mixed. When one eats meat, it depends on where the meat comes from, which food was given to the animal, in which environment the animal was growing, and finally in which conditions was the animal killed. If the animal received many antibiotics, grow in a closed big industrial factory, with some GMO soy coming from huge plantations where the forest has been destroyed for such plantation, the 3 points of view are mixed. First, it will not be good for your health and from an ethical point of view the person will participate in climate change, the change of organic agriculture to industrial agriculture, the destruction of nature, etc. Young people are very concerned with climate change. Humanity should immediately change radically many things and not wait. Maybe it is already too late. Eating meat instead of vegetables, even if all conditions for the animal are good, is sending much more carbon into the atmosphere compared to eating vegetables.

The last two days, all Bordeaux vineyards were affected by the freeze. This is a classical phenomenon. But what is new, was the hot and beautiful weather the last 3 weeks, cold nights with a lot of winds. Most the vegetable gardens, fruit trees, ..., were frozen, the wind increasing the bad effect of the freeze. My neighbor (84 years old) who teach me many things about vineyards and vegetable gardens, told me that it was the first time in his life he has seen such a disaster. During the night many winegrowers light some fires inside the vineyard to slow down the freezing. I was concerned with my vineyard and told you that my report will be a little late. Living constantly in nature, you can really see and feel the changes.

But there are other concerns than climate change. Intensive industrial farms for growing animals make the propagation of possible viruses much more efficient. This is why bats create more often viruses because some fifty thousand can live in the same grotto. Industrial farms for animals is a good vector for the new virus. For example, with the Covid-19 virus, in Denmark, there are intensive mink breedings, some get a form of Covid. The government was afraid that this virus makes mutations inside these breedings, and go back to humans in a variant form where the actual Covid vaccines will not be efficient. They ask to kill the millions of minks! Of course, we do not eat minks, but it may be the same in industrial farms for meat. Thus not eating meat, or at a least few, and checking how it has bred can give more chances to avoid another virus after Covid.

Now another point of view is the respect of nature and of the "well-being" of the animal. You must know how the animal travel to the slaughterhouse and how it is killed, in which conditions. This is for the respect of the animal-life if it is going to give you its meat. Even if you do not care about the animal and just want to have good meat, this is important. Here we have lampreys, a kind of ugly prehistoric fish which look like meat (we cook it in red wine, usually white wine is associated with eating fish). It is very expensive to have. The first time I taste this special plate, it was in a restaurant and I did not really find it exceptional, even not so good. In my village there is a fisherman who goes fishing in spring for lampreys, there are no more in the Garonne river, but there are some in the Dordogne river. His wife told us that the animal is very afraid. Once they are fished, they put the animals in calm water for half a day, so that the animal became calm. The preparation is a whole week. The taste was completely different. A scientist will say that when animals are afraid, they will secrete some chemical and that you will eat this chemical making the taste different. Philosophers will say that you "eat the fear" of the animal when it is going to be killed, which I suppose is not good for your mental health.

I can continue on the spiritual side, but I think it is already too long. I hope this will contribute to the discussion with your daughter.

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Viennot: Try to construct your glasses, to look at the problem with your personal feeling, your taste.

Try to reprove known facts in your own way, do not wait to begin researches once you know everything in your domain.

Do not choose a field because it is fashionable, or there are more chances to get a position.

Choose what you like the best, even if your career will be slower at the beginning.

If you work on a hard problem, work very hard constantly on it several days (and nights) and then stop and go back to one of your other favorite problems.

Do not have only one problem in your mind. Work on several problems, but one by one.

Have conversations with friends doing researches, compare your researches, maybe a combination of both will leads to a solution. See my rencontre with L.-F. Préville-Ratelle about the discovery of ν-Tamari ^{f}.

Sometimes working hard on a problem you will get the idea to solve another of you current problems.

Maybe you will get prices, medals, and success, but it may be dangerous for your ego. Try to keep your usual modesty.

Listen to the advice of Joseph-Louis Lagrange encouraging colleagues and students to study Leonhard Euler: *"Lisez Euler, dans ses écrits tout est clair, bien calculé, ils regorgent de beaux exemples et parce que l'on doit toujours étudier les sources"*.

My friend Alain Lascoux was always going back to the sources. We spent one month together at Mittag-Leffler Institute. He was sad to see young students spending most of their time on the screen of their computer without exploring the fantastic historical library of old books. He told me to look at the book of Arbogast[87] (1800). I was very surprised to discover Catalan numbers, the so-called ballot numbers, and the formula for them, which everybody believe it was given by Witworth[88] (1879), Bertrand[89] (1887), André[90] (1887). But few people know that it was also given earlier by Delannoy[91] (1886) with what he called "méthode de l'échiquier", i.e. movement of a tower on a chessboard.

Going back in history is really pleasant, especially for combinatorics. At Mittag-Leffler I was also surprised to open a book of

[87] L. F. A. Arbogast, *Du calcul des derivations* (in French), Strasbourg: Levrault, 1800.
[88] W. A. Whitworth, *Choice and Chance*, 4th ed., Deighton Bell, Cambridge, 1879.
[89] J. Bertrand, *Solution d'un problème*, C. R. Acad. Sci. (1887), 369.
[90] D. André, *Solution directe du problème résolu par M. Bertrand*, C. R. Acad. Sci. (1887), 436.
[91] H. Delannoy, *Emploi de l'échiquier pour la résolution de problèmes arithmétiques*, Assoc. Franç. Nancy XV, 1886.

Eugène Catalan, with a handwritten dedication to Mittag-Leffler, and where I found a page where Catalan introduced some numbers named "Catalan numbers" (!) (by Amiral De Jonquière).

I will finish with two more advices.

Keep contact with Nature, view the beauty of Maths and Nature as one, get your inspiration by walking in forests, mountains, in front of a lake, the ocean, or under the stars by a beautiful night.

And finally, do mathematics with your heart.

Mansour: Would you tell us about your interests besides mathematics?

Viennot: I like to be in contact with Nature, to be embedded in wild and remote places. I used to do climbing, mountaineering, trekking, and sometimes some expeditions in various mountains of the globe. With Pierre Leroux, almost every year we were going in winter for one-week trekking with skis in some mountains of Québec, Rocky mountains and Pyrénées. With Pierre we have done an unforgettable expedition in canoe: 3 weeks in Nunavik (ᓄᓇᕕᒃ), (the Inuit country far North of Québec). Now, no more expeditions. I enjoy going with my québecquois canoe on the Garonne river very close to my village. The rapids are replaced by this amazing wave called *"mascaret"* when the river reverses the way it's going. Also, I like sailing. For my postdoc at UCSD in San Diego, I went by crossing the Atlantic Ocean with a sailing boat. Again an unforgettable experience (and also for the administration at CNRS when they had to fill up the administrative papers). I love going on a bicycle, hearing classical music, spending hours under the sky by night, and doing photography, especially in Nature.

After my postdoc, when I came back from California, I was dreaming to live in a village, taking care of fruit trees, growing my own vegetables, near a big city having a university with a combinatorial group. Bordeaux was the perfect place, with a small combinatorial group around Robert Cori, with young people looking for a "Thèse d'Etat" like Dominique Gouyou-Beauchamps, Bernard Vauquelin, Maylis Delest, Jean-Guy Penaud, or younger for a "thèse 3ème cycle" Serge Dulucq, Myriam DeSainte-Catherine, Mireille Vauchaussade de Chaumont, etc. Moreover, my favorite mountains, the Pyrénées were not too far. I settle in Isle-Saint-Georges, a village 20 km south of Bordeaux. Near the village, I found a piece of land to make a vegetable garden, but it was a vineyard and I did not want to take off everything for the garden. This is the reason why I start to learn to make wine, take care of the vineyard.

I also practice some yoga and I am interested in oriental spirituality. When I do intensive research, I superpose in my mind the combinatorial objects and bijections involved, and concentrate on them, trying not to think, keeping the mind in stillness. Now I understand that this is a kind of meditation.

Mansour: You also dedicate your time for the popularization of science in general and mathematics in particular. What do you aim for in these activities?

Viennot: I like to share the beauty of science and mathematics with students and the general public. With bijective combinatorics you can really touch a wide audience from elementary schools to people who hate mathematics after some trauma during their scholar education.

For example I have explained Young tableaux, Schützenberger "jeu de taquin" and the geometric construction of Robinson-Schensted[j] at elementary schools and in some colleges. It will take more than one session, but starting from a permutation, kids can play with these very concrete constructions on a grid and feel the magic coincidence of the two constructions. Trees and branching structures in Nature, in relation with the Strahler analysis of trees mentioned in [g] is a perfect theme for a wide audience in order to show the mathematics interacting in different domains of science.

François Hollande's socialist government a few years ago, introduced the so-called "TAPS" (in French: *Temps d'Activité Péri-Scolaire*) at primary schools. With my wife Marcia Pig Lagos, storyteller, we went every week to the primary school of our village Isle-Saint-Georges and try to introduce children to science with astronomy, with some evenings outside under the stars (with the parents). Children loved that.

In these activities, I aim to attract students to mathematics with the beauty of bijective combinatorics. With Marcia Pig Lagos and Gérard Duchamp (mathematician, computer scientist, and violinist) we have created the association "Cont'science"[92] in which its aim is the popularization of science, combining science (especially combinatorics), tales, and music. The motto of our association is *"Science avec Cont'science pour l'élévation de l'âme"*, which is a paraphrase of the famous Rabelais *"Science sans conscience n'est que ruine de l'âme"*. This reminds the thesis defended by Christian Krattenthaler *"Both mathematics AND music are food for the soul AND the brain"*, in a talk-performance [93] he has given in 2013 in Vienna.

Mansour: The term *bijective paradigm* is often associated to your work. How would you describe it?

Viennot: In a first approach, this paradigm is an attitude for many parts of mathematics, proving a fact, an identity, or a theorem with the construction of bijections.

The most simple case is starting from an identity like $A = B$ where both sides are some formal power series in one variable. A bijective proof of the identity is in two steps. First, give a combinatorial interpretation of both sides, that is invent some combinatorial objects whose generating functions are A and B. This can be very difficult and strong intuition can be needed. Then the second step will be to find a bijection (one-to-one correspondence) between the two classes of objects, which will prove the identity.

At another level, A and B can be formal power series with many variables, functions coming from analysis, or algebra. The extra variables will be considered as parameters for the underlying combinatorial objects interpreting A and B and the identity will be a consequence of a weighted preserving bijection between both sides.

More generally, the analytic expressions for A and B can be decomposed into some smaller pieces, each piece having its own combinatorial interpretation. The combination of these pieces is made with some analytic operation (sum, product, integration, derivation, etc) which will be interpreted by some combinatorial constructions at the level of the combinatorial objects. It is the so-called *symbolic calculus* dear to Philipe Flajolet[81] or in another point of view the *species* introduced by André Joyal and dear to the Québec school at LaCIM. You need to define operations at the level of the combinatorial objects, which are a mirror of the operations at the analytic level. All this philosophy is now very classical and is nothing but bijective combinatorics.

This attitude can be summarized by a beautiful sentence suggested to me by Karine Chemla, CNRS historian of ancient Chinese mathematics: (in French) *"Dessiner des calculs, calculer avec des dessins"*. ("Drawing calculus, making calculus from some drawings"). Of course, by drawings, it is suggested (visual) combinatorial objects. There is a return of figures in mathematics[85]. Pierre Laplace was very proud of his book on mechanics. In the introduction, he claimed that you will not find a single figure. The bijective paradigm is also the return of figures, replacing calculus, but taking "figures" at another level, not the same as the figures illustrating for example euclidian geometry. Finally, going back to [a], the bijective paradigm is at the basis of combinatorics viewed as an *attitude* transverse to mathematics, theoretical physics, and other sciences.

Mansour:[i] One of the interdisciplinary research projects you led was project MARS (mathematics, computer science, physics) "Combinatorial Physics: Around alternating sign matrices and the Razumov-Stroganov conjecture on spins chains model". Would you tell us the project? What were the main outcomes of it?

Viennot: I am a researcher at LaBRI, a computer science department. When I started as a CNRS researcher, combinatorics was considered badly, especially in France with the influence of Bourbaki. Thus, it was better to go under the umbrella of theoretical computer science. In the old-time, we did not need to submit individual research projects, as in North America. Then it was time to group different teams under a common research theme, in particular, international and European (many thanks to Christian Krattenthaler) projects which means a huge amount of energy to submit and handle such projects. I am not good

[92] See http://www.contscience.org/.
[93] See http://www.mat.univie.ac.at/~kratt/artikel/musimate.pdf.

at doing such works. But a director of ANR (analog of NSF in France) came in LaBRI and said that projects grouping very few people to solve a main conjecture would be encouraged. Thus I decided to submit something around the Razumov-Stroganov (RS) conjecture coming from Physics with spins chains model. RS conjecture was discovered in 2001 with experimental combinatorics and use of OEIS[94] (the On-line Encylopaedia of Integers Sequences). It describes a kind of Catalan refinement of the enumeration of ASM with chords diagrams. I called this project MARS.

Why MARS ? First because "RS" for Razumov and Stroganov, and "MA" because the conjecture is related to ASM (alternating sign matrices), see [d]. Second, we submit the project in March (Mars in French) (just half an hour before the electronic deadline!). And third, there is the expression in French "promettre la Lune" (= promise the moon), promising something which seems impossible to do or reach. What about "promettre Mars"! It fits much better in the context of ASM, TSSCPP, DPP, and other fascinating combinatorial objects. Finally the project[95] was accepted (2007-2010). We were four researchers from the combinatorial group in LaBRI working on ASM and related topics (Aval, Duchon, Le Borgne, Viennot, together with Lascoux from the algebraic combinatorial team in University Paris-Est-Marne-La-Vallée (now called Université Gustave Eiffel). We did not land on Mars and did not prove the RS conjecture. In fact, each member of our team did not work hard on this conjecture. Since the discovery of RS, physicists like Di Francesco, Zinn-Justin or Zuber, in their attempt to prove the conjecture were turning around, producing many deep papers, with each time many new and exciting conjectures. In our project, each of us did in the same way, continuing the exploration of the amazing world around RS and ASM. A list of papers and conferences, including subjects such as Six vertex model, Schubert, Jack, Macdonald polynomials, symmetries in ASM, alternative approach to ASM, and alternative tableaux for the PASEP, is on the website [95].

The RS conjecture was proved just after the end of the MARS project by physicists Sportiello and Cantini[25] in 2010 with bijective techniques. We can proudly say that our team gave a contribution to this success: before 2010, Sprotiello came to LaBRI and gave a talk in the warm combinatorial ambiance of our weekly "GT" (seminar). To my knowledge this is the first time that a problem coming from physics is solved with bijective combinatorics and that no proof has been founded using the big machinery of theoretical physics such as Yang-Baxter equation, transfer matrices, etc.

Mansour: One of your awards was for your work in combinatorics and control theory. How is combinatorics related to control theory? What are the most important results in this direction? How did combinatorics help solve problems in control theory?

Viennot: The relation between combinatorics and control theory is based on two methodologies: the combinatorial theory of differential equations developed with my beloved friend Pierre Leroux, and the approach of control theory with non-commutative formal power series developed by Michel Fliess and his school.

This second approach fits very well in the spirit of Schützenberger's school in theoretical computer science, replacing formal languages by non-commutative power series, automaton by the representation of free monoids by matrices, rational languages (languages accepted by finite automaton) by the so-called rational power series in non-commutative variables. In control theory you have to solve differential equations with some "entries" which are some functions. For example the movements of a boat in the ocean are some functions given by a system of differential equations, the "entries" are some functions that are a modelization of the waves and you want to know how the boat reacts. You have to solve this system as some functions of the entries that are themselves functions.

Forget the boat and let's take the following simple basic example with the non-linear differential equation $y' = y^2 + u(t)$. This example is given as a toy example in a paper of Fliess, Lamnabhi, and Lamnabhi-Lagarrigue[96]

[94] See `https://oeis.org/`.

[95] See `http://mars.xavierviennot.org/`.

[96] M. Fliess, M. Lamnabhi, and F. Lamnabhi-Lagarrigue, *An algebraic approach to nonlinear functional expansions*, IEEE Trans. Circuits Systems, 30, (1983), 554-570.

and express the behavior of a certain non-linear electrical circuit. We want to express $y(t)$ as a function of $u(t)$. The classical analysis gives the solution as an infinite sum where each term is a multiple integral (the Voltera kernels). This series can be rewritten as an infinite sum in terms of the so-called (Chern) iterated integrals. Solving the equation means to find the coefficient in front of each iterated integral. Now come Fliess methodology. Each iterated integral is in bijection with words in two letters. In the case of linear equations, you are in the world of Schützenberger rational non-commutative power series. With this methodology, the differential system of equations becomes a system of non-commutative power series, where the product is replaced by the shuffle product. This system can be solved on a computer by iteration.

If you replace the entry $u(t)$ by 1, you get the differential equation for the function $\tan t$. The expansion into Chern iterated integrals become a formal power series, which since Désiré André, we know is the generating function for alternating permutations on an odd number of elements.

Now comes the combinatorial theory developed with Pierre Leroux for (system of) differential equations, written in the language of species, "enriched" increasing trees, *"éclosions combinatoires"*, etc, dear to Gilbert Labelle. In the case of the simple differential equation $y' = y^2$, we get the well-known increasing binary trees, enumerated by $n!$. The crucial argument is that this interpretation *contains* the interpretation for the equation $y' = y^2 + u(t)$. The entry $u(t)$ is attached to each external vertex of the binary tree. If $u(t) = 1$ you get the "complete" increasing binary trees (in bijection with alternating permutations). This crucial argument can be developed for general systems of differential equations with some entries. Sorry to make this a little long, here is the relation between combinatorics and control theory.

Now you ask how combinatorics help solve problems in control theory? With the combinatorial approach the resolution by computer is much more efficient. In the case of the equation $y' = y^2 + u(t)$ you do not need to use the algorithm. One get an explicit formula for the coefficients. The iterated integrals with non-zero coefficients are in bijection with Dyck words (!), and the corresponding coefficient is the product of the height of each vertex of the Dyck word (!). I remember the reaction of Michel Fliess when I showed him that. He said "I am speechless".

But there is more than your simple question of how combinatorics help to solve problems in control theory. It can be used to invent some new notions in control theory. I cannot resist the pleasure of telling you a funny story.

We have seen in a particular case the connection between increasing binary trees and weighted Dyck paths[97,98]. This can be done in general. The resolution of equations in control theory can be express with weighted paths. By taking some bounds in the family of paths, you can define an approximant of the solution of the equations in control theory, analog to the convergent of the Jacobi continued fraction associated to weighted Motzkin paths, extended to a theory of Padé approximants in Emmanuel Roblet[99] thesis. In a join work with Pierre Leroux and Françoise Lamanabhi-Lagarrigue, we have developed this idea and define a kind of approximants in control theory, analog to classical Padé approximants in analysis. We were supposed to present these new approximants in a meeting in control theory and Lamanabhi-Lagarrigue was supposed to give the talk. Just before the meeting, she discovered that these approximants were already known and asked what we should do. No problems we told her with Pierre. Replace the title "A new approximants for ..." by "A combinatorial interpretation of the approximants defined by", and change a few words inside the paper for the Proceedings.

Your third subquestion in your question is "What are the most important results in this direction?". During my (permanent) combinatorial life I just did an incursion in control theory for a few years 30 years ago. I am not at all qualified to give you an answer, even if the

[97]X. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, Lecture Notes, UQAM (1983), 219 (pages II-14,15).

[98]See `http://www.viennot.org/abjc4-ch2.html`.

[99]E. Roblet, *Une interprétation combinatoire des approximants de Padé*, Publications du LACIM 17, UQAM, Montréal, (1994), 213.

award in your question was the *"CNRS silver medal"* in 1994.

Mansour:[j] In combinatorics, the Viennot's geometric construction is named after you. What is it about? Would you mention some of its applications?

Viennot: The Robinson-Schensted correspondence is a bijection between permutations and pairs (P, Q) of Young tableaux of the same shape. I presented a geometric version of the correspondence[100] at a meeting at Strasbourg organized by Dominique Foata in 1976. This was my first paper in combinatorics. The idea is to represent the permutation as a set of points on a square lattice with one (and only one) point in each row and column and to put some "light" from one of the corners of the figure. This creates successive "shadows" of the set of points, a set of non-intersecting "zig-zag lines" appear with a new set of points (the red points) having the same initial property (at most one point in each row and column) and giving the first row of P and Q. By repeating the "light and shadow" construction you get a set of blue points and the second row of P and Q, etc .. until getting an empty set of colored points and the pair (P, Q).

This construction "explains" the nice symmetry property: exchanging P and Q correspond to take the inverse of the permutation.

Maybe it is of interest for the reader to learn about the genesis of this geometric construction. In 1974 I finished my "Thèse d'Etat" about factorizations of free monoids in relation with free Lie algebra. I learn with great pleasure the RS correspondence and try to look at it through my combinatorial algebra glasses, i.e. free monoids and Lie algebra. Young tableaux are encoded by Yamanouchi words and these words are one of the basic family (i.e. code in the theory of variable length code) of a factorization of free monoids with only two factors $X^* = A^*B^*$ (called bisection).

I played with the insertion process of RS taking Yamanouchi words instead of Young tableaux. Displaying these words on the grid, playing with examples, one day, suddenly the zig-zag lines of different colors appear to me. In 1976 I presented this geometric version of RS at a meeting in Strasbourg organized by Dominique Foata. Gilbert de Beauregard Robinson himself was there! Curtis Green presented his famous interpretation of the shape of P and Q. They were surprised. My Master Schützenberger presented for the first time the *"jeu de taquin"* version of RS. For me, I was going from the world of combinatorial algebra to the world of algebraic combinatorics.

Many years after, I learned from some comments in mathoverflow, that this construction was equivalent, written in another language, to a construction described in 1970 Knuth's paper extending RS to matrices, (now called RSK correspondence). Morality: it happens many times in mathematics that the name of a new notion is not given to the original inventor but to somebody who will present it in a different equivalent form, maybe more beautiful, and will popularize that notion. Another example is the so-called LGV Lemma[101,102], relating determinants and configurations of non-intersecting paths. Linström discovered it and write it in terms of matroids. Gessel and myself independently rediscovered it and joined our efforts by introducing bijections between non-crossing paths and Young tableaux, interpretation of Hankel determinants combining LGV with the interpretation of moment of orthogonal polynomials, and many other determinants, which made this Lemma well known and very popular.

The second part of your question is about applications. In a strict sense, I do not really see other "applications" that the one explaining the symmetry (P, Q) and (Q, P). But in a broad sense, that is putting some "light" in some planar combinatorial object, analogous constructions can be done and lead to new results.

Let's mention the nice construction of Stefan Felsner given at FPSAC'00. Edelman and Green[103] gave a bijective proof of Stanley's conjecture about the number of maximal chains in the weak Bruhat order of S_n. Fel-

[100] X. Viennot, *Une forme géométrique de la correspondance de Robinson–Schensted*, in: D. Foata (Ed.), Combinatoire et Représenatation du Groupe Symétrique, in: Lecture Notes in Math., vol. 579, Springer, (1977), 29–58.

[101] I. Gessel and X. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. 58 (1985), 300–321.

[102] B. Lindström, *On the vector representations of induced matroids*, Bull. London Math. Soc. 5 (1973), 85–90.

[103] P. Edelman and C. Green, *Balanced tableaux*, Adv. Math. 63 (1987), 42–99.

[104] S. Felsner, *The Skeleton of a Reduced Word and a Correspondence of Edelman and Greene*, Electron. J. Combin. 8:1 (2001), Article #R10.

sner[104] gave an extension and said "As it is the case with the classical correspondence the planarized proofs have their own beauty and simplicity".

Another example is given by Burstein and Lankham[105] with the Patience Sorting algorithm, having some similarities with the Robinson-Schensted correspondence. The authors use this similarity and gave a nice correspondence between the piles formed under Patience Sorting and the shadow lines of the geometric construction of RS. This was presented at FPSAC'05.

The construction with light and zig-zag lines also appears in some works related to statistical mechanics.

In fact, here I should talk about the notion of *"essence"* of bijections, that is something, an idea, common to different bijections, even if there are operating on different combinatorial objects, enumerated by different numbers. An idea of this concept is given in the talk I gave for the 60th birthday of Christian Krattenthaler, SLC81 (Séminaire Lotharigien de Combinatoire) at Strobl. Video, slides, and some extraits of Christian concert are available at [83]. Thus it is the *"essence"* of the geometric RS construction which gives nice "applications".

Mansour: One of your areas of interest is statistical physics. We have seen great progress in many statistical mechanical models in recent years. Would you list a couple of open questions of great importance from statistical physics which you think combinatorics would play an important role in their solutions?

Viennot: I am not an expert in statistical physics and I am embarrassed to give you this list of open questions you are asking. I feel like a bee going from flowers to flowers in the garden of statistical mechanics models, taking some pollen here and there, and going back to the combinatorial hive making honey. Then the bee will come back in the garden for bijective and fruitful pollination. How to choose which flower is more important than another? I am rather interested in the beauty.

In this spirit, I would list the hard square gas model. Is there an explicit expression for the density of the model, in the same way as the hard hexagons model? Is it algebraic? A solution may come from a bijective explanation of the algebricity of the hard hexagons model, see [e]. Amazingly, the model on the hexagonal lattice can be solved[106], but is completely open on the square lattice.

Another example is the Ising model. This is far from being open and hundred of solutions giving the Onsager partition function (1944) has been given. Well-known combinatorial solutions have been given by Kastelyn[107] and Fisher-Temperley[108] in the '60s, giving bijection between the configurations of loops in a rectangle of the square lattice (interpreting the partition function of the model) and perfect matchings on a planar graph whose solutions is given by a Pfaffian and thus a determinant.

There is a connection between the Ising model on a planar graph and non-backtracking walks introduced by Kac and Ward[109] in 1952 and make rigorous by Sherman[110] in 1960. Tyler Helmuth[111] presented an illuminating alternating proof using the theory of heaps of pieces. The paper gives a new self-contained and simple proof that the partition function of the Ising model on a finite graph can be expressed in terms of weighted non-backtracking walks. The core of this work is the use of heaps of pieces (the logarithmic lemma) in the standard theory of the so-called Mayer expansion.

Such connections between heaps theory and Ising model, or the use of heaps as an efficient source of formulas for Mayer coefficients should be used to solve some open problems in this domain.

Surprise, surprise, in this story Ising, Feynman and Sherman came Schützenberger, who made to Sherman the remark that one of his

[105]A. Burstein and I. Lankham, *A geometric form for the extended patience sorting algorithm*, Adv. in Appl. Math. 36:2 (2006), 106–117.

[106]R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London, 1989.

[107]P. W. Kastelyn, *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice*, Physica 27:12 (1961), 1209–1225.

[108]H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics-an exact result*, The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics 6:68 (1961), 1061–1063.

[109]M. Kac and J. Ward, *A combinatorial solution of the two-dimensional Ising model*, Phys. Rev. 88 (1952), 1332–1337.

[110]S. Sherman, *Combinatorial aspects of the Ising model for ferromagnetism. I. A conjecture of Feynman on paths and graphs*, J. Math. Phys. 1(3), 202 (1960), Article 202.

[111]T. Helmuth, *Ising model observables and non-backtracking walks*, J. Math. Phys. 55 (2014), Article 083304.

identity in his paper is the same as an identity of Witt related to the dimension of some elements in free Lie algebra. They published a paper together[112]. We are back to the beginning of this interview with free groups and free Lie algebra and so.

Mansour: The title of one of your 2015 talks was *The birth of new domain: Combinatorial Physics*. Would you tell us about this new field?

Viennot: In fact the relation between combinatorics and physics is not new. I just related in the previous question to the work of Kastelyn and Fisher-Temperley in the '60s giving a combinatorial resolution of the Ising model in two-dimension with no external field.

The purpose of the talk[113] was to present with some examples this new field putting in strong relation combinatorics and physics (statistical, quantum mechanics, quantum gravity, etc). What is new is the first journal devoted to this field, created by its founding editors Gérard Duchamp, Vincent Rivasseau, Alan Sokal and Adrian Tanasă, in the prestigious series of the Annals of the Poincaré Institute.

Physicists like Philippe Di Francesco calls the domain *"combinatoire intégrable"* "= solvable combinatorics" about systems in physics where you can find an explicit expression for the partition function (or other similar function "solving" the model).

In this review, there are some examples of this relation between combinatorics and physics, PASEP[d], Razumov-Stroganov conjecture[i], heaps of pieces[k], and directed animals.

Mansour:[k] *Theory of heaps of pieces* is a theme that appears in many of your papers. What is this theory about? What are connections with Young tableaux, Cartier-Foata commutation monoids, statistical physics, $2D$ Lorentzian quantum gravity, and chromatic polynomials of graphs?

Viennot: Heaps of pieces[114] were introduced in 1985 as a "geometric" interpretation of the so-called Cartier-Foata[115] monoids introduced in 1969 as an "algebrization" of the work of D. Foata going back to his thesis in 1965 about combinatorial problems of *rearrangements* and *permutations with repetitions*, with applications to probability theory.

Cartier-Foata monoids (or free partially commutative monoids) have been used in computer science as a theoretical model for parallelism and concurrency. They have been introduced by Mazurkiewicz[116] in 1977 under the name trace monoids. A trace is just an equivalence of words, i.e. words up to commutation of some letters. The commutations are defined by a symmetric relation on the letters of the alphabet.

This equivalence class is interpreted by a heap of pieces. Thus heaps of pieces are equivalent to commutation monoids. The (big) advantage is that you get a powerful spatial representation which is missing in the linear vision of words up to the commutation of letters. Imagine some solid dimers falling on a chessboard, one by one, such that the projection of the dimers on the chessboard is two adjacent cells. When a dimer falls down it may fall on the floor (the chessboard) or on another dimer and will glue to that dimer. The heap is what you see at the end.

The theory is based on 3 basic lemmas: the inversion lemma N/D, the logarithmic lemma, and the lemma "paths = heaps". With these 3 lemmas a huge variety of classical and new results can be proved bijectively. Some classical theories get a new life as most of linear algebra (See Chapter 2b and 2c of ABjC, Part II).

I do not see connections with Young tableaux, except a connection about the Temperley-Lieb algebra. You give me the opportunity to say a few words about this beautiful topic. The dimension is the Catalan number. There are many Catalan bijections and relations between the following concepts: fully commutative elements in Coxeter groups, (321)-avoiding permutations, paral-

[112] M. P. Schützenberger and S. Sherman, *On a formal product over the conjugate classes in a free group*, J. Math. Anal. Appl. 7:3 (1963), 482–488.

[113] See `https://www.youtube.com/watch?v=tcGNLbRR4KU`.

[114] X. Viennot, *Heaps of pieces. I. Basic definitions and combinatorial lemmas*, In Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985), Lecture Notes in Math. 1234, 321–350. Springer, Berlin, 1986.

[115] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics 85, Springer-Verlag, Berlin, 1969.

[116] A. Mazurkiewicz, *Introduction to Trace Theory*, 3–41, in The Book of Traces, V. Diekert, G. Rozenberg, eds., 1995, World Scientific, Singapore.

lelogram polyominoes (= staircase polygons), heaps of dimers. In ABjC, A (new) bijection between parallelogram polyominoes and Kauffman generators of the Temperley-Lieb algebra appears naturally in the context of heaps of dimers. An exercise given to students is to prove that this bijection is nothing but the Robinson-Schensted correspondence in the case of (321)-avoiding permutations. See Chapter 6b, slide 58 of ABjC, Part II, `http://www.viennot.org/abjc2-ch6.html` where you will also find the link to the video.

Temperley-Lieb algebra[117] are related to the so-called totally commutative elements in a Coxeter group. A fundamental characterization is due to John Stembridge[118]. We take heaps[114] on a Dynkin diagram. Here it is better to take an equivalent definition of heaps. The binary relation defining the commutations is defined by a graph. A heap is a certain poset "above" the graph satisfying some axioms. Many papers have been written, such as the recent one quoted in [f]. see Chapter 6 of ABjC Part II.

Chromatic polynomials are replaced by chromatic power series and correspond to the intuitive idea of painting several times the vertices of a graph with successive layers (in each layer some vertices are colored with the usual condition that no two adjacent vertices are in the same layer). Thus a vertex can receive different color during the process. In particular one can deduce the very classical theorem of Stanley[119] relating acyclic orientations and chromatic polynomials at value -1. This proof with heaps has been done by Gessel[120] in terms of commutation monoids. see Chapter 5a of ABjC Part II `http://www.viennot.org/abjc2-ch5.html`

About Lorentzian quantum gravity: Quantum gravity is a very active field in theoretical physics. It is a tentative to unify two incompatible theories: general relativity and quantum mechanics. Various candidates are well known such as string theory or Alain Connes's approach with non-commutative geometry. Other approaches, such as loop quantum gravity, are based on a discrete vision, which I like much more, see [a]. Another approach is with a discrete triangulation of space-time. "Lorentzian" means you make a distinction between time and space, in opposite to "Euclidian".

One day my friend Deepak Dhar told me: Xavier you should look at a paper of Ambjorn and Loll[121] about "non-perturbative Lorentzian quantum gravity" published in 1998 in Nuclear Physics B in a journal I would never look at. In that paper, Catalan numbers appear (!) related to the enumeration of the so-called Lorentzian triangulations. In the paper of Ambjorn and Loll, the model can be "solved", but only in 2 dimensions (one for the time and one for the space). Combinatorial proofs were given by the physicists Di Francesco, Guitter, and Kristjansen[122] in 2000 using bijection with heaps of dimers on a line. Thus I became really excited about the connection between heaps and 2D quantum gravity, see Chapter 7b of ABjC `http://www.viennot.org/abjc2-ch7.html`

Mansour:[l] Enumeration of polyominoes is an interesting topic in enumerative combinatorics. But we still do not know how to enumerate them for the general case? Why is this problem very difficult? Which results do you consider the best so far? Are you still interested in polyomino enumeration?

Viennot: Underlying your question about the difficulty of the problem of enumerating polyominoes is the question: what means to solve the enumeration problem? In general, we think about finding a formula for the number of polyominoes according to some parameters (number of squares, perimeter, length, height, ...). But what means an "enumerative formula?" I will not discuss this interesting debate but I will go back to the most central problem: enumerate polyominoes on a square lattice according to the area (number of squares).

[117] H. N. V. Temperley and E. H. Lieb, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A, 322 (1549), 251–280, 1971.

[118] J. R. Stembridge, *The enumeration of fully commutative elements of Coxeter groups*, J. Algebraic Combin. 7:3 (1998), 291–320.

[119] R. P. Stanley, *Acyclic orientations of graphs*, Discrete Math. 5 (1973), 171–178.

[120] I. M. Gessel, *Acyclic orientations and chromatic generating functions*, Discrete Math. 232:1-3 (2001), 119–130.

[121] J. Ambjorn and R. Loll, *non-perturbative Lorentzian quantum gravity*, Nuclear Physics B 536:1-2 (1998), 407–434.

[122] P. Di Francesco, E. Guitter, and C. Kristjansen, *Generalized Lorentzian gravity in (1+1)D and the Calogero Hamiltonian*, Nucl. Phys. B608 (2001), 485–526.

The problem seems to be very difficult because maybe a "formula" does not exist. In that case the problem in your question is solved. I do not mean that the difficulty to solve a combinatorial problem increases with the complexity of the formula. See the "extreme" Example [m].

As for the question, which result I consider the best is very subjective. For me the "best" means a result solving a difficult problem with beautiful proof using new tools which can be interesting elsewhere. Thus I have chosen, this is a purely personal choice which I hope is objective, even if one of the two authors is one of my former Ph.D. students and the other a good friend which I met several times in meetings all over the world and also spend some time in his Melbourne department and apartment.

The story starts in 1984 with a paper by Delest and Viennot[123]. where we give a formula for the number of convex polyominoes with given perimeter $p_{2n+8} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}$. The proof is in the spirit of what Schützenberger called the DSV methodology.

This methodology is the following. In the case where the generating function is algebraic, the algebricity is explained by coding the combinatorial objects by some words of (non-ambiguous) algebraic languages, a notion coming from theoretical computer science, language which are words accepted by a pushdown automaton or an algebraic grammar from which you get a system of equations in non-commutative variables. In case the grammar is not "ambiguous", sending everything to commutative variables, you get the proof of the algebricity (with the system of equations) for the ordinary generating function. This DSV methodology has been used by Cori and Vauquelin[46] for the bijective proof of the equations enumerating planar maps, see [e]. I asked Schützenberger the meaning of DSV. He told me that "D" was for "Dyck" (language) the basic example of algebraic language, "S" was for Schützenberger, and he let me guess the meaning of "V". He wanted to introduce a name on a methodology, analog to some principles in physics. But the name DSV did not survive after that. Even after Mireille Bousquet-Mélou thesis, when Marco asked (he was one of the referees) Mireille to retype her whole manuscript with the name DSV.

Going back to the enumerative problem for convex polyominoes we can cut a convex polyomino in 3 parts: a parallelogram polyominoes (equivalently called staircase polygon by physicists). The middle part is enumerated by Catalan numbers and the two other parts are enumerated by Fibonacci numbers, each part has an algebraic or rational generating function. The problem will be solved if you invent an "algebraic glue" to encode the whole polyomino. It is not so easy, and we introduce a new bijection between parallelogram polyominoes and Dyck paths[123], and after long calculus by computer we get the surprising formula for the number of such polyominoes according to the perimeter. This example of the resolution of an enumerative problem using some bijections gives me a nice argument when colleagues in physics ask for the interest of bijective proofs. When we start with Maylis working on the problem, we did not know the existence of an exact formula. But following our knowledge of Catalan bijections, we construct some new bijections and arrive at a system of equations, via the coding with algebraic languages, and finally find and prove the formula. We did not know at the same time, Guttmann and Enting were doing some experiments, and from the first values of the sequence, applying techniques of approximants, guess (and conjecture) the formula[124]. Because of DSV methodology, our paper was published in a journal in theoretical computer science and could not be seen by our colleagues in statistical mechanics. This was the starting point of nice and fruitful cooperation between our combinatorial group in LaBRI and the department of statistical mechanics at Melbourne University.

It may qualify to the prize for the "best" result at that time: a nice amazing formula, difficult to prove, and with a new bijection, from parallelogram polyominoes to binary trees and Dyck paths[39], which has played a key role much later in defining an extension of the Tamari lattice to ν-Tamari lattice (work with Préville-Ratelle). By modesty, I cannot give

[123] M.-P. Delest and X. Viennot, *Algebraic languages and polyominoes enumeration*, Theoretical Computer Science, 34 (1984) 169-206.

[124] A. J. Guttmann and I. G. Enting, *The number of convex polygons on the square and honeycomb lattices*, J. Phys. A: Math. Gen. 21:8 (1988), L467–L474.

the prize to this formula. At that time, for a bijective proof of the formula for the number of convex polyominoes with given perimeter, I gave a prize of ten bottles of wine I was making in my village Isle-Saint-Georges South of Bordeaux ("Domaine des Mattes" 1981). The name "les Mattes" (!) is a pure coincidence with my domain of research. It comes from the local name place where is my vineyard, close to the Garonne river, which means it is at the lower level in the classification of Bordeaux wines. (= appellation *"Bordeaux supérieure"*). Offering bottles of wine for a problem is better than offering money. With time, the wine becomes older and quality improves (*"il se bonifie"*), but after a certain number of years, it becomes too old, and the quality decrease. Time for solving the problem has a limit.

The first solution to my problem came and the authors asked for their ten bottles of wine. It was a short paper with a series of identities and calculus. Not a single bijection explained! The authors claim that each of these identities or line of calculus had a bijective proof. The above paper gave a "bijective" proof, but is it really what we expect of "nice" bijective proof which explains the formula?

From this experience, I learn that is dangerous to give some price for a bijective proof of an identity. I should add "nice" bijective proof and a jury would meet and decide if the bijection is really "nice". In the ambiguity of the question I gave only 5 bottles of wine and 5 other bottles were still remaining for a really "nice" bijective proof.

Then came the paper[125] of M. Bousquet-Mélou and Tony Guttmann. The paper contains many enumerative formulas for 3-dimensional paths, and as a by-product give, maybe not the expected bijection, but an "explanation" of the rational part of the formula, of the algebraic part, and of the difference between the two. Thus, after a long deliberation the jury composed of Gérard Viennot and Xavier Viennot, decided to give the last other 5 bottles of wine. (In fact time to solved the problem was too long, the bottles became not good, and my colleagues received some good quality official Bordeaux wine)

My answer is maybe already too long but let me take this opportunity to mention Mireille Bousquet-Mélou for a digression about prize, medals and the progress of combinatorics, especially in France. In this interview, you mention the CNRS silver medal for myslef in "Combinatorics and Control Theory". The CNRS is divided in institutes, there is one for Physics, one for Computer Science, one for Mathematics. As explained above combinatorists get a decent life as being under the umbrella of "mathematics for computer science" also called "theoretical computer science". I considered the medal I received a medal for the recognization of combinatorics in Computer Science. Mireille, CNRS research director at LaBRI combinatorial group, also received the silver medal. My surprise was, not that she received this prestigious medal, but the fact that the medal was received from the "Institut National des Sciences Mathématiques et de leurs Interactions" of CNRS. Of course, this medal is an official reconnaissance of the fact that she is an international star in the different facets of combinatorics (enumerative, algebraic, asymptotic, and physics, etc). From my point of view, these two medals in the same combinatorial group at LaBRI, one in computer science and one in mathematics is the recognition of the efforts of many colleagues to elevate the field of combinatorics to one of the major domain at the frontier in computer science, maths (and physics). Soon after Mireille was admitted ro the French Academy of Science (in Mathematics). Philippe Flajolet also get the CNRS silver medal of CNRS in 2004 and Michel Fliess in 1991. Another recognition of Schützenberger's school!

This section would not be complete without mentioning that the famous "Collège de France" open a *"chaire de Combinatoire"*. Timoty Gowers, fields medalist, was invited to begin this *"chaire"*. This is another side of combinatorics called "Hungarian combinatorics" after Paul Erdös, Béla Bollobás, László Lovász, etc.

The last part of your question is about my present interest in polyominoes enumeration. Polyominoes are equivalent to animals, squares are replaced by points. The last time I did something on that topic was to give a bijective

[125] M. Bousquet-Mélou and A. J. Guttmann, *Enumeration of three-dimensional convex polygons*, Ann. Combin. 1 (1997), 27–53.

[126] X. Viennot, *Multi-directed animals, connected heaps of dimers and Lorentzian triangulations*, J. Phys. Conf. Ser. 42 (2006), 268–280.

proof[126] of an amazing formula (the generating function is not D-finite) of Rechnitzer and Bousquet-Mélou about the so-called multidirected animals, which are equivalent to enumerate connected heaps of dimers on a line, equivalent to enumerate (general) Lorentzian triangulations. This was presented at the conference "Counting complexity" in honor of Tony Guttmann for his 60th birthday, Dunk Island, Australia.

Mansour:[m] Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have an "eureka moment"?

Viennot: I will choose the enumeration of compact source directed animals on a square lattice. At that time the problem was open and not easy at all. It comes from statistical physics in relation to percolation model, branch polymer, etc. A directed animal is a set of points on the square lattice such that each point can be reached from the origin (the source point) by a path included in the animal with an elementary step North or East.

The starting point are the papers of Dhar, Phani, and Barma[127] and Nadal, Derrida, and Vannimenus[128] in 1982. In the first paper, a conjectured enumeration formula was given, in the second an extension conjecture formula for the number of such animal bounded on the circular strip. There are also the so-called critical exponent for the average with 1/2 and a conjecture for the length with 9/11. At that time I knew nothing about statistical physics. The three physicists were at the Physics laboratory of ENS Ulm and asked for a solution of their conjecture to various mathematicians of this same ENS, in particular Adrien Douady.

The problem arrived in Bordeaux and with Dominique Gouyou-Beauchamps we start immediately to work on these combinatorial conjectures. The formula for directed animals is simple and is algebraic. The problem was also related to the width and length of the animal. After some manipulation and experiment, Dominique extended the class of directed animals to a larger class: compact source directed animals (animals with several source points, but consecutive on a line perpendicular to the diagonal). I do not remember how he got this awesome idea. Enumerating by experiment, we get another amazing formula. No need to look at the Sloane book encyclopedia of sequences, the number of such animals is 3^n (!!). We work hard to find a bijection between this class of animals and words in 3 letters A, B, C. Equivalently defining 3 operators acting on animals, adding a new point in the animal, which you can reverse the construction.

We start by defining some natural parameters on both objects: compact source directed animal and words in 3 letters viewed as a path with North-East, East, and South-East steps. For two parameters on both classes of objects, we found by a computer that the distribution was the same for the first value of n. For directed animals the first parameter was, of course, the number of source points and for the second parameter, we invented a new parameter we called it "lower width".

Experimental combinatorics lead to a search for 3 operators A, B, C with the following property. An operator A will add a source point, an operator C will reduce by one the number of source points, except if there is only one source point, in that case, the number of source point remains one. For the operator B, the number of source points should be invariant. By experiment, we know what should be the behavior of this second parameter with the operators A, B, C. Finally, we get easily a possible definition for operators A and C, but operator B strongly resists unveilling his mystery.

I stay several days, isolated in my village, completely concentrated on B. One day the "eureka moment" arrives and long after, the writing of the paper[2]. The length of the proof is inversely proportional to the simplicity of the formula 3^n. Finally, a two-dimensional problem is put in bijective correspondence with a one-dimensional path problem.

From this bijection, we get a proof that the critical exponent for the width is 1/2. To my knowledge, the critical exponent for the length is still open (but by experimentation seems not

[127] D. Dhar, M. K. Phani, and M. Barma, *Enumeration of directed site animals on two-dimensional lattices*, J. Phys. A 15 (1982), L279.

[128] J. P. Nadal, B. Derrida, and J. Vannimenus, *Directed lattice animals in 2 dimensions: numerical and exact results*, J. Phys. (Paris) 43 (1982), 1561.

to be 9/11). At the same time, the idea of heaps of pieces emerge in my mind, directed animals are in bijection with some pyramids of heaps of dimers, which gives simpler proof and bijection for the number of directed animals on a square lattice. From the basic lemma on heaps of pieces, it is easy to get a proof of the formula of Nadal, Derrida, and Vannimenus[128] for the number of directed animals on a circular strip, involving the zeros of the Tchebycheff polynomials (first kind).

Betrema and Penaud[129] gave a simpler proof of the formula 3^n with heaps of dimers on a line, which is deduced from a system of equations. Long after, Zeiberger[130] transformed this system into a system involving equations with only positive coefficients, from which a "bijectivisation" is possible, leaving the task to the reader. I guess that this "bijectivisation" would give the same bijection as the one we discovered with Gouyou-Beauchamps.

More anectodes: [about experimental researches 40 years ago] Jean Vannimenus told me that they made a computer experiment to compute the biggest eigenvalue of the transition matrix related to directed animals on a bounded circular strip of given diameter. Some real numbers appear and were close to 3 when the diameter increase. Next, another researcher was doing some experiments about zeros of chromatic polynomials of a sequence of graphs, which limit was 4. (the 4 colors theorem is not true in the "infinite limit"). For joking, they compare their sequence of real numbers, they were the same up to an addition by 1! From that, the physicist guessed the other eigenvalues and conjectured the formula which is proved with dimers.

A second anectode [about the position of combinatorics among the French mathematicians]. To my surprise, after this resolution of the directed animal problem with the combinatorial tool, I was invited to give a talk in 1984 on the subject at the famous "Séminaire Bourbaki" at the Institut Poincarré in the center of Paris. After the talk, I heard a famous mathematician asking Adrian Douady: Is the speaker a mathematician or a physicist? Answer: neither, he is a computer scientist! This talk[131] has been published in the proceeding in the category "statistical physics".

Mansour: Professor Xavier Viennot, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

[129] J. Betrema and J.-G. Penaud, *Modèles avec particules dures, animaux dirigés et séries en variables partiellement commutatives*, arXiv:math/0106210 [math CO], 2001 (LaBRI Report, May 1993).

[130] See `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bordelaise.html`.

[131] G. Viennot, *Problèmes combinatoires posés par la physique statistique*, Astérisque, SMF, tome 121–122 (1985), Séminaire Bourbaki, exposé 626, 225–246. Available at `http://www.numdam.org/article/SB_1983-1984__26__225_0.pdf`.