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Hausdorff Moment Problem for Combinatorial Numbers of Brown and Tutte: Exact Solution

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Abstract: We investigate the combinatorial sequences $A(M,n)$ introduced by W. G. Brown (1964) and W. T. Tutte (1980) appearing in the enumeration of convex polyhedra. Their formula is

$$A(M,n) = \frac{2(2M+3)!}{(M+2)!M!}\,\frac{(4n+2M+1)!}{n!(3n+2M+3)!}$$

with $n, M = 0, 1, 2, \ldots$, and we conceive it as Hausdorff moments, where M is a parameter and n enumerates the moments. We solve exactly the corresponding Hausdorff moment problem: $A(M,n) = \int_0^R x^n W_M(x)\,\mathrm{d}x$ on the natural support $(0,R)$, $R = 4^4/3^3$, using the method of inverse Mellin transform. We provide explicitly the weight functions $W_M(x)$ in terms of the Meijer G-functions $G^{4,0}_{4,4}$, or equivalently, the generalized hypergeometric functions ${}_3F_2$ (for $M = 0, 1$) and ${}_4F_3$ (for $M \geq 2$). For $M = 0, 1$, we prove that $W_M(x)$ are non-negative and normalizable, thus they are probability distributions. For $M \geq 2$, $W_M(x)$ are signed functions vanishing on the extremities of the support. By encoding this problem entirely in terms of Meijer G-representations we reveal an integral relation that directly furnishes $W_M(x)$ based on the ordinary generating function of $A(M,n)$ as an input. All the results are studied analytically as well as graphically.

Keywords: Combinatorial numbers; Hausdorff moment problem; Meijer G-functions; Probability distributions
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1. Introduction

Combinatorial numbers, which are by necessity positive integers, turn out very often to be related to probability, as they can be identified as power moments of positive and normalizable functions, i.e. the probability distributions. In most known cases the support of these distributions are either sets of positive integers, the positive half-axis, or finite segments of the positive axis in the form $(0,R)$. For instance, many combinatorial numbers characterizing set partitions [11, 12] turn out to be moments of positive functions. Certain sequences of numbers contain parameters that permit us to relate them to probability distributions only for limited values of parameters, see for instance [15]. (As we shall see later, the sequence of (1) also belongs to this category.) In this work we concentrate on a sequence of combinatorial numbers appearing in the counting of bisections of convex polyhedra [5, 22], which reads:

$$A(M,n) = \frac{2(2M+3)!}{(M+2)!M!}\,\frac{(4n+2M+1)!}{n!(3n+2M+3)!}, \tag{1}$$

where $M, n = 0, 1, \ldots$. $A(M, n)$ are integers for all M and n. We enumerate below the initial values $n = 0, \ldots, 5$ of $A(M, n)$ for $0 \leq M \leq 4$:

$$
\begin{aligned}
A(0, n) &= 1, 1, 13, 68, 399, 2530, \ldots \\
A(1, n) &= 2, 5, 20, 100, 570, 3542, \ldots \\
A(2, n) &= 5, 21, 105, 595, 3675, 24150, \ldots \\
A(3, n) &= 14, 84, 504, 3192, 21252, 147420, \ldots \\
A(4, n) &= 42, 330, 2310, 16170, 115500, 844074, \ldots.
\end{aligned}
$$

The sequences $A(M, n)$ for $0 \leq M \leq 3$ are documented and discussed in N. J. A. Sloane's Online Encyclopedia of Integer Sequences (OEIS) [19]: $A(0, n) =$ A000260(n), $A(1, n) =$ A197271(n), $A(2, n) =$ A341853(n), and $A(3, n) =$ A341854(n). However, notice that

$$A(M, 0) = \mathrm{Cat}(M + 1), \tag{2}$$

where $\mathrm{Cat}(n) = \binom{2n}{n}\frac{1}{n+1}$ are Catalan numbers. We set out to solve the following Hausdorff power moment problem: find $W_M(x)$ satisfying the infinite set of equations:

$$A(M, n) = \int_0^R x^n \, W_M(x) \, \mathrm{d}x, \qquad n = 0, 1, \ldots, \tag{3}$$

where R is given by the known formula $R = \lim_{n\to\infty}[A(M, n)]^{1/n} = 4^4/3^3$, i.e. is independent on M. We shall employ the method of inverse Mellin transform, which implies for $n = s - 1$ that

$$W_M(x) = \mathcal{M}^{-1}[A(M, s - 1); x].$$

In Section 2 we shall enumerate and present the conventional tools applied in calculating the inverse Mellin transforms, including the Meijer G-functions, the generalized hypergeometric functions, and some of their properties. With the above tools at hand, in Section 3 we shall perform in detail the Mellin inversion and obtain the explicit closed-form solutions for $W_M(x)$ in terms of generalized hypergeometric functions ${}_3F_2$ and ${}_4F_3$. We prove the positivity of $W_M(x)$ for $M = 0, 1$ only, whereas for $M \geq 2$ we demonstrate that $W_M(x)$ are signed functions. $W_M(x)$ are discussed graphically for $0 \leq M \leq 4$. In Section 4 we derive the closed-form expression for the ordinary generating function (ogf) of $A(M, n)$, i.e. for $G(M, z) = \sum_{n=0}^{\infty} A(M, n)z^n$, and we establish a relationship between $W_M(x)$ and $G(M, z)$, by rephrasing both of them in the common language of Meijer G-functions. The aforementioned relation allows constructing explicitly the function $W_M(x)$ using *solely* the hypergeometric representation of $G(M, z)$. This procedure is strongly evocative of the inversion of the one-sided, finite Hilbert transform. Section 5 contains the discussion and final remarks.

2. Definitions and preliminaries

The main tool that we employ in the treatment of various moment problems is the Mellin transform $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$. In the following, we group several definitions and information about the Mellin transform of a function $f(x)$ defined for $x \geq 0$. The Mellin transform is defined for complex s as [20]

$$\mathcal{M}[f(x); s] = f^\star(s) = \int_0^\infty x^{s-1} f(x) dx,$$

along with its inverse

$$\mathcal{M}^{-1}[f^\star(s); x] = f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} f^\star(s) ds.$$

For the role of constant c consult [20].

If $\mathcal{M}[f(x); s] = f^\star(s)$ and $\mathcal{M}[g(x); s] = g^\star(s)$ then

$$\mathcal{M}^{-1}[f^\star(s) g^\star(s); x] = \int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{1}{t} \, \mathrm{d}t = \int_0^\infty g\left(\frac{x}{t}\right) f(t) \frac{1}{t} \, \mathrm{d}t. \tag{4}$$

The last two integrals are called Mellin (i.e. multiplicative) convolutions of $f(x)$ with $g(x)$. For fixed $a > 0$, $h \neq 0$, the Mellin transform satisfies the following scaling property:

$$\mathcal{M}[x^b f(a x^h); s] = \frac{1}{|h|} a^{-\frac{s+b}{h}} f^\star(\tfrac{s+b}{h}). \tag{5}$$

Among known Mellin transforms a very special role is played by those entirely expressible through products and ratios of Euler's gamma functions. The Meijer G-function is defined as an inverse Mellin transform [18]:

$$G^{m,n}_{p,q}\left(x\Big|\begin{matrix}\alpha_1\dots\alpha_p\\ \beta_1\dots\beta_q\end{matrix}\right) = \mathcal{M}^{-1}\left[\frac{\prod_{j=1}^{m}\Gamma(\beta_j+s)\,\prod_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-s)\,\prod_{j=n+1}^{p}\Gamma(\alpha_j+s)};x\right] \tag{6}$$

$$= \mathrm{MeijerG}([[\alpha_1,\dots,\alpha_n],[\alpha_{n+1},\dots,\alpha_p]],[[\beta_1,\dots,\beta_m],[\beta_{m+1},\dots,\beta_q]],x). \tag{7}$$

The notation for $G^{m,n}_{p,q}$ in (7) is motivated by Maple and Mathematica notation*. We will consequently use both notations throughout this paper. In (6) empty products are taken to be equal to 1. In (6) and (7) the parameters are subject of conditions:

$$z\neq 0,\quad 0\le m\le q,\quad 0\le n\le p,$$
$$\alpha_j\in\mathbb{C},\quad j=1,\dots,p;\quad \beta_j\in\mathbb{C},\quad j=1,\dots,q.$$

See [7,18] for a full description of integration contours in (6), general properties and special cases of the Meijer G-functions. The convergence of the Mellin inversion in (6) and (7) is conditioned upon specific requirements involving both chains of parameters (a_p) and (b_q). The aforementioned conditions will be quoted and checked in Section 3 on the example of the Mellin inversion derived from (1).

The generalized hypergeometric function ${}_pF_q$ is defined as:

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} = {}_pF_q([a_1,\dots,a_p],[b_1,\dots,b_q];z),$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$ is called the Pochhammer symbol, and neither of b_j, $j=1,\dots,q$, is a negative integer, see [21].

We shall also use the following relation linking one ${}_pF_q$ with one $G^{m,n}_{p,q}$, for $p\le q+1$:

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right) = \left(\prod_{k=1}^{q}\Gamma(b_k)/\prod_{k=1}^{p}\Gamma(a_k)\right)G^{1,p}_{p,q+1}\left(-z\Big|\begin{matrix}1-a_1,\dots,1-a_p\\ 0,1-b_1,\dots,1-b_q\end{matrix}\right), \tag{8}$$

see Equation (16.18.1) of [16], where particular attention should be paid to the position of 0 in the lower list of parameters. For the proof of (8) see Equation (12.3.18) on page 317 of [2]. Additional identities satisfied by $G^{m,n}_{p,q}$ and used in this work are

$$G^{m,n}_{p,q}\left(\frac{1}{z}\Big|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) = G^{n,m}_{q,p}\left(z\Big|\begin{matrix}1-b_1,\dots,1-b_q\\ 1-a_1,\dots,1-a_p\end{matrix}\right) \tag{9}$$

$$z^{\mu}\,G^{m,n}_{p,q}\left(z\Big|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(z\Big|\begin{matrix}a_1+\mu,\dots,a_p+\mu\\ b_1+\mu,\dots,b_q+\mu\end{matrix}\right), \tag{10}$$

see Equations (16.19.1) and (16.19.2) of [16], correspondingly.

For certain conditions satisfied by the parameter lists, the functions $G^{m,n}_{p,q}$ can be represented as a finite sum of hypergeometric function, see Equation (16.17.2) of [16] and/or Equation (8.2.2.3) of [18], which are sometimes referred to as Slater relations.

We quote for reference the Gauss-Legendre multiplication formula for the gamma function encountered in this work:

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}}\,n^{nz-\frac{1}{2}}\prod_{j=0}^{n-1}\Gamma\left(z+\frac{j}{n}\right),\qquad z\neq 0,-1,-2,\dots,\quad n=1,2,\dots. \tag{11}$$

We also introduce a short notation for a special list of k elements:

$$\Delta(k,a) = \frac{a}{k},\frac{a+1}{k},\dots,\frac{a+k-1}{k},\qquad k\neq 0. \tag{12}$$

3. Solving the moment problem

In this section, we shall derive the exact and explicit forms of the solutions $W_M(x)$ of the Hausdorff moment problem of (3) where $A(M,n)$ is given by (1). Denote

$$P(M) = \frac{2(2M+3)!}{(M+2)!M!},$$

*Notice that in Mathematica notation the Meijer G-function is represented in the form

$$\mathrm{MeijerG}[\{\{\alpha_1,\dots,\alpha_n\},\{\alpha_{n+1},\dots,\alpha_p\}\},\{\{\beta_1,\dots,\beta_m\},\{\beta_{m+1},\dots,\beta_q\}\},x].$$

set in (3) $n = s-1$, and use twice the Gauss-Legendre formula (11) in transforming (3) to obtain $A(M, s-1) \equiv \tilde{A}(M, s)$:

$$\tilde{A}(M,s) = r_W(M)R^s\,\frac{\Gamma(s-\frac{1}{2}+\frac{M}{2})\Gamma(s-\frac{1}{4}+\frac{M}{2})\Gamma(s+\frac{M}{2})\Gamma(s+\frac{1}{4}+\frac{M}{2})}{\Gamma(s+\frac{1}{3}+\frac{2M}{3})\Gamma(s+\frac{2}{3}+\frac{2M}{3})\Gamma(s+1+\frac{2M}{3})\Gamma(s)}, \tag{13}$$

where

$$r_W(M) = \frac{3^{\frac{1}{2}-2M}2^{4M+\frac{1}{2}}}{192\sqrt{\pi}}P(M).$$

We apply now the scaling property (5) along with the definitions of the Meijer G-function (6) and (7) in order to write the final form of $W_M(x) = \mathcal{M}^{-1}[\tilde{A}(M,s); x]$:

$$\begin{aligned} W_M(x) &= r_W(M)\, G^{4,0}_{4,4}\left(\frac{x}{R}\middle|\begin{matrix}0, \Delta(3, 2M+1)\\ \Delta(4, 2M-2)\end{matrix}\right) \qquad (14)\\ &= r_W(M)\,\mathrm{MeijerG}\Big([[\ ], [0, \tfrac{2M}{3}+\tfrac{1}{3}, \tfrac{2M}{3}+\tfrac{2}{3}, \tfrac{2M}{3}+1]], [[\tfrac{M}{2}-\tfrac{1}{2}, \tfrac{M}{2}-\tfrac{1}{4}, \tfrac{M}{2}, \tfrac{M}{2}+\tfrac{1}{4}], [\ ]], \frac{x}{R}\Big). \end{aligned}$$

The solutions in (14) are unique. According to the definitions of (6) and (7) the parameter lists $(\alpha_1, \ldots, \alpha_4)$ and $(\beta_1, \ldots, \beta_4)$ for $p = q = 4$ in (14) can be read off as $(\alpha_p) = (0, \Delta(3, 2M+1))$ and $(\beta_q) = (\Delta(4, 2M-2))$, using (12). We can now extract the conditions for convergence of integral (6) as a function of (α_p) and (β_q). They define the range of variable s for which the convergence is assured with the formula (2.24.2.1) of [18]. Here $m = 4$, $n = 0$, and $p = q = 4$, and the auxiliary parameter $c^\star \equiv m + n - (p+q)/2 = 0$. Thus, the range of real s is determined from the inequality:

$$\begin{aligned} -\min_{1\le j\le m}(\beta_j) &\le s \le 1 - \max_{1\le j\le n}(\alpha_j), \quad \text{which reads}\\ \frac{1}{4}-\frac{M}{2} &\le s \le 1-(-\infty), \qquad \text{or for} \quad s = n'+1\\ \frac{1}{4}-\frac{M}{2} &\le n'+1 \le \infty, \qquad \text{and finally}\\ -\frac{3}{4}-\frac{M}{2} &\le n' \le \infty, \end{aligned}$$

where n' enumerates the moments. We conclude that for $W_M(x)$ all the moments $\int_0^R x^{n'} W_M(x)\,\mathrm{d}x$, for $0 \le n' < \infty$ are legitimate and converging.

Before embarking on detailed evaluation of (14) we claim that for $M = 0, 1$ the weight function $W_M(x)$ will be a positive function on $x \in (0, R)$. This is based on the Mellin convolution property of (4) which shows that if two individual functions are positive, then for positive arguments, their Mellin convolution is also positive. The second element of this reasoning tells us that

$$\mathcal{M}^{-1}\left[\frac{\Gamma(s+a)}{\Gamma(s+b)}; x\right] = \frac{(1-x)^{1-a+b}x^a}{\Gamma(b-a)} > 0, \quad \text{for} \quad 0 < x < 1, \;\; b > a, \tag{15}$$

which is the direct consequence of Equation (8.4.2.3) on page 631 of [18]. Equation (15) is strongly reminiscent of the classical Euler Beta function. Moreover, the Beta distribution is the probability measure characterized by the density function $g_{\alpha,\beta}(x) = \{\Gamma(\alpha+\beta)/[\Gamma(\alpha)\Gamma(\beta)]\}\, x^{\alpha-1}(1-x)^{\beta-1}$ [1]. The r.h.s. of (15) for $0 < x < 1$ and $b > a$ is a positive function.

Suppose that we will be able to order the shifts in four gamma ratios in (13) in such a way that for every ratio $b > a$, as in (15). Then the resulting weight function will be a *threefold* Mellin convolution of positive functions, and, through the above argument, will itself be positive. Let us first enumerate the gamma shifts for $M = 0$ in (14), with u = upper and l = lower shifts. In the formulas (16), (17), and (18) below, the arrow " $\Longrightarrow$ " should be understood as: "can be reordered as".

$$M=0: \quad \left\{\begin{matrix} u: 0, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}\\ l: 0, 1, \frac{2}{3}, \frac{1}{3}\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} u: -\frac{1}{4}, -\frac{1}{2}, 0, \frac{1}{4}\\ l: 0, \frac{1}{3}, \frac{2}{3}, 1\end{matrix}\right\} \tag{16}$$

resulting in the gamma ratios:

$$\frac{\Gamma(s-\frac{1}{4})}{\Gamma(s+0)}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{1}{3})}\,\frac{\Gamma(s+0)}{\Gamma(s+\frac{2}{3})}\,\frac{\Gamma(s+\frac{1}{4})}{\Gamma(s+1)}.$$

Then the resulting $W_0(x)$ will be a positive function. We continue with the same argument for $M = 1$:

$$M=1: \quad \left\{\begin{matrix} u: \frac{1}{2}, 0, \frac{3}{4}, -\frac{1}{4}\\ l: 0, \frac{5}{3}, \frac{4}{3}, 1\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} u: -\frac{1}{4}, 0, \frac{1}{2}, \frac{3}{4}\\ l: 0, 1, \frac{4}{3}, \frac{5}{3}\end{matrix}\right\}, \tag{17}$$

resulting in the gamma ratios:

$$\frac{\Gamma(s-\frac{1}{4})}{\Gamma(s+0)}\,\frac{\Gamma(s+0)}{\Gamma(s+1)}\,\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{4}{3})}\,\frac{\Gamma(s+\frac{3}{4})}{\Gamma(s+\frac{5}{3})}.$$

Then again, the resulting $W_1(x)$ will be a positive function. The situation changes for $M=2$, as then

$$M=2: \quad \begin{Bmatrix} u: 1, \frac{1}{2}, \frac{5}{4}, \frac{3}{4} \\ l: 0, \frac{7}{3}, 2, \frac{5}{3} \end{Bmatrix} \Longrightarrow \begin{Bmatrix} u: \frac{1}{2}, 1, \frac{3}{4}, \frac{5}{4} \\ l: 0, \frac{5}{3}, 2, \frac{7}{3} \end{Bmatrix}, \tag{18}$$

and the resulting gamma ratio

$$\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+0)}\,\frac{\Gamma(s+1)}{\Gamma(s+\frac{5}{3})}\,\frac{\Gamma(s+\frac{3}{4})}{\Gamma(s+2)}\,\frac{\Gamma(s+\frac{5}{4})}{\Gamma(s+\frac{7}{3})}$$

excludes the positivity of $W_2(x)$ as here

$$\mathcal{M}^{-1}\left[\Gamma\Big(s+\frac{1}{2}\Big)/\Gamma(s); x\right] \sim -\sqrt{x}/(1-x)^{3/2} < 0$$

for $0<x<1$. Similar arguments exclude the positivity for $M>2$. The method of studying the positivity via multiple Mellin convolution was initiated in [17] and further applied in [4, 6, 7, 13, 14] to various sequences of combinatorial numbers.

Since $A(M,0)\neq 1$, see (2), it is reasonable not to compare $W_M(x)$ for different M, but rather to consider $\tilde{W}_M(x) = W_M(x)/A(M,0)$, "normalized" weight functions for different M. Note that zeroth moments of $\tilde{W}_M(x)$ are equal to 1, but higher moments of $\tilde{W}_M(x)$ are not any more integers but are rationals. In order to do so, we have chosen to represent the Meijer G-functions of (14) as a finite sum of three generalized hypergeometric functions ${}_3F_2$ (for $M=0,1$), and ${}_4F_3$ (for $M\geq 2$), employing Equation (8.2.2.3) of [18]. This last formula also permits writing down the general expression for $\tilde{W}_M(x) = W_M(x)/A(M,0)$ for arbitrary integer M with the help of generalized hypergeometric functions. However, due to its complexity, we shall not reproduce this last formula here. Instead we quote below the explicit forms for $\tilde{W}_M(x)$ for $0\leq M\leq 3$, with $R=4^4/3^3$:

$$\begin{aligned}
\tilde{W}_0(x) &= \frac{2}{\pi\sqrt{x}}\,{}_3F_2\left(\begin{matrix}-\frac{1}{2}, -\frac{1}{6}, \frac{1}{6} \\ \frac{1}{4}, \frac{3}{4}\end{matrix}; \frac{x}{R}\right) - \frac{\sqrt{2}}{\pi x^{1/4}}\,{}_3F_2\left(\begin{matrix}-\frac{1}{4}, \frac{1}{12}, \frac{5}{12} \\ \frac{1}{2}, \frac{5}{4}\end{matrix}; \frac{x}{R}\right) + \frac{\sqrt{2}\,x^{1/4}}{32\pi}\,{}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{7}{12}, \frac{11}{12} \\ \frac{3}{2}, \frac{7}{4}\end{matrix}; \frac{x}{R}\right) \\
&= \frac{2}{\pi\sqrt{x}}\,{}_3F_2\big([-\tfrac{1}{2}, -\tfrac{1}{6}, \tfrac{1}{6}], [\tfrac{1}{4}, \tfrac{3}{4}], \tfrac{x}{R}\big) - \frac{\sqrt{2}}{\pi x^{1/4}}\,{}_3F_2\big([-\tfrac{1}{4}, \tfrac{1}{12}, \tfrac{5}{12}], [\tfrac{1}{2}, \tfrac{5}{4}], \tfrac{x}{R}\big) \\
&+ \frac{\sqrt{2}\,x^{1/4}}{32\pi}\,{}_3F_2\big([\tfrac{1}{4}, \tfrac{7}{12}, \tfrac{11}{12}], [\tfrac{3}{2}, \tfrac{7}{4}], \tfrac{x}{R}\big).
\end{aligned}$$

In the following three equations we skip the Maple notation.

$$\tilde{W}_1(x) = \frac{2\sqrt{2}}{\pi}x^{1/4}\,{}_3F_2\left(\begin{matrix}-\frac{5}{12}, -\frac{1}{12}, \frac{1}{4} \\ \frac{1}{2}, \frac{3}{4}\end{matrix}; \frac{x}{R}\right) - \frac{5\sqrt{x}}{2\pi}\,{}_3F_2\left(\begin{matrix}-\frac{1}{6}, \frac{1}{6}, \frac{1}{2} \\ \frac{3}{4}, \frac{5}{4}\end{matrix}; \frac{x}{R}\right) + \frac{5\sqrt{2}}{16\pi}x^{3/4}\,{}_3F_2\left(\begin{matrix}\frac{1}{12}, \frac{5}{12}, \frac{3}{4} \\ \frac{5}{4}, \frac{3}{2}\end{matrix}; \frac{x}{R}\right),$$

$$\begin{aligned}
\tilde{W}_2(x) &= -\frac{14\sqrt{x}}{5\pi}\,{}_4F_3\left(\begin{matrix}-\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{3}{2} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\end{matrix}; \frac{x}{R}\right) + \frac{3\sqrt{2}}{\pi}x^{3/4}\,{}_4F_3\left(\begin{matrix}-\frac{7}{12}, -\frac{1}{4}, \frac{1}{12}, \frac{7}{4} \\ \frac{1}{2}, \frac{3}{4}, \frac{5}{4}\end{matrix}; \frac{x}{R}\right) \\
&- \frac{35\sqrt{2}}{32\pi}x^{5/4}\,{}_4F_3\left(\begin{matrix}-\frac{1}{12}, \frac{1}{4}, \frac{7}{12}, \frac{9}{4} \\ \frac{5}{4}, \frac{3}{2}, \frac{7}{4}\end{matrix}; \frac{x}{R}\right),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{W}_3(x) &= -\frac{4\sqrt{2}}{\pi}x^{5/4}\,{}_4F_3\left(\begin{matrix}-\frac{3}{4}, -\frac{5}{12}, -\frac{1}{12}, \frac{9}{4} \\ \frac{1}{2}, \frac{3}{4}, \frac{5}{4}\end{matrix}; \frac{x}{R}\right) + \frac{9}{\pi}x^{3/2}\,{}_4F_3\left(\begin{matrix}-\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{5}{2} \\ \frac{3}{4}, \frac{5}{4}, \frac{3}{2}\end{matrix}; \frac{x}{R}\right) \\
&- \frac{21\sqrt{2}}{8\pi}x^{7/4}\,{}_4F_3\left(\begin{matrix}-\frac{1}{4}, \frac{1}{12}, \frac{5}{12}, \frac{11}{4} \\ \frac{5}{4}, \frac{3}{2}, \frac{7}{4}\end{matrix}; \frac{x}{R}\right).
\end{aligned}$$

We display graphically $\tilde{W}_M(x)$ for $M=0,1$ on Figure 1, and for $M=2,3,4$ on Figure 2.

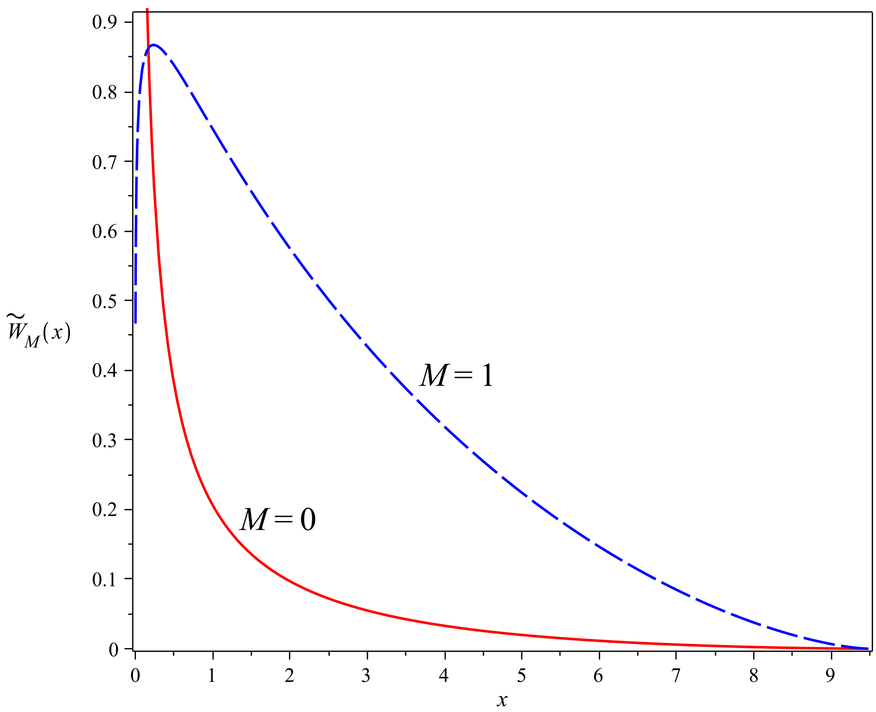

Figure 1: (Color online) Plot of $\tilde{W}_M(x)$ for $M = 0$ (red continuous curve) and $M = 1$ (blue dashed curve) for $x \in (0, R)$. Notice that $\tilde{W}_0(x)$ tends to infinity at $x = 0$ whereas $\tilde{W}_1(x)$ approaches zero at $x = 0$. $\tilde{W}_0(x)$ and $\tilde{W}_1(x)$ are normalized probability distributions.

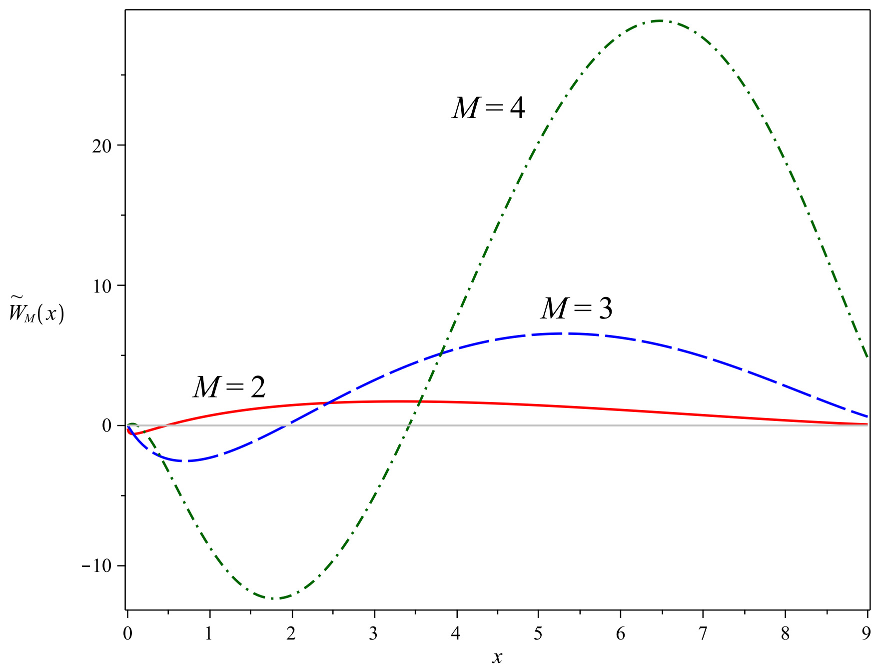

Figure 2: (Color online) Plot of $\tilde{W}_M(x)$ for $M = 2$ (red continuous curve), $M = 3$ (blue dashed curve), and $M = 4$ (green dashed-dotted curve) for $x \in (0, R - 0.48)$. Notice that $\tilde{W}_M(x)$ for $M \geq 2$ have a negative part and tend to zero at $x = 0$.

4. Linking generating and weight functions: encoding via Meijer G-functions

We start with a general Hausdorff moment problem in the form of (3). Solving (3) means to obtain $W(x)$ given the set $\rho(n)$, $n = 0, 1, \ldots$. We define the ordinary generating function (ogf) of moments $\rho(n)$ as

$$G(z) = \sum_{n=0}^{\infty} \rho(n) z^n,$$

with the radius of convergence equal to $1/R$, i.e. $z < 1/R$. We classically observe that

$$G(z) = \sum_{n=0}^{\infty} z^n \left[\int_0^R x^n W(x)\,\mathrm{d}x\right] = \int_0^R W(x) \left[\sum_{n=0}^{\infty} (xz)^n\right] \mathrm{d}x = \int_0^R \frac{W(x)}{1 - zx}\,\mathrm{d}x, \tag{19}$$

with $zx < 1$. Since in (19) $0 \le x \le R$, it implies $z < 1/R$. From (19) it follows that

$$\frac{1}{z} G\left(\frac{1}{z}\right) = \int_0^R \frac{W(x)}{z - x}\,\mathrm{d}x, \qquad \text{with} \quad z > R. \tag{20}$$

The above equations constitute the seed of the inversion procedure by Stieltjes to solve (20) using the complex analysis. For singularity analysis of $G(z)$ in complex plane see [10]. For a recent detailed application of Stieltjes method, along with the exhaustive reference list, see [3]. A very complete exposition of the Stieltjes method can be found in [8].

The above transformations are fairly standard, however, in view of the results of Section 3, a certain pattern does appear that permits one to deduce $W(x)$ directly from $G(z)$, via (20). In order to make this pattern explicit several manipulations with $G(z) \equiv G(M, z)$ are needed.

We use the definition of Pochhammer symbols to write down the ogf $G(M, z)$ of the moments $A(M, n)$, and it reads

$$G(M, z) = \frac{2(2M+1)!}{(M+2)!M!}\, {}_4F_3\left(\begin{matrix}\Delta(4, 2M+2) \\ \Delta(3, 2M+4)\end{matrix}; Rz\right) \tag{21}$$

$$= \frac{2(2M+1)!}{(M+2)!M!}\, {}_4F_3\big([1 + \tfrac{M}{2}, \tfrac{3}{4} + \tfrac{M}{2}, \tfrac{1}{2} + \tfrac{M}{2}, \tfrac{5}{4} + \tfrac{M}{2}], [2 + \tfrac{2M}{3}, \tfrac{5}{3} + \tfrac{2M}{3}, \tfrac{4}{3} + \tfrac{2M}{3}]; Rz\big), \quad z < R. \tag{22}$$

We come back to (8) in order to frame (21) in the Meijer G-notation. Carrying out the products of gamma functions in (8) this furnishes:

$$G(M, z) = r_G(M) G^{1,4}_{4,4}\left(-Rz \Big| \begin{matrix}\Delta(4, -2M-1) \\ 0, \Delta(3, -2M-3)\end{matrix}\right) \tag{23}$$

$$= r_G(M) \mathrm{MeijerG}\big([[-\tfrac{M}{2}, \tfrac{1}{2} - \tfrac{M}{2}, \tfrac{1}{4} - \tfrac{M}{2}, -\tfrac{1}{4} - \tfrac{M}{2}], [\ ]], [[0], [-\tfrac{2}{3} - \tfrac{2M}{3}, -\tfrac{1}{3} - \tfrac{2M}{3}, -1 - \tfrac{2M}{3}]], -Rz\big)$$

with

$$r_G(M) = \frac{4}{81\sqrt{\pi}} 3^{\frac{1}{2} - 2M} 2^{4M + \frac{1}{2}} P(M), \tag{24}$$

where in obtaining (24) the use of (11) was again made.

Further transformations of (23) are necessary in order to take full advantage of (20). For that purpose we apply (9) to (23):

$$G^{1,4}_{4,4}\left(\frac{1}{z} \Big| \begin{matrix} a_1, \ldots, a_4 \\ b_1, \ldots, b_4\end{matrix}\right) = G^{4,1}_{4,4}\left(z \Big| \begin{matrix} 1 - b_1, \ldots, 1 - b_4 \\ 1 - a_1, \ldots, 1 - a_4\end{matrix}\right),$$

where (a_p), $p = 4$, and (b_q), $q = 4$ can be read off (23). Then $G(M, z)$ becomes

$$G(M, z) = r_G(M)\, G^{4,1}_{4,4}\left(-\frac{1}{Rz} \Big| \begin{matrix} 1, \frac{4}{3} + \frac{2M}{3}, \frac{5}{3} + \frac{2M}{3}, 2 + \frac{2M}{3} \\ \frac{3}{4} + \frac{M}{2}, 1 + \frac{M}{2}, \frac{5}{4} + \frac{M}{2}, \frac{3}{2} + \frac{M}{2}\end{matrix}\right),$$

which permits to evaluate

$$\frac{1}{z} G\left(M, \frac{1}{z}\right) = r_G(M)\, \frac{1}{z}\, G^{4,1}_{4,4}\left(-\frac{z}{R} \Big| \begin{matrix} 1, \frac{4}{3} + \frac{2M}{3}, \frac{5}{3} + \frac{2M}{3}, 2 + \frac{2M}{3} \\ \frac{3}{4} + \frac{M}{2}, 1 + \frac{M}{2}, \frac{5}{4} + \frac{M}{2}, \frac{3}{2} + \frac{M}{2}\end{matrix}\right). \tag{25}$$

Equation (25) will be again transformed now, using (10), where $\mu = -1$:

$$\frac{1}{z} G\left(M, \frac{1}{z}\right) = r_G(M)\, G^{4,1}_{4,4}\left(-\frac{z}{R}\middle| \begin{matrix} a'_1 - 1, \ldots, a'_4 - 1 \\ b'_1 - 1, \ldots, b'_4 - 1 \end{matrix}\right),$$

where now (a'_4) and (b'_4) are read off the lists in (25); it gives finally

$$\frac{1}{z} G\left(M, \frac{1}{z}\right) = -\frac{r_G(M)}{R}\, G^{4,1}_{4,4}\left(-\frac{z}{R}\middle| \begin{matrix} 0, \frac{1}{3} + \frac{2M}{3}, \frac{2}{3} + \frac{2M}{3}, 1 + \frac{2M}{3} \\ \frac{M}{2} - \frac{1}{2}, \frac{M}{2} - \frac{1}{4}, \frac{M}{2}, \frac{M}{2} + \frac{1}{4} \end{matrix}\right).$$

It is instructive to write now (20) exclusively using the Meijer G-function in Maple notation and *retaining* both the multiplicative constants $r_G(M)$ and $r_W(M)$:

$$\begin{aligned} -\frac{r_G(M)}{R}\, &\mathrm{MeijerG}\big([[0], [\tfrac{1}{3} + \tfrac{2M}{3}, \tfrac{2}{3} + \tfrac{2M}{3}, 1 + \tfrac{2M}{3}]], [[\tfrac{M}{2} - \tfrac{1}{2}, \tfrac{M}{2} - \tfrac{1}{4}, \tfrac{M}{2}, \tfrac{M}{2} + \tfrac{1}{4}], [\ ]], -\tfrac{z}{R}\big) \\ &= r_W(M) \int_0^R \frac{\mathrm{d}x}{z - x}\, \mathrm{MeijerG}\big([[\ ], [0, \tfrac{1}{3} + \tfrac{2M}{3}, \tfrac{2}{3} + \tfrac{2M}{3}, 1 + \tfrac{2M}{3}]], \\ &\qquad [[\tfrac{M}{2} - \tfrac{1}{2}, \tfrac{M}{2} - \tfrac{1}{4}, \tfrac{M}{2}, \tfrac{M}{2} + \tfrac{1}{4}], [\ ]], \tfrac{x}{R}\big), \qquad z > R. \end{aligned} \tag{26}$$

The same formula is presented in the traditional notation, i.e.:

$$\begin{aligned} \frac{r_G(M)}{R} &G^{4,1}_{4,4}\left(-\frac{z}{R}\middle| \begin{matrix} 0; \frac{1}{3} + \frac{2M}{3}, \frac{2}{3} + \frac{2M}{3}, 1 + \frac{2M}{3} \\ \frac{M}{2} - \frac{1}{2}, \frac{M}{2} - \frac{1}{4}, \frac{M}{2}, \frac{M}{2} + \frac{1}{4} \end{matrix}\right) \\ &= r_W(M) \int_0^R \frac{\mathrm{d}x}{x - z}\, G^{4,0}_{4,4}\left(\frac{x}{R}\middle| \begin{matrix} ; 0, \frac{1}{3} + \frac{2M}{3}, \frac{2}{3} + \frac{2M}{3}, 1 + \frac{2M}{3} \\ \frac{M}{2} - \frac{1}{2}, \frac{M}{2} - \frac{1}{4}, \frac{M}{2}, \frac{M}{2} + \frac{1}{4} \end{matrix}\right), \quad z > R, \end{aligned} \tag{27}$$

and in shorter notation of (12), (27) becomes

$$\frac{r_G(M)}{R} G^{4,1}_{4,4}\left(-\frac{z}{R}\middle| \begin{matrix} 0; \Delta(3, 2M+1) \\ \Delta(4, 2M-2) \end{matrix}\right) = r_W(M) \int_0^R \frac{\mathrm{d}x}{x - z} G^{4,0}_{4,4}\left(\frac{x}{R}\middle| \begin{matrix} ; 0, \Delta(3, 2M+1) \\ \Delta(4, 2M-2) \end{matrix}\right), \tag{28}$$

where $z > R$. The validity of (27) has been independently verified numerically for various values M. Equations (27) and (28) appear to be less transparent than (26) and are rather more error-prone. We slightly overstretched the notation of (6) in (27) by (temporarily) introducing the semicolons to explain the correct position of 0 in the coefficient lists. We stress that it is essential to keep the multiplicative constants $r_G(M)$ and $r_W(M)$ on both sides of (26) and (27) in order to consider these equations as full solutions of (3). The attentive reader will rapidly notice that $r_G(M)/r_W(M) = R$, and after this simplification (26) and (27) become "bare" relations between Meijer G-functions.

For reader's convenience we quote below the formula which results from the functional composition of (8) - (10) which allows quasi-automatically to arrive at the coefficient lists appearing in $\frac{1}{z}G(M, \frac{1}{z})$ here, as well as serving for studying related problems.
Starting with ${}_pF_q\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q}; Rz\right)$ as in (21) and (22), one obtains:

$$\frac{1}{z}\, {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; \frac{R}{z}\right) = -\frac{\prod_{k=1}^q \Gamma(b_k)}{\prod_{k=1}^p \Gamma(a_k)}\, \frac{1}{R}\, G^{p,1}_{q+1,p}\left(-\frac{z}{R}\middle| \begin{matrix} 0, b_1 - 1, \ldots, b_q - 1 \\ a_1 - 1, \ldots, a_p - 1 \end{matrix}\right)$$

for $p \leq q + 1$, applicable in our context only for the cases when the ogf is a *single* generalized hypergeometric function ${}_pF_q$.

In the language of Meijer G-functions (26) and (27) display a visibly regular scheme, which can be symbolically written down if we define the lists $\mathbf{L1} = (\frac{1}{3} + \frac{2M}{3}, \frac{2}{3} + \frac{2M}{3}, 1 + \frac{2M}{3})$ and $\mathbf{L2} = (\frac{M}{2} - \frac{1}{2}, \frac{M}{2} - \frac{1}{4}, \frac{M}{2}, \frac{M}{2} + \frac{1}{4})$. Then, neglecting, for now, the multiplicative constants we observe that

$$\frac{1}{z} G\left(M, \frac{1}{z}\right) \cong \mathrm{MeijerG}\left([[\mathbf{0}], [\mathbf{L1}]], [[\mathbf{L2}], [\ ]], -\frac{z}{R}\right) \tag{29}$$

and

$$W_M(x) \cong \mathrm{MeijerG}\left([[\ ], [\mathbf{0}, \mathbf{L1}]], [[\mathbf{L2}], [\ ]], \frac{x}{R}\right) \tag{30}$$

are related through the integral formula of (20) whose specific realizations are (26) and (27). From two previous equations, we note that by reinserting the multiplicative constants one can construct the weight $W_M(x)$ by simply moving the number $\mathbf{0}$ from the first bracket in (29) to the second bracket in (30), where $\mathbf{0}$ joins the list

$$\frac{1}{z}G\left(\frac{1}{z}\right) = \mathrm{MeijerG}\left([[\,0\,],[\mathbf{L1}]],[[\mathbf{L2}],[\;]],\ -\frac{z}{R}\right)$$

$$W(x) = \mathrm{MeijerG}\left([[\;],[\,0,\mathbf{L1}]],[[\mathbf{L2}],[\;]],\ \frac{x}{R}\right)$$

Figure 3: (Color online) Schematic illustration of relations of (20) for specific case of Tutte numbers of (1) and (26). The lists **L1** and **L2** are defined before (29) in the text. We emphasize that both functions illustrated here are of Meijer G-type, but they are different functions. We neglect any multiplicative numerical constants in this illustration.

L1. The position of the list **L2** stays unchanged in the third bracket, and the argument of $W_M(x)$ becomes x/R. Schematic display of ingredients of (20) are presented in Figure 3.

We believe that the moments $A(M,n)$ belong to a larger family of similar types of moments, for which the aforementioned reshuffling of lists gives explicitly the weight $W(x)$ from the data of the appropriate ogf $G(z)$, as in (29) and (30). If so, then there is no need to perform the inverse Mellin transform from the moments, since all the information are already contained in the ogf $G(z)$. We are searching for possible candidates to extend the sequence $A(M,n)$ studied here. It should be kept in mind that the analysis of other possible sequences obeying relations of type (29) and (30) should also allow for exact determination of respective (multiplicative) constant r_G and r_W. The integral relation (20) rewritten as (26) can be viewed as a variant of one-sided, finite Hilbert transform [9]. However, the strict condition $z > R$ imposed by convergence requires special care in all the manipulations.

5. Discussion and Conclusions

We have exactly solved the moment problem of (3) following two different, and seemingly unrelated paths. The first method used was the inverse Mellin transform which resulted in exact and explicit expression for the weight functions $W_M(x)$ formulated in the language of Meijer G-functions. The second, less orthodox approach, consists in "upgrading" the notation for the ogf of moments $G(M,z)$, which initially was a generalized hypergeometric function ${}_4F_3$, to encode it in terms of a Meijer G-function. This procedure has revealed a hitherto hidden relation between the parameter lists of $G(M,z)$ and the parameter lists of solutions $W_M(x)$. That observation is quite fertile, as it allows one, almost automatically, to obtain explicit forms of $W_M(x)$, without having recourse to any further manipulations. We believe that the sequence $A(M,n)$ of (1) belongs to a larger family of moment sequences for which analogous relations of type (29) and (30) hold. This feature is under active consideration.

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