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Enumeration of Partial Łukasiewicz Paths

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Abstract: Łukasiewicz paths are lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis, and consisting of steps in the set $\{(1,k), k \geq -1\}$. We give bivariate generating functions and exact values for the number of n-length prefixes (resp. suffixes) of these paths ending (resp. starting) at height $k \geq 0$ with a given type of step. We make a similar study for paths of bounded height, and we prove that the average height of n-length paths ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$. Finally, we study prefixes of alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do not contain two consecutive steps in the same direction.

Keywords: Asymptotics; Enumeration; Partial Łukasiewicz paths
2020 Mathematics Subject Classification: 05A05; 05A15; 05A15

1. Introduction

A *Łukasiewicz path* of length $n \geq 0$ is a lattice path in $\mathbb{N}^2$ starting at the origin $(0,0)$, ending on the x-axis, consisting of n steps lying in $S = \{(1,k), k \geq -1\}$. We denote by ϵ the empty path, i.e., the path of length zero. These paths constitute a natural generalization of Dyck and Motzkin paths (see [3,4]), which are made using steps into the sets $\{(1,1),(1,-1)\}$ and $\{(1,1),(1,0),(1,-1)\}$, respectively. We refer to [1,8,14,17–19] for some combinatorial studies on Łukasiewicz paths. Let $\mathcal{L}_n$, $n \geq 0$, be the set of Łukasiewicz paths of length n, and $\mathcal{L} = \bigcup_{n\geq 0} \mathcal{L}_n$. For convenience, we set $D = (1,-1)$, $F = (1,0)$, $U_k = (1,k)$ for $k \geq 1$. See Figure 1 for an illustration of a Łukasiewicz path of length 18. Note that Łukasiewicz paths can be interpreted as an algebraic language of words $w \in \{x_0, x_1, x_2, \ldots\}^\star$ such that $\delta(w) = -1$ and $\delta(w') \geq 0$ for any proper prefix w' of w where δ is the map from $\{x_0, x_1, x_2, \ldots\}^\star$ to $\mathbb{Z}$ defined by $\delta(w_1 w_2 \ldots w_n) = \sum_{i=1}^{n} \delta(w_i)$ with $\delta(x_i) = i - 1$ (see [13,15]).

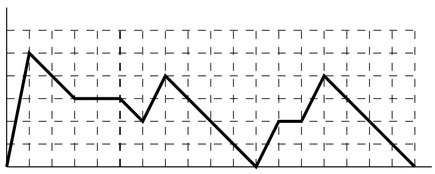

Figure 1: A Łukasiewicz path of length 18: $U_5DDFFDU_2DDDDU_2FU_2DDDD$.

Any non-empty Łukasiewicz path $L \in \mathcal{L}$ can be decomposed (see [6]) into one of the two following forms: (1) $L = FL'$ with $L' \in \mathcal{L}$, or (2) $L = U_k L_1 D L_2 D \ldots L_k D L'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$ (see Figure 2). Due to this decomposition, the generating function $L(z) = \sum_{n\geq 0} a_n z^n$ where a_n is the cardinality of $\mathcal{L}_n$, satisfies the functional equation $L(z) = 1 + zL(z) + \sum_{k\geq 1} z^{k+1} L(z)^{k+1}$, or equivalently, $L(z) = \frac{1}{1 - zL(z)}$. Then, $L(z) = \frac{1-\sqrt{1-4z}}{2z}$. Therefore, a_n is the n-th Catalan number $a_n = \frac{1}{n+1}\binom{2n}{n}$ (see sequence A000108 in [16]).

In this paper, we provide enumerating results for several classes of partial Łukasiewicz paths (prefixes and suffixes of Łukasiewicz paths, partial alternate Łukasiewicz paths). More precisely, in Sections 2 and 3, we give bivariate generating functions and exact values for the number of n-length prefixes (resp. suffixes) of these paths ending at height $k \geq 0$ with a given type of step (down, up, or horizontal step). In Sections 4 and 5, we make

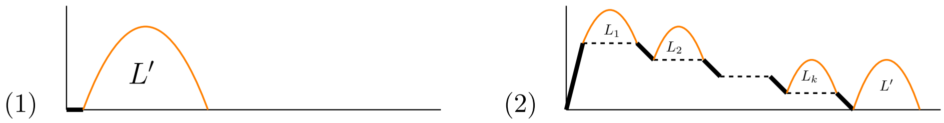

Figure 2: The two forms of a non-empty Łukasiewicz path.

a similar study for paths of bounded height. In Section 6, we prove that the average height of n-length paths ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$. In Section 7, we focus on partial alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do contain two consecutive steps with the same direction.

All our *explicit* formulæ follow from the standard identity

$$[z^n]\left(\frac{1-\sqrt{1-4z}}{2z}\right)^k = \binom{2n-1+k}{n} - \binom{2n-1+k}{n}.$$

2. Enumeration of Partial Lukasiewicz paths

Partial Łukasiewicz paths of length n (i.e., n-length prefixes of Łukasiewicz paths) ending at height k can be constructed through the following state diagram (Figure 3). The diagram has three types of states ranging from 0 to infinity on three layers; in the drawing, only the first fifth states of each type are shown. The first type of states (top layer) refers to an up-step leading to a state, the second type (middle layer) refers to a horizontal step leading to a state, and the third type (bottom layer) refers to a down-step leading to a state. Any path from the origin to a state of rank k of a layer represents a partial Łukasiewicz path ending at height k.

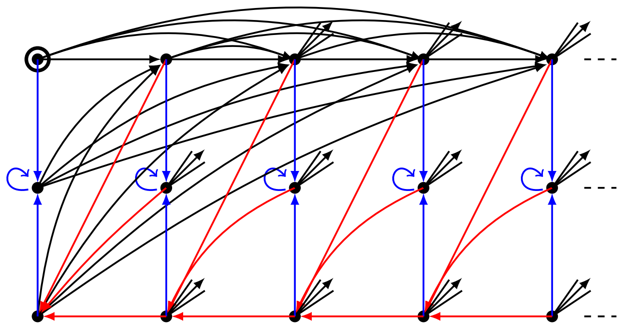

Figure 3: The state diagram for the generation of partial Łukasiewicz paths. Black (resp. red, blue) arrows correspond to up-steps (resp. down-steps, horizontal steps).

For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Łukasiewicz paths of length n ending at height k with an up-step U_k, $k \geq 1$, (resp. with a down-step D, resp. with a horizontal step F). Considering the state diagram in Figure 3, f_k (resp. g_k, h_k) is the generating function in the variable z marking the length of the paths ending on the $(k+1)$-th state of the top (resp. middle, bottom) layer. So, we easily obtain the following equations:

$$\begin{aligned}
&f_0 = 1, \text{ and } f_k = z\sum_{\ell=0}^{k-1} f_\ell + z\sum_{\ell=0}^{k-1} g_\ell + z\sum_{\ell=0}^{k-1} h_\ell, \quad k \geq 1,\\
&g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geq 0,\\
&h_k = zf_k + zg_k + zh_k, \quad k \geq 0.
\end{aligned} \tag{1}$$

Now, we introduce bivariate generating functions

$$F(u,z) = \sum_{k\geq 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geq 0} u^k g_k(z), \text{ and } H(u,z) = \sum_{k\geq 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$ and $H(u)$ for these functions. Summing the recursions in (1), we have:

$$F(u) = 1 + z\sum_{k\geq 1} u^k \left(\sum_{\ell=0}^{k-1} f_\ell + \sum_{\ell=0}^{k-1} g_\ell + \sum_{\ell=0}^{k-1} h_\ell\right)$$

$$\begin{aligned}
&= 1 + z\sum_{k\geq 0}\frac{u^{k+1}}{1-u}f_k + z\sum_{k\geq 0}\frac{u^{k+1}}{1-u}g_k + z\sum_{k\geq 0}\frac{u^{k+1}}{1-u}h_k\\
&= 1 + \frac{uz}{1-u}(F(u)+G(u)+H(u)),\\
G(u) &= z\sum_{k\geq 0}u^k\Big(f_{k+1}+g_{k+1}+h_{k+1}\Big)\\
&= \frac{z}{u}(F(u)+G(u)+H(u)-F(0)-G(0)-H(0)),\\
H(u) &= \frac{z}{1-z}(F(u)+G(u)),
\end{aligned}$$

where $F(0)+G(0)+H(0)$ is the number of Łukasiewicz paths of length n, i.e.,

$$F(0)+G(0)+H(0) = L(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Solving these functional equations, we deduce

$$F(u) = 1 - z - \frac{z\left(1+\sqrt{1-4z}\right)}{2u-1-\sqrt{1-4z}}, \quad G(u) = \frac{\sqrt{1-4z}+2z-1}{2u-1-\sqrt{1-4z}}, \text{ and}$$

$$H(u) = z + \frac{z\left(\sqrt{1-4z}-1\right)}{2u-1-\sqrt{1-4z}},$$

which implies that

$$f_k = [u^k]F(u) = \frac{2^k z}{(1+\sqrt{1-4z})^k} = z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^k, \tag{2}$$

$$\begin{aligned}
g_k = [u^k]G(u) &= \frac{2^k(1-2z-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}}\\
&= \frac{(1-2z-\sqrt{1-4z})}{2}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1}\\
&= z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+2} - z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1},
\end{aligned}\tag{3}$$

and

$$\begin{aligned}
h_k = [u^k]H(u) &= \frac{2^k z(1-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}} = \frac{z(1-\sqrt{1-4z})}{2}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1}\\
&= z^2\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+2}.
\end{aligned}\tag{4}$$

Theorem 1. *The bivariate generating function for the total number of partial Łukasiewicz paths of length n with respect to the height of the end-point is given by*

$$Total(z,u) = 1 + \frac{-1+\sqrt{1-4z}}{2u-1-\sqrt{1-4z}},$$

and we have

$$[u^k]\,Total(z,u) = [k=0] + zL(z)^{k+2}.$$

Finally, we have for $n\geq 1$,

$$\begin{aligned}
[z^n][u^k]\,Total(z,u) &= \frac{k+2}{n+k+1}\binom{2n+k-1}{n-1},\\
[z^n][u^k]F(u) &= \frac{k}{n+k-1}\binom{2n+k-3}{n-1},\\
[z^n][u^k]G(u) &= \frac{k+3}{n+k+1}\binom{2n+k-2}{n-2},\\
[z^n][u^k]H(u) &= \frac{k+2}{n+k}\binom{2n+k-3}{n-2}.
\end{aligned}$$

Here are examples of the series expansions of $[u^k]\,Total(z,u)$ for $k = 0, 1, 2, 3$ (leading terms):

- $1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9$,
- $z + 3z^2 + 9z^3 + 28z^4 + 90z^5 + 297z^6 + 1001z^7 + 3432z^8 + 11934z^9$,
- $z + 4z^2 + 14z^3 + 48z^4 + 165z^5 + 572z^6 + 2002z^7 + 7072z^8 + 25194z^9$,
- $z + 5z^2 + 20z^3 + 75z^4 + 275z^5 + 1001z^6 + 3640z^7 + 13260z^8 + 48450z^9$,

which correspond respectively to A000108, A000245, A002057, and A000344 in [16].

According to Theorem 3.1 and Theorem 3.3 in [9], $[z^n][u^k]\,Total(z,u)$ counts also standard Young tableaux of shape $(n+2, n-k+1)$ (see [11,20] for the definition of a standard Young tableau), and Dyck paths of semilength $n + k$ starting with at least k up-steps and touching the x-axis somewhere between the two end-points.

3. Partial Łukasiewicz paths from right to left

In this section, we count partial Łukasiewicz paths that read from right to left, i.e., paths in $\mathbb{N}^2$ starting at the origin, consisting of steps $(1, k)$, $k \leq 1$, and ending at a given height with a given type of step. Of course, this study is completely equivalent to counting suffixes of Łukasiewicz paths starting at a given height with a given type of step. We denote here by f_k, g_k, and h_k the generating functions for the number of these paths (with respect to the length) ending at height k with an up-step, down-step, or a horizontal step, respectively.

Then we have

$$\begin{aligned}
&f_0 = 1, \quad \text{and} \quad f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geq 1,\\
&g_k = z\sum_{\ell\geq k+1} f_\ell + z\sum_{\ell\geq k+1} g_\ell + z\sum_{\ell\geq k+1} h_\ell, \quad k \geq 0,\\
&h_k = zf_k + zg_k + zh_k, \quad k \geq 0.
\end{aligned} \tag{5}$$

Considering the bivariate generating functions

$$F(u) = \sum_{k\geq 0} u^k f_k(z), \quad G(u) = \sum_{k\geq 0} u^k g_k(z), \text{ and } H(u) = \sum_{k\geq 0} u^k h_k(z),$$

and summing the recursions in (5), we obtain:

$$\begin{aligned}
F(u) &= 1 + z\sum_{k\geq 1} u^k \left(f_{k-1} + g_{k-1} + h_{k-1}\right)\\
&= 1 + zuF(u) + zuG(u) + zuH(u),\\
G(u) &= z\sum_{k\geq 0} u^k \left(\sum_{\ell\geq k+1} f_\ell + \sum_{\ell\geq k+1} g_\ell + \sum_{\ell\geq k+1} h_\ell\right)\\
&= z\sum_{k\geq 1} \frac{1-u^k}{1-u} f_k + z\sum_{k\geq 1} \frac{1-u^k}{1-u} g_k + z\sum_{k\geq 1} \frac{1-u^k}{1-u} h_k\\
&= \frac{z}{1-u}(F(1) + G(1) + H(1) - F(u) - G(u) - H(u)),\\
H(u) &= \frac{z}{1-z}(F(u) + G(u)),
\end{aligned}$$

with

$$F(0) + G(0) + H(0) = L(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Moreover, we have

$$F(1) + G(1) + H(1) = \frac{L(z) - 1}{z},$$

since there is a bijection between all partial Łukasiewicz paths of length n that read from right to left and Łukasiewicz paths that read from left to right of length $n + 1$ (from a Łukasiewicz path, we remove the first step, and we read it from right to left).

Solving these functional equations, we deduce

$$F(u) = -\frac{1+\sqrt{1-4z}}{2zu - \sqrt{1-4z} - 1}, \qquad G(u) = \frac{-1+\sqrt{1-4z}+2z}{2zu - \sqrt{1-4z} - 1},$$

$$H(u) = -\frac{2z}{2zu - \sqrt{1-4z} - 1},$$

which implies that

$$f_k = [u^k]F(u) = \frac{2^k z^k}{(1+\sqrt{1-4z})^k}, \tag{6}$$

$$g_k = [u^k]G(u) = \frac{2^k z^k(1-2z-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}}, \text{ and} \tag{7}$$

$$h_k = [u^k]H(u) = \frac{2^{k+1} z^{k+1}}{(1+\sqrt{1-4z})^{k+1}}. \tag{8}$$

Theorem 2. *The bivariate generating function for the total number of partial Łukasiewicz paths of length n (from right to left) with respect to the height of the end-point is given by*

$$\mathit{Total}(z,u) = 1 + \frac{2}{1-2zu+\sqrt{1-4z}},$$

and we have

$$[u^k]\,\mathit{Total}(z,u) = z^k L(z)^{k+1}.$$

Finally, for $n \geq 1$, we obtain:

$$[z^n][u^k]\mathit{Total}(z,u) = \frac{k+1}{n+1}\binom{2n-k}{n},$$

$$[z^n][u^k]F(u) = \frac{k}{n}\binom{2n-k-1}{n-1},$$

$$[z^n][u^k]G(u) = \frac{k+3}{n+1}\binom{2n-k-2}{n},$$

$$[z^n][u^k]H(u) = \frac{k+1}{n}\binom{2n-k-2}{n-1}.$$

Here are examples of the series expansions of $[u^k]\,\mathit{Total}(z,u)$ for $k = 0, 1, 2, 3$ (leading terms):

- $1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9$,
- $z^2 + 3z^3 + 9z^4 + 28z^5 + 90z^6 + 297z^7 + 1001z^8 + 3432z^9$,
- $z^3 + 4z^4 + 14z^5 + 48z^6 + 165z^7 + 572z^8 + 2002z^9$,

which correspond to shifts of A000108, A000245, A002057, and A000344 in [16].

4. Partial Łukasiewicz paths constrained by height

In this section, we count partial Łukasiewicz paths bounded by a given height $t \geq 0$. We introduce the notation f_k^t, g_k^t, h_k^t for $0 \leq k \leq t$, $F^t(u)$, $G^t(u)$ and $H^t(u)$, which are the counterparts of f_k, g_k, h_k, $F(u)$, $G(u)$ and $H(u)$. Considering the state diagram of Figure 3 where each layer consists of only $t+1$ states, we deduce the following system of equations:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & z & z & z & 0 & 0 & 0 & \cdots \\ z & z & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ z & z & z & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -1 & 0 & z & z & z & \cdots \\ 0 & 0 & 0 & z & z & z-1 & 0 & 0 & 0 & \cdots \\ z & z & z & z & z & z & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & z & z & z-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \cdot \begin{bmatrix} f_0^t \\ g_0^t \\ h_0^t \\ f_1^t \\ g_1^t \\ h_1^t \\ f_2^t \\ g_2^t \\ h_2^t \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

For a given height $t \geq 0$, the previous matrix (denoted A_t) is square with $3(t+1)$ rows. Using classical properties of the determinant, we can prove that $D_t = \det(A_t)$ satisfies

$$D_{t+2} + D_{t+1} + zD_t = 0,$$

anchored with $D_0 = z-1$, and $D_1 = 1-2z$. Then we deduce,

$$D_t = \frac{z(-1)^{t+2}\left(1-\sqrt{1-4z}\right)^{t+2}}{2^{t+1}\sqrt{1-4z}\left(1+\sqrt{1-4z}\right)} + \frac{z(-1)^{t+1}\left(1+\sqrt{1-4z}\right)^{t+2}}{2^{t+1}\sqrt{1-4z}\left(1-\sqrt{1-4z}\right)},$$

which corresponds to

$$D_t = (-1)^{t+1}\cdot F_t,$$

where F_t is the generalized Fibonacci polynomial (see [7,12]):

$$F_t = 1 - \binom{t+1}{1}z + \binom{t}{2}z^2 - \binom{t-1}{3}z^3 + \dots.$$

For instance, we have $D_3 = F_3 = 1-4z+3z^2$, and $D_4 = -F_4 = -1+5z-6z^2+z^3$.

Using Cramer's rule to solve the system, for $0 \le k \le t$, we have

$$f_k^t = \frac{N_{3k+1}^t}{D_t}, \quad g_k^t = \frac{N_{3k+2}^t}{D_t}, \quad h_k^t = \frac{N_{3k+3}^t}{D_t}, \tag{9}$$

where N_k^t is the determinant of the matrix $A_t(k)$ obtained from A_t by replacing the $(k+1)$-th column with the vector $(-1,0,0,0,\dots,0,0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the two recurrence relations

$$N_k^{t+2} + N_k^{t+1} + zN_k^t = 0, \quad 1 \le k,\ 1 + \left\lceil \frac{k}{3} \right\rceil \le t, \text{ and}$$

$$N_{k+3}^{t+1} = -N_k^t, \quad 4 \le k,\ 1 \le t.$$

Calculating N_k^t for $(t,k) \in \{0,1,2\}\times\{1,2,3\}$, and for $(t,k)\in\{1,2,3\}\times\{4,5,6\}$, we can easily obtain a closed form for N_{3k+i}^t, $1 \le i \le 3$. See Table 1 for exact values of N_k^t when $0 \le t \le 4$ and $1 \le k \le 12$.

k/t	0	1	2	3	4
1	$z-1$	$1-2z$	$-(z^2-3z+1)$	$3z^2-4z+1$	z^3-6z^2+5z-1
2	0	z^2	$-z^2$	$z^2(1-z)$	$-z^2(1-2z)$
3	$-z$	$z(1-z)$	$-z(1-2z)$	$z(z^2-3z+1)$	$-z(3z^2-4z+1)$
4		$z(1-z)$	$-z(1-2z)$	$z(z^2-3z+1)$	$-z(3z^2-4z+1)$
5		0	$-z^2$	z^2	$-z^2(1-z)$
6		z^2	$-z^2$	$z^2(1-z)$	$-z^2(1-2z)$
7			$-z(1-z)$	$z(1-2z)$	$-z(z^2-3z+1)$
8			0	z^2	$-z^2$
9			$-z^2$	z^2	$-z^2(1-z)$
10				$z(1-z)$	$-z(1-2z)$
11				0	$-z^2$
12				z^2	$-z^2$
13					$\dots$

Table 1: The first values of N_k^t for $0 \le t \le 4$ and $1 \le k \le 12$.

In particular, for $t \ge 0$, we have $N_1^t = D_t$ (see above for a closed form),

$$N_2^t = \frac{z^2\left(-\frac{2z}{1+\sqrt{1-4z}}\right)^t}{\sqrt{1-4z}} - \frac{z^2\left(-\frac{2z}{-\sqrt{1-4z}+1}\right)^t}{\sqrt{1-4z}},$$

$$N_3^t = -\frac{z^2\left(-1+\sqrt{1-4z}\right)\left(-\frac{2z}{1+\sqrt{1-4z}}\right)^t}{\sqrt{1-4z}\left(1+\sqrt{1-4z}\right)} - \frac{z^2\left(1+\sqrt{1-4z}\right)\left(-\frac{2z}{-\sqrt{1-4z}+1}\right)^t}{\sqrt{1-4z}\left(-\sqrt{1-4z}+1\right)},$$

and for $t \ge 1$,

$$N_4^t = N_3^t,\ N_5^t = -N_2^{t-1}, \text{ and } N_6^t = N_2^t.$$

Using (9), we can deduce closed forms for f_k^t, g_k^t, h_k^t, $0 \le k \le 1$, and $k \le t$. Using the above recurrence relations for N_k^t, we deduce closed forms for f_k^t, g_k^t, h_k^t, $2 \le k \le t$.

Theorem 3. *For $2 \leq k \leq t$, we have*

$$f_k = [u^k]F(u) = \frac{N_3^{t-k+1}}{D_t}(-1)^{k-1}, \tag{10}$$

$$g_k = [u^k]G(u) = \frac{N_2^{t-k}}{D_t}(-1)^{k}, \text{ and} \tag{11}$$

$$h_k = [u^k]H(u) = \frac{N_2^{t-k+1}}{D_t}(-1)^{k-1}. \tag{12}$$

For $t = 2, 3, 4$, the first terms of the series expansion of f_2 are

- $z + 2z^2 + 5z^3 + 13z^4 + 34z^5 + 89z^6 + 233z^7 + 610z^8 + 1597z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 41z^5 + 122z^6 + 365z^7 + 1094z^8 + 3281z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 131z^6 + 417z^7 + 1341z^8 + 4334z^9$, which correspond to the sequences A001519, A007051, A080937 in [16], that also count Dyck paths of semilength n of height at most $t + 1$.

Theorem 4. *The generating function $[u^k]\,Total_t(z, u)$ for the number of partial Łukasiewicz paths of height at most $t \geq 0$, ending at height $k \geq 1$, is given by*

$$(-1)^{k-1}\frac{N_3^{t-k+1} - N_2^{t-k} + N_2^{t-k+1}}{D_t}.$$

Moreover, we have

$$[u^0]\,Total_t(z, u) = \frac{D_t + N_2^t + N_3^t}{D_t}.$$

The generating function for the total number of partial Łukasiewicz paths of height at most $t \geq 0$ is given by

$$Total_t(z, 1) = (-1)^{t+1} \cdot D_t^{-1} = F_t^{-1}.$$

For $t = 0, 1, 2, 3, 4$, the first terms of the series expansion of $Total_t(z, 1)$ are

- $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9$,
- $1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + 64z^6 + 128z^7 + 256z^8 + 512z^9$,
- $1 + 3z + 8z^2 + 21z^3 + 55z^4 + 144z^5 + 377z^6 + 987z^7 + 2584z^8 + 6765z^9$,
- $1 + 4z + 13z^2 + 40z^3 + 121z^4 + 364z^5 + 1093z^6 + 3280z^7 + 9841z^8 + 29524z^9$,
- $1 + 5z + 19z^2 + 66z^3 + 221z^4 + 728z^5 + 2380z^6 + 7753z^7 + 25213z^8 + 81927z^9$, which correspond to A000012, A000079, A001906, A003462, A005021 in [16].

Using [12], $[z^n]\ Total_t(z, 1)$ counts also paths of length $2n + 1 + t$ in $\mathbb{N}^2$ starting at the origin, ending at $(n + t + 1, n)$, consisting of steps $(0, 1)$, and $(1, 0)$, and such that all its points (x, y) satisfy $x - t - 1 \leq y \leq x$. It would be interesting to exhibit a constructive bijection between these paths and partial Łukasiewicz paths of height at most $t \geq 0$.

5. Partial Łukasiewicz paths constrained by height from right-to-left

In this section, we count partial Łukasiewicz paths from right-to-left bounded by a given height $t \geq 0$. We denote here by f_k^t, g_k^t, h_k^t for $0 \leq k \leq t$, $F^t(u)$, $G^t(u)$ and $H^t(u)$, the generating functions in the same way as for Section 4. We deduce the following system of equations:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & z & z & z & z & z & z & \cdots \\ z & z & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ z & z & z & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -1 & 0 & z & z & z & \cdots \\ 0 & 0 & 0 & z & z & z-1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & z & z & z & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & z & z & z-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \cdot \begin{bmatrix} f_0^t \\ g_0^t \\ h_0^t \\ f_1^t \\ g_1^t \\ h_1^t \\ f_2^t \\ g_2^t \\ h_2^t \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

For a given height $t \geq 0$, the previous matrix (denoted A'_t) is square with $3(t+1)$ rows. Using classical properties of the determinant, we can prove that

$$\det(A'_t) = \det(A_t) = D_t, \quad \text{for } t \geq 0.$$

Using Cramer's rule to solve the system, for $0 \leq k \leq t$, we have

$$f_k^t = \frac{N_{3k+1}^t}{D_t}, \quad g_k^t = \frac{N_{3k+2}^t}{D_t}, \quad h_k^t = \frac{N_{3k+3}^t}{D_t}, \tag{13}$$

where N_k^t is the determinant of the matrix $A'_t(k)$ obtained from A'_t by replacing the k-th column with the vector $(-1,0,0,0,\dots,0,0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the two recurrence relations

$$N_k^{t+2} + N_k^{t+1} + zN_k^t = 0, \quad 1 \leq k \leq 3,\ 2 \leq t, \text{ and}$$

$$N_k^{t+1} = -zN_{k-3}^t, \quad 4 \leq k,\ 0 \leq t,$$

where N_k^t is the same as in Section 4 whenever $(t,k) \in \mathbb{N} \times \{1,2,3\}$.

Theorem 5. *For $0 \leq k \leq t$, we have*

$$f_k = [u^k]F(u) = \frac{N_1^{t-k}}{D_t}(-1)^k z^k, \tag{14}$$

$$g_k = [u^k]G(u) = \frac{N_2^{t-k}}{D_t}(-1)^k z^k, \text{ and} \tag{15}$$

$$h_k = [u^k]H(u) = \frac{N_3^{t-k}}{D_t}(-1)^k z^k. \tag{16}$$

Theorem 6. *The generating function $[u^k]\,Total_t(z,u)$ for the number of partial Łukasiewicz paths (from right to left) of height at most $t \geq 0$, ending at height $k \geq 0$, is given by*

$$(-1)^k z^k \frac{N_1^{t-k} + N_2^{t-k} + N_3^{t-k}}{D_t}.$$

The generating function for the total number of partial Łukasiewicz paths (from right to left) of height at most $t \geq 2$ is given by

$$Total_t(z,1) = \frac{D_{t-2}}{D_t}.$$

Moreover, we have

$$Total_0(z,1) = \frac{1}{1-z}, \text{ and } Total_1(z,1) = \frac{1}{1-2z}.$$

For $t = 0,1,2,3$, the first terms of the series expansion of $Total_t(z,1)$ are

- $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9$,
- $1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + 64z^6 + 128z^7 + 256z^8 + 512z^9$,
- $1 + 2z + 5z^2 + 13z^3 + 34z^4 + 89z^5 + 233z^6 + 610z^7 + 1597z^8 + 4181z^9$,
- $1 + 2z + 5z^2 + 14z^3 + 41z^4 + 122z^5 + 365z^6 + 1094z^7 + 3281z^8 + 9842z^9$,

which correspond to A000012, A000079, A001519, A007051 in [16].

Note that the two series in Theorem 4 and Theorem 6 coincide when $t = 0,1$ since, in these cases, partial Łukasiewicz paths bounded by the height t are identical from left-to-right and from right-to-left.

6. The average height of Łukasiewicz paths

In this section, we prove that the average height of n-length partial Łukasiewicz paths (from left to right, and from right to left) ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$

6.1 The left-to-right model

We simplify the expressions given in Section 4 using the substitution $z = \frac{u}{(1+u)^2}$, first used in [2]. Then, we find

$$D_t = (-1)^{t+3}\frac{1-u^{t+3}}{(1-u)(1+u)^{t+2}},$$
$$N_2^t = \frac{(-1)^{t+1}u^2(1-u^t)}{(1+u)^{t+3}(1-u)},$$
$$N_3^t = \frac{(-1)^{t+1}u(1-u^{t+2})}{(1-u)(1+u)^{t+3}}.$$

We start with paths ending on the x-axis, as the formula is (slightly) different. The generating function of these paths bounded by t is

$$\mathscr{L}^{[\le t]} = \frac{D_t + N_2^t + N_3^t}{D_t} = \frac{(1+u) - u^{t+2}(1+u^2)}{1-u^{t+3}}.$$

Taking the limit when $t \to \infty$, we retrieve, as expected,

$$\mathscr{L}^{[\le\infty]} = 1 + u = \frac{1-\sqrt{1-4z}}{2z}.$$

Then, the generating function of paths of height at least $t+1$ is

$$\mathscr{L}^{[>t]} = \mathscr{L}^{[\le\infty]} - \mathscr{L}^{[\le t]} = \frac{u^{t+2}(1-u^2)}{1-u^{t+3}}.$$

We refer to [10] where a similar instance is worked out with an extensive amount of detail. For the average height, we have (before normalizing by the Catalan numbers) to compute

$$\sum_{t\ge0} t\cdot\mathscr{L}^{[=t]} = \sum_{t\ge0}\mathscr{L}^{[>t]} = \frac{1-u^2}{u}\sum_{t\ge3}\frac{u^t}{1-u^t}.$$

The goal is to find the local behavior of $u \sim 1$ since it translates to the local behavior of $z \sim \frac{1}{4}$. To find this, we set $u = e^{-t}$ and we use the Mellin transform. We do not need to do the actual computation, since we just cite the result from [10]. First, the factor is simple since we have

$$\frac{1-u^2}{u} \sim 2(1-u).$$

Since we only compute the leading term of the asymptotic expansion, we use

$$\sum_{t\ge1}\frac{u^t}{1-u^t} \sim -\frac{\log(1-u)}{1-u},$$

found in [10] for instance. So, we obtain

$$\sum_{t\ge0} t\cdot\mathscr{L}^{[=t]} \sim -2\log(1-u) \sim -2\log(2\sqrt{1-4z}) \sim -\log(1-4z).$$

Singularity analysis of generating functions (see [5]) allows us to translate this to the coefficients of z^n, with the result $\sim \frac{4^n}{n}$. For Catalan numbers, we have the well-known

$$\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Finally, the average height of paths ending on the x-axis behaves as

$$\frac{\frac{4^n}{n}}{\frac{4^n}{n^{3/2}\sqrt{\pi}}} = \sqrt{\pi n}.$$

For path ending at height $k \ge 1$, the generating function of paths bounded by t is

$$\mathscr{L}_k^{[\le t]} = (-1)^{k-1}\frac{N_3^{t-k+1} - N_2^{t-k} + N_2^{t-k+1}}{D_t} = u(1+u)^k\frac{1-u^{t-k+1}}{1-u^{t+3}}.$$

The limit for $t \to \infty$ is $u(1+u)^k$, and $\mathscr{L}_k^{[>t]} = \mathscr{L}_k^{[\le\infty]} - \mathscr{L}_k^{[\le t]}$ is

$$u(1+u)^k - u(1+u)^k \frac{1-u^{t-k+1}}{1-u^{t+3}} = u^{-k-1}(1+u)^k(1-u^{k+2})\frac{u^{t+3}}{1-u^{t+3}}.$$

Locally, we have $(u \sim 1)$

$$u^{-k-1}(1+u)^k(1-u^{k+2}) \sim 2^k(k+2)(1-u).$$

For the average height (the leading term only, before normalization), we compute

$$2^k(k+2)(1-u)\sum_{t\ge1}\frac{u^t}{1-u^t} \sim -2^k(k+2)\log(1-u),$$

and

$$[z^n]\big(-2^k(k+2)\log(1-u)\big) \sim 2^{k-1}(k+2)\frac{4^n}{n}.$$

For the total number of paths ending at height k, we have

$$[z^n]u(1+u)^k = \binom{2n-1+k}{n-1} - \binom{2n-1+k}{n-2} \sim \frac{4^n}{\sqrt{\pi}n^{3/2}}(k+2)2^{k-1},$$

and the average height (k fixed, $n \to \infty$) is asymptotic to

$$\frac{2^{k-1}(k+2)\frac{4^n}{n}}{\frac{4^n}{\sqrt{\pi}n^{3/2}}(k+2)2^{k-1}} = \sqrt{\pi n},$$

as before.

To compute the average height of paths with unspecified endpoints makes no sense in this model since the number of such paths of length n is infinite.

6.2 The right-to left model

We simplify the expressions given in Section 5 using the substitution $z = \frac{u}{(1+u)^2}$. We have to analyze

$$\mathscr{R}_k^{[\le t]} = (-1)^k z^k \frac{N_1^{t-k} + N_2^{t-k} + N_3^{t-k}}{D_t} = \frac{u^k}{(1+u)^{k-1}}\frac{1-u^{t+2-k}}{1-u^{t+3}}.$$

Taking the limit when $t \to \infty$, we obtain

$$\mathscr{R}_k^{[\le\infty]} = \frac{u^k}{(1+u)^{k-1}},$$

and

$$\mathscr{R}_k^{[>t]} = \mathscr{R}_k^{[\le\infty]} - \mathscr{R}_k^{[\le t]} = \frac{u^{t+2}(1-u^{k+1})}{(1+u)^{k-1}(1-u^{t+3})}.$$

For the average, we must compute

$$\frac{(k+1)(1-u)}{2^{k-1}}\sum_{t\ge1}\frac{u^t}{1-u^t},$$

where we took liberties about two missing terms, which do not influence the main term of the average height. As before, we get the asymptotic behavior

$$\frac{(k+1)}{2^k}\frac{4^n}{n}.$$

For the total number of paths ending at height k, we get

$$\frac{k+1}{2^k}\frac{4^n}{\sqrt{\pi}n^{3/2}},$$

and the average height (k fixed, $n \to \infty$) is again asymptotic to $\sqrt{\pi n}$.

Now we move to the Łukasiewicz paths with unspecified end and have to consider

$$\mathscr{R}^{\le t} = \frac{D_{t-2}}{D_t} = \frac{(1+u)^3(1-u)}{u^2}\frac{u^{t+3}}{1-u^{t+3}}.$$

The fact that the formula is different for small values of t and that we start the sum at 1 and not at 3, does not change the main term. We get

$$\mathscr{R}^{>t} \sim \frac{(1+u)^3(1-u)}{u^2} \sum_{t\geq 1} \frac{u^t}{1-u^t} \sim -8\log(1-u),$$

and the coefficient of z^n in it is asymptotic to

$$\frac{4^{n+1}}{n}.$$

For the total number of paths, we get the asymptotic formula

$$\frac{4^{n+1}}{\sqrt{\pi} n^{3/2}},$$

and the average height is again asymptotic to $\sqrt{\pi n}$.

7. Partial alternate Łukasiewicz paths

In this section, we study prefixes of alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do not contain two consecutive steps with the same direction (or equivalently, walks in the state diagram of Figure 3 without two consecutive arrows of the same color). We refer to Figure 4 for the state diagram associated with these paths. We denote here by f_k, g_k, and h_k the generating functions for the number of these paths (with respect to the length) ending at height k with an up-step, down-step, or a horizontal step, respectively.

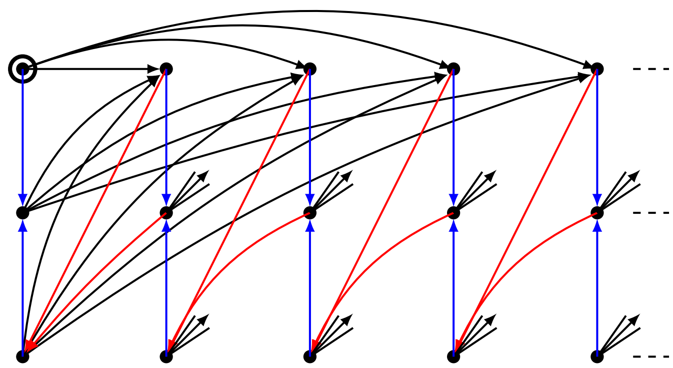

Figure 4: The state diagram for partial alternate Łukasiewicz paths. Black (resp. red, blue) arrows correspond to up-steps (resp. down-steps, horizontal steps).

We have the following equations:

$$\begin{aligned}
&f_0 = 1, \quad \text{and} \quad f_k = zf_0 + z\sum_{\ell=0}^{k-1} g_\ell + z\sum_{\ell=0}^{k-1} h_\ell, \quad k \geq 1,\\
&g_k = zf_{k+1} + zh_{k+1}, \quad k \geq 0,\\
&h_k = zf_k + zg_k, \quad k \geq 0.
\end{aligned} \tag{17}$$

Considering the bivariate generating functions

$$F(u,z) = \sum_{k\geq 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geq 0} u^k g_k(z), \text{ and } H(u,z) = \sum_{k\geq 0} u^k h_k(z),$$

and summing the recursions in (17), we obtain

$$\begin{aligned}
F(u) &= 1 + z\sum_{k\geq 1} u^k \left(1 + \sum_{\ell=0}^{k-1} g_\ell + \sum_{\ell=0}^{k-1} h_\ell\right)\\
&= 1 + \frac{zu}{1-u} + z\sum_{k\geq 0} \frac{u^{k+1}}{1-u} g_k + z\sum_{k\geq 0} \frac{u^{k+1}}{1-u} h_k
\end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{zu}{1-u}(1 + G(u) + H(u)),\\
G(u) &= \frac{z}{u}(F(u) + H(u) - 1 - H(0)),\\
H(u) &= z(F(u) + G(u)).
\end{aligned}$$

Solving these functional equations, we deduce

$$\begin{aligned}
F(u) &= \frac{uz^2(1+z)H(0) + 2uz^3 - u^2z + u^2 + z^2 - u}{u^2z^2 + 2uz^3 + u^2 + z^2 - u},\\
G(u) &= -\frac{z\left(H(0)(uz^2 + u - 1) + 2uz^2 + z\right)}{u^2z^2 + 2uz^3 + u^2 + z^2 - u},\\
H(u) &= \frac{z\left(zH(0)(uz - u + 1) - u^2z + u^2 - u\right)}{u^2z^2 + 2uz^3 + u^2 + z^2 - u}.
\end{aligned}$$

Now we apply the kernel method on $H(u)$. We have

$$H(u) = \frac{z\left(zH(0)(uz - u + 1) - u^2z + u^2 - u\right)}{(1+z^2)(u-s_1)(u-s_2)}, \tag{18}$$

with

$$\begin{aligned}
s_1 &= \frac{1 - 2z^3 + \sqrt{4z^6 - 4z^4 - 4z^3 - 4z^2 + 1}}{2z^2 + 2},\\
s_2 &= \frac{1 - 2z^3 - \sqrt{4z^6 - 4z^4 - 4z^3 - 4z^2 + 1}}{2z^2 + 2}.
\end{aligned}$$

In order to compute $H(0)$, it suffices to plug $u = s_2$ in the numerator of (18). Then, $H(0)$ satisfies $zH(0)(s_2z - s_2 + 1) - s_2^2z + s_2^2 - s_2 = 0$, which implies that

$$H(0) = \frac{s_2}{z}.$$

After this, and using $s_1s_2(1+z^2) = z^2$, we simplify of both, numerators and denominators, in $F(u)$, $G(u)$, $H(u)$ by factorizing them with $(u - s_2)$.

$$\begin{aligned}
F(u) &= \frac{(1-z)u - (1+z^2)s_1}{(1+z^2)(u-s_1)} = -\frac{z-1}{z^2+1} - \frac{s_1z\,(z+1)}{(u-s_1)\,(z^2+1)},\\
G(u) &= -\frac{z^2s_1^{-1} + 2z^3}{(1+z^2)(u-s_1)} = -\frac{z^2(2zs_1+1)}{(1+z^2)s_1(u-s_1)},\\
H(u) &= \frac{z((1-z)u - 1)}{(1+z^2)(u-s_1)} = \frac{(1-z)\,z}{z^2+1} - \frac{(zs_1 - s_1 + 1)\,z}{(u-s_1)\,(z^2+1)}.
\end{aligned}$$

Finally, we easily obtain

$$f_k = [u^k]F(u) = \frac{z(z+1)}{(1+z^2)s_1^k}, \tag{19}$$

$$g_k = [u^k]G(u) = \frac{z^2(2zs_1+1)}{(1+z^2)s_1^{k+2}}, \quad \text{and} \tag{20}$$

$$h_k = [u^k]H(u) = \frac{z((z-1)s_1+1)}{(1+z^2)s_1^{k+1}}. \tag{21}$$

Since the series expansion of s_1 does not have pretty coefficients, we cannot expect this from our final answers.

Theorem 7. *The bivariate generating function for the total number of partial alternate Lukasiewicz paths with respect to the length and the height of the end-point is given by*

$$\mathit{Total}(z,u) = \frac{s_1^2z^2 + s_1uz^2 + 2s_1z^3 + s_1^2 - s_1u + zs_1 + z^2}{(z^2+1)\,(-u+s_1)s_1}.$$

Moreover, we have

$$[u^0]\,\mathit{Total}(z,u) = \frac{s_1^2z^2 + 2s_1z^3 + s_1^2 + s_1z + z^2}{(z^2+1)s_1^2},$$

and for $k \geq 1$,

$$[u^k]\,\mathit{Total}(z,u) = \frac{z(2s_1^2z + 2s_1z^2 + s_1 + z)}{(z^2+1)s_1^{k+2}}.$$

Here are examples of the series expansions of $[u^k]\,Total(z,u)$ for $k = 0,1,2,3$ (leading terms):

- $1 + z + z^2 + 3z^3 + 5z^4 + 9z^5 + 19z^6 + 39z^7 + 81z^8 + 173z^9$,
- $z + 3z^2 + 5z^3 + 11z^4 + 25z^5 + 53z^6 + 115z^7 + 255z^8 + 565z^9$;
- $z + 3z^2 + 7z^3 + 19z^4 + 45z^5 + 105z^6 + 247z^7 + 575z^8 + 1333z^9$,
- $z + 3z^2 + 9z^3 + 27z^4 + 69z^5 + 177z^6 + 443z^7 + 1087z^8 + 2645z^9$,

which do not appear in [16]. The first terms of the series expansion of the generating function for the number of alternate Łukasiewicz paths are

$$1 + z + z^2 + 3z^3 + 5z^4 + 9z^5 + 19z^6 + 39z^7 + 81z^8 + 173z^9.$$

A singularity analysis of the generating function $[u^0]\,Total(z,u)$ gives

$$[z^n][u^0]\,Total(z,u) \sim \frac{\sqrt{-6a^6+4a^4+3a^3+2a^2}(a+1)2^n\left(-a^2+a+1\right)^n}{\sqrt{\pi}a^2\left(a^2+1\right)n^{\frac{3}{2}}},$$

with

$$a = \frac{1}{3} - \frac{2\cos\left(\frac{\arctan\left(\frac{15\sqrt{111}}{487}\right)}{6}+\frac{\pi}{6}\right)}{3} + \frac{2\sin\left(\frac{\arctan\left(\frac{15\sqrt{111}}{487}\right)}{6}+\frac{\pi}{6}\right)\sqrt{3}}{3}.$$

The reason that this constant appears, results from the singularity analysis. Indeed, one needs the solution closest to the origin of $4z^6 - 4z^4 - 4z^3 - 4z^2 + 1 = 0$, which is $a = 0.403031716762\ldots$. Maple provides the curious explicit version if one asks for a *simplification*.

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