ecajournal.haifa.ac.il
Enumerative Combinatorics and Applications
ECA **3:3** (2023) Article #S2R23
https://doi.org/10.54550/ECA2023V3S3R23

On the Number of Bracelets Whose Co-periods Divide a Given Integer

Petros Hadjicostas

Department of Mathematical Sciences, University of Nevada (Las Vegas), Las Vegas, Nevada 89154, USA
Email: peterhadji1@gmail.com

Received: March 22, 2023, **Accepted**: June 2, 2023, **Published**: June 9, 2023

Abstract: Inspired by a 1979 conjecture of former English professor Richard H. Reis, we give formulas for the number of symmetric q-ary necklaces and the number of q-ary bracelets over the color set $\{a_1, \dots, a_q\}$ with n_i beads of color a_i for $i = 1, \dots, q$ and co-periods dividing a nonnegative integer v. (The co-period of a necklace or a bracelet is its length divided by its period.) The proof of the formula for the bracelets utilizes an earlier formula of the author for the number of q-ary (fixed) necklaces over the color set $\{a_1, \dots, a_q\}$ with n_i beads of color a_i for $i = 1, \dots, q$ and co-periods dividing v.

Keywords: Bracelet; Co-period; Necklace; Palindromic word; Ramanujan sum; Symmetric necklace
2020 Mathematics Subject Classification: 05A15; 05A19

1. Introduction

The *period* of a (linear) string or word $(b_1, \dots, b_n)$ over an alphabet $\mathcal{A}$ is the smallest positive divisor d of n with the property $b_{kd+j} = b_j$ for $k = 0, \dots, \frac{n}{d} - 1$ and $j = 1, \dots, d$.

According to Hadjicostas and Zhang [7], a *Type I linear palindromic string* $(b_1, \dots, b_n)$ over $\mathcal{A}$ with length n is such that $b_i = b_{n+1-i}$ for $i = 1, \dots, n$, while a *Type II linear palindromic string* $(b_1, \dots, b_n)$ over $\mathcal{A}$ with length n is such that either $n = 1$, or $n \geq 2$ and $b_{i+1} = b_{n+1-i}$ for $i = 1, \dots, n-1$. (Only strings of period $d = 1$ are of both types.)

A *(fixed) necklace* over $\mathcal{A}$ with length n is the orbit of a word over $\mathcal{A}$ with length n under the action of the cyclic group C_n of order n. The orbit of a necklace is also called the equivalence class or the conjugacy class of the necklace. All the (linear) strings in the orbit of a necklace have the same period d, and we call d the *period of the necklace*, while we call the number $\frac{n}{d}$ the *co-period of the necklace*. If the period is $d = n$ (and the co-period is $n/d = 1$), then the necklace is called *aperiodic*.

According to Hadjicostas and Zhang [7], a *symmetric necklace* over $\mathcal{A}$ with length n is a necklace whose equivalence class (under C_n) contains at least one palindromic linear string either of Type I or of Type II. If the period d of a symmetric necklace is greater than 1, then Hadjicostas and Zhang [7] proved that its orbit contains exactly two palindromic strings of either type.

If we put the letters of a symmetric necklace on a circle, then the necklace will have *one or two axes of symmetry*. If n is odd (and greater than 1), then an axis of symmetry should pass through exactly one of the letters and through the middle of two consecutive and identical letters on the circle. If n is even, however, then an axis of symmetry may pass through two (not necessarily identical) antipodal letters on the circle or through the middle of two antipodal pairs (a, a) and (b, b) of consecutive and identical letters. (If a symmetric necklace has two axes of symmetry, then it cannot be aperiodic, in which case its co-period is greater than 1.)

If we assume some lexicographic order for the elements of an alphabet $\mathcal{A}$, a *Lyndon word* over $\mathcal{A}$ with length n is any (linear) word over $\mathcal{A}$ with length n that is strictly smaller than all the other elements in its orbit under C_n. In other words, a Lyndon word is strictly smaller than all its cyclic shifts.

A *free necklace or bracelet* over $\mathcal{A}$ with length n is the orbit of a word over $\mathcal{A}$ with length n under the action of the dihedral group D_n of order $2n$. Thus, each of the necklaces mentioned in the previous paragraphs is called a *fixed necklace* and comprises the set of all circular shifts of a word, while a free necklace (i.e., a bracelet) is the set of all circular shifts or reflections (with respect to an axis of symmetry) of a word. (Terminology, unfortunately, is not standard on this subject. For example, Zagaglia Salvi [20] calls a fixed necklace a 'cycle' and a bracelet a 'necklace'.)

All the (linear) strings in the orbit of a bracelet have the same period d, and we call d the *period of the bracelet*, while we call the number $\frac{n}{d}$ the *co-period of the bracelet*.

Note that a (fixed) necklace that is symmetric around an axis is also a bracelet (free necklace) that is symmetric around the same axis, and vice versa. Thus, we may interchangeably use the terms *symmetric necklace* and *symmetric bracelet*.

If $d > 1$ is the period of (linear) string $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\lambda}$ is a palindromic string, then $(\lambda_1, \ldots, \lambda_d)$ is a palindromic string of the same type (either I or II). This implies that the two palindromic strings $\boldsymbol{\lambda}^A = (\lambda_1^A, \ldots, \lambda_n^A)$ and $\boldsymbol{\lambda}^B = (\lambda_1^B, \ldots, \lambda_n^B)$ in the equivalence class of a symmetric necklace of length n and period $d > 1$ both have period d, and the sets of cyclic shifts of the two substrings $(\lambda_1^A, \ldots, \lambda_d^A)$ and $(\lambda_1^B, \ldots, \lambda_d^B)$ are identical and correspond to an aperiodic symmetric necklace of length d.

Given $q \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}_{\geq 0}$, and $\mathbf{n} = (n_1, \ldots, n_q) \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| := \sum_{i=1}^q n_i > 0$, denote by $M_v(\mathbf{n})$ the number of q-ary (fixed) necklaces of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$ and co-periods dividing v.

Hence, for $v = 1$, $M_1(\mathbf{n})$ is the number of aperiodic q-ary (fixed) necklaces of length $|\mathbf{n}|$ with n_i beads of color a_i for $i = 1, \ldots, q$. In addition, $M_1(\mathbf{n})$ is also the number of Lyndon words over the q-ary alphabet $\{a_1, \ldots, a_q\}$ of length $|\mathbf{n}|$ with n_i symbols a_i for $i = 1, \ldots, q$ (assuming some lexicographic order, say $a_1 < a_2 < \cdots < a_q$).

It is well-known (e.g., see Moree [9]) that

$$M_1(\mathbf{n}) = \frac{1}{|\mathbf{n}|} \sum_{d |\, \gcd(\mathbf{n})} \mu(d) \binom{\frac{|\mathbf{n}|}{d}}{\frac{\mathbf{n}}{d}}, \tag{1}$$

where $\mu(d)$ is the *Möbius function* evaluated at the positive integer d, and the sum above ranges over all positive divisors d of $\gcd(\mathbf{n}) = \gcd(n_1, \ldots, n_q)$. Here, for $\mathbf{m} = (m_1, \ldots, m_q) \in (\mathbb{Z}_{\geq 0})^q$, the quantity $\binom{|\mathbf{m}|}{\mathbf{m}} = (\sum_{i=1}^q m_i)!/\prod_{i=1}^q m_i!$ is a multinomial coefficient.

Witt [19] proved the identity

$$1 - \sum_{i=1}^q x_i = \prod_{\mathbf{n} \neq \mathbf{0}} \left(1 - \prod_{i=1}^q x_i^{n_i}\right)^{M_1(\mathbf{n})}. \tag{2}$$

The outer product on the RHS of Eq. (2) is over the set

$$\{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^q : |\mathbf{n}| > 0\}.$$

See also da Costa and Zimmermann [2] and Sherman [14]. These authors study the *Sherman identity*, which is a non-trivial special case of the *Feynman identity* for graphs and is related to the Ising model in Physics. They also examine how the Sherman identity is related to the above Witt identity (2).

Furthermore, if $v = 0$, $M_0(\mathbf{n})$ is the total number of q-ary (fixed) necklaces over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$, and it is known that

$$M_0(\mathbf{n}) = \sum_{d |\, \gcd(\mathbf{n})} M_1\left(\frac{\mathbf{n}}{d}\right) = \frac{1}{|\mathbf{n}|} \sum_{d |\, \gcd(\mathbf{n})} \phi(d) \binom{\frac{|\mathbf{n}|}{d}}{\frac{\mathbf{n}}{d}}, \tag{3}$$

where $\phi(d)$ is Euler's *totient function* evaluated at the positive integer d. See, for example, Brysiewicz [1, Section 2] and Reutenauer [13, Corollary 7.3, p. 157].

Using Eqs. (2) and (3), Hadjicostas [6] proved that

$$\prod_{d \geq 1} \left(1 - \sum_{i=1}^q x_i^d\right) = \prod_{\mathbf{n} \neq \mathbf{0}} \left(1 - \prod_{i=1}^q x_i^{n_i}\right)^{M_0(\mathbf{n})}. \tag{4}$$

For a general $v \in \mathbb{Z}_{\geq 0}$, Hadjicostas [6] provided an explicit formula for $M_v(\mathbf{n})$ and generalized Eqs. (2)–(4) using *Ramanujan sums*. The Ramanujan sum $c_n(m)$ is the sum of the m^{th} powers of the n^{th} primitive roots of unity:

$$c_n(m) = \sum_{\substack{1 \leq s \leq n \\ \gcd(s,n)=1}} \left(e^{\frac{s}{n}(2\pi\sqrt{-1})}\right)^m \quad \text{for } n \in \mathbb{Z}_{>0} \text{ and } m \in \mathbb{Z}_{\geq 0}.$$

These sums were originally defined by Kluyver [8] and independently (later) by Ramanujan [11]. Various values of the Ramanujan sums can be found at the OEIS [16] sequences A054532, A054533, A054534, and A054535.

The Ramanujan sums satisfy the following properties (e.g., see Wintner [18]):

$$c_n(m) = \sum_{d |\, \gcd(n,m)} \mu\left(\frac{n}{d}\right) d \quad \text{and} \quad \sum_{d|n} c_d(m) = nI(n|m). \tag{5}$$

Here (and throughout the paper), $I(\text{condition}) = 1$, if the 'condition' holds, and 0 otherwise. Note that $c_n(0) = \phi(n)$ and $c_n(1) = \mu(n)$.

More specifically, Hadjicostas [6] proved Theorem 1.1 below. Special cases of the theorem were proved by Elashvili *et al.* [3], Fredman [4], and Panyushev [10]. (In Eqs. (7) and (8) below, we use the principal branch of the logarithm.)

Theorem 1.1. *For $q \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}_{\geq 0}$, and $\mathbf{n} = (n_1, \ldots, n_q) \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| = \sum_{i=1}^q n_i > 0$, we have*

$$M_v(\mathbf{n}) = \sum_{d|\gcd(\mathbf{n},v)} M_1\left(\frac{\mathbf{n}}{d}\right) = \frac{1}{|\mathbf{n}|} \sum_{d|\gcd(\mathbf{n})} c_d(v) \binom{\frac{|\mathbf{n}|}{d}}{\frac{\mathbf{n}}{d}}. \tag{6}$$

In addition, for all $x_1, \ldots, x_q \in \mathbb{C}$ with $\sum_{i=1}^q |x_i| < 1$,

$$\sum_{\mathbf{n} \neq \mathbf{0}} M_v(\mathbf{n}) \prod_{i=1}^q x_i^{n_i} = -\sum_{m \geq 1} \frac{c_m(v)}{m} \log\left(1 - \sum_{i=1}^q x_i^m\right), \tag{7}$$

$$\sum_{\mathbf{n} \neq \mathbf{0}} |\mathbf{n}| M_v(\mathbf{n}) \prod_{i=1}^q x_i^{n_i} = \sum_{m \geq 1} c_m(v) \frac{\sum_{j=1}^q x_j^m}{1 - \sum_{j=1}^q x_j^m},$$

and

$$\prod_{\mathbf{n} \neq \mathbf{0}} \left(1 - \prod_{i=1}^q x_i^{n_i}\right)^{M_v(\mathbf{n})} = \prod_{d|v} \left(1 - \sum_{i=1}^q x_i^d\right). \tag{8}$$

For $q \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}_{\geq 0}$, and $\mathbf{n} = (n_1, \ldots, n_q) \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| = \sum_{i=1}^q n_i > 0$, let $R_v(\mathbf{n})$ and $B_v(\mathbf{n})$ be the numbers of symmetric q-ary necklaces and q-ary bracelets, respectively, of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$ and co-periods dividing v.

In particular, $R_0(\mathbf{n})$ counts symmetric q-ary necklaces while $B_0(\mathbf{n})$ counts q-ary bracelets of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$. On the other hand, $R_1(\mathbf{n})$ counts aperiodic symmetric q-ary necklaces while $B_1(\mathbf{n})$ counts aperiodic q-ary bracelets of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$.

In this paper, inspired by a conjecture in Reis [12] *, we derive explicit formulas for $R_v(\mathbf{n})$ and $B_v(\mathbf{n})$. (Reis's [12] original conjecture was for the case $v = 0$.) See Theorem 1.2 below. The proof of the theorem appears in Section 3.

For $q \in \mathbb{Z}_{>0}$ and $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| > 0$, let

$$J(\mathbf{n}) := \begin{cases} 1, & \text{if the list } \mathbf{n} \text{ has at most two} \\ & \text{odd components;} \\ 0, & \text{otherwise.} \end{cases}$$

(For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ be the *floor* of x, i.e., the greatest integer less than or equal to x.)

Theorem 1.2. *For $q \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}_{\geq 0}$, and $\mathbf{n} = (n_1, \ldots, n_q) \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| = \sum_{i=1}^q n_i > 0$, we have*

$$R_0(\mathbf{n}) = J(\mathbf{n}) \binom{\sum_{i=1}^q \lfloor \frac{n_i}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor, \ldots, \lfloor \frac{n_q}{2} \rfloor}, \tag{9}$$

$$R_1(\mathbf{n}) = \sum_{d|\gcd(\mathbf{n})} \mu(d) R_0\left(\frac{\mathbf{n}}{d}\right), \tag{10}$$

$$R_v(\mathbf{n}) = \sum_{d|\gcd(\mathbf{n},v)} R_1\left(\frac{\mathbf{n}}{d}\right), \tag{11}$$

$$B_v(\mathbf{n}) = \frac{M_v(\mathbf{n}) + R_v(\mathbf{n})}{2}, \tag{12}$$

and

$$B_v(\mathbf{n}) = \sum_{d|\gcd(\mathbf{n},v)} B_1\left(\frac{\mathbf{n}}{d}\right). \tag{13}$$

*Richard H. Reis (1930–2008) was a Professor of English at Southeastern Massachusetts University in N. Darmouth, USA. He was a friend of Professor Hansraj Gupta with whom he was discussing the enumeration of necklaces and bracelets. See Gupta [5] and Shevelev [15].

Corollary 1.1. *Consider the notation in Theorem 1.2. If the list* $\mathbf{n}$ *contains at least three odd components, then*

$$J(\mathbf{n}) = R_0(\mathbf{n}) = R_1(\mathbf{n}) = R_v(\mathbf{n}) = 0,$$

and the number of q*-ary bracelets of length* $|\mathbf{n}|$ *over the color set* $\{a_1, \ldots, a_q\}$ *with* n_i *beads of color* a_i *for* $i = 1, \ldots, q$ *and co-periods dividing* v *is equal to*

$$B_v(\mathbf{n}) = \frac{1}{2} M_v(\mathbf{n}).$$

In other words, in such a case, all such bracelets are not symmetric.

Remark 1.1. If $q = 1$, then

$$R_0(n_1) = J(n_1) = 1 \quad \text{and} \quad R_1(n_1) = \sum_{d | n_1} \mu(d) = I(n_1 = 1).$$

Therefore, the number of 1-ary symmetric necklaces/bracelets of length n_1 with n_1 beads of color a_1 and co-period (in this case, $n_1/1 = n_1$) dividing v is

$$R_v(n_1) = \sum_{d | \gcd(n_1, v)} I\left(\frac{n_1}{d} = 1\right) = I(n_1 | v).$$

Using Eqs. (5) and (6), we find that

$$M_v(n_1) = I(n_1 | v),$$

which is the number of 1-ary (fixed) necklaces over the color set $\{a_1\}$ with n_1 beads of color a_1 whose co-period (in this case, n_1) divides v. It then follows from Eq. (12) that the number of 1-ary bracelets over the color set $\{a_1\}$ with n_1 beads of color a_1 whose co-period (in this case, n_1) divides v is

$$B_v(n_1) = \frac{1}{2}(M_v(n_1) + R_v(n_1)) = I(n_1 | v).$$

Remark 1.2. For $q = 2$ and $0 \leq k \leq n$, we have that

$$R_0(k, n-k) = \text{A119963}(n, k) = \binom{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{n-k}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor}$$

in the OEIS [16]. This is the number of symmetric cyclic compositions of n into k parts; see Hadjicostas and Zhang [7, Section 1]. This result was originally proved by Sommerville [17, pp. 301–304].

2. Examples

The three examples below illustrate Theorems 1.1 and 1.2. We denote by $S^{\mathrm{I}}(\mathbf{n})$ and $S^{\mathrm{II}}(\mathbf{n})$ the numbers of Type I and Type II palindromic q-ary strings, respectively, of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$. In the first example, we underline those linear strings that are palindromic of either type. In the second example, there are no palindromic strings.

Example 2.1. *Suppose* $q = 2$*,* $n_1 = 4$*,* $n_2 = 2$*, and* $v = 2$*. We enumerate all symmetric necklaces/bracelets and all bracelets of length* $n_1 + n_2 = 6$ *with* $n_1 = 4$ *beads of color* B *and* $n_2 = 2$ *beads of color* W *with co-period that divides* $v = 2$ *(i.e., co-period either 1 or 2).*

By Theorem 1.1, the total number of (fixed) necklaces of length $n_1 + n_2 = 6$ *with* $n_1 = 4$ *beads of color* B *and* $n_2 = 2$ *beads of color* W *with co-period that divides* $v = 2$ *is*

$$M_2(4,2) = \frac{1}{6} \sum_{d | \gcd(4,2)} c_d(2) \binom{\frac{6}{d}}{\frac{4}{d}, \frac{2}{d}} = \frac{1}{6}\left(c_1(2)\binom{6}{4,2} + c_2(2)\binom{3}{2,1}\right) = \frac{1}{6}\left((1)(15) + (1)(3)\right) = 3.$$

Indeed, here are the equivalence classes (or conjugacy classes) of the above necklaces:

$$\begin{aligned} C_1 &= \{BBWBBW, \underline{BWBBWB}, \underline{WBBWBB}\}, \\ C_2 &= \{BBBBWW, BBBWWB, \underline{BBWWBB}, BWWBBB, WWBBBB, \underline{WBBBBW}\}, \\ C_3 &= \{BBBWBW, \underline{BBWBWB}, BWBWBB, WBWBBB, \underline{BWBBBW}, WBBBWB\}. \end{aligned}$$

Necklace C_1 *has period 3 and co-period 2, while each of the necklaces* C_2 *and* C_3 *has period 6 and co-period 1.*

Note that $J(4,2) = 1 = J(2,1)$. *Therefore, by Theorem 1.2,*

$$R_0(4,2) = \binom{3}{2,1} = 3 \quad \textit{and} \quad R_0(2,1) = \binom{1}{1,0} = 1.$$

In addition,

$$R_1(4,2) = \sum_{d|\gcd(4,2)} \mu(d) R_0\left(\frac{4}{d}, \frac{2}{d}\right) = \mu(1)R_0(4,2) + \mu(2)R_0(2,1) = (1)(3) + (-1)(1) = 2 \quad \textit{and}$$

$$R_1(2,1) = \sum_{d|\gcd(2,1)} \mu(d) R_0\left(\frac{2}{d}, \frac{1}{d}\right) = \mu(1)R_0(2,1) = (1)(1) = 1.$$

Therefore, the number of symmetric necklaces/bracelets of length $n_1 + n_2 = 6$ *with* $n_1 = 4$ *beads of color B and* $n_2 = 2$ *beads of color W with co-period that divides* $v = 2$ *is*

$$R_2(4,2) = \sum_{d|\gcd(4,2,2)} R_1\left(\frac{4}{d}, \frac{2}{d}\right) = R_1(4,2) + R_1(2,1) = 2 + 1 = 3.$$

Indeed, all three necklaces above are symmetric:

- *Necklace* C_1 *contains the 'palindromic' words BWBBWB and WBBWBB, which when placed on a circle have an axis of symmetry through the two W's and another one through the middle points of the two pairs of consecutive B's (The first palindromic string is of Type I while the second one is of Type II.)*
- *Necklace* C_2 *contains the 'palindromic' words BBWWBB and WBBBBW, which when placed on a circle have a single axis of symmetry that passes at a point between two consecutive B's and a point between two consecutive W's. (Both palindromic strings are of Type I.)*
- *Necklace* C_3 *contains the 'palindromic' words BBWBWB and BWBBBW, which when placed on a circle have a single axis of symmetry that passes through two B's. (Both palindromic strings are of Type II.)*

By Theorem 1.2, the number of bracelets of length $n_1 + n_2 = 6$ *with* $n_1 = 4$ *beads of color B and* $n_2 = 2$ *beads of color W with co-period that divides* $v = 2$ *is*

$$B_2(4,2) = \frac{1}{2}(M_2(4,2) + R_2(4,2)) = \frac{3+3}{2} = 3.$$

Indeed, the equivalence classes of these three bracelets are identical to the equivalence classes of necklaces C_1, C_2, *and* C_3 *above.*

Finally, from the above discussion, we see that $S^{\mathrm{I}}(4,2) = 3 = S^{\mathrm{II}}(4,2)$.

Example 2.2. *Suppose* $q = 3$, $n_1 = n_2 = n_3 = 1$, *and* $v = 2$. *We enumerate all symmetric necklaces/bracelets and all bracelets of length* $n_1 + n_2 + n_3 = 3$ *with* $n_1 = 1$ *bead of color B,* $n_2 = 1$ *bead of color W, and* $n_3 = 1$ *bead of color Y with co-period that divides* $v = 2$ *(i.e., co-period either 1 or 2).*

By Theorem 1.1, the total number of (fixed) necklaces of length $n_1 + n_2 + n_3 = 3$ *with* $n_1 = 1$ *beads of color B,* $n_2 = 1$ *bead of color W, and* $n_3 = 1$ *bead of color Y with co-period that divides* $v = 2$ *is*

$$M_2(1,1,1) = \frac{1}{3} \sum_{d|\gcd(1,1,1)} c_d(2) \binom{\frac{3}{d}}{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}} = \frac{1}{3} c_1(2) \binom{3}{1,1,1} = \frac{(1)(6)}{3} = 2.$$

Indeed, here are the equivalence classes (or conjugacy classes) of the above necklaces:

$$D_1 = \{BWY, WYB, YBW\} \quad \textit{and} \quad D_2 = \{YWB, BYW, WBY\}.$$

Each of the necklaces D_1 *and* D_2 *has period 3 and co-period 1.*

By Theorem 1.2,

$$R_1(1,1,1) = \sum_{d|\gcd(1,1,1)} \mu(d) R_0\left(\frac{1}{d}, \frac{1}{d}, \frac{1}{d}\right) = R_0(1,1,1) = J(1,1,1)\binom{0}{0,0,0} = 0.$$

Therefore, the number of symmetric necklaces/bracelets of length $n_1 + n_2 + n_3 = 3$ *with* $n_1 = 1$ *bead of color B,* $n_2 = 1$ *bead of color W, and* $n_3 = 1$ *bead of color Y with co-period that divides* $v = 2$ *is*

$$R_2(1,1,1) = \sum_{d|\gcd(1,1,1,2)} R_1\left(\frac{1}{d}, \frac{1}{d}, \frac{1}{d}\right) = R_1(1,1,1) = 0.$$

Table 1: Values of the quantities $M_v(\mathbf{n})$, $R_v(\mathbf{n})$, and $B_v(\mathbf{n})$ for $\mathbf{n} = (6, 12, 6)$ and $v = 0, 1, \ldots, 5$

v	$M_v(\mathbf{n})$	$R_v(\mathbf{n})$	$B_v(\mathbf{n})$
0	104110812	18480	52064646
1	104109219	18449	52063834
2	104110758	18468	52064613
3	104109270	18460	52063865
4	104110758	18468	52064613
5	104109219	18449	52063834

By Theorem 1.2, the number of bracelets of length $n_1 + n_2 + n_3 = 3$ with $n_1 = 1$ bead of color B, $n_2 = 1$ bead of color W, and $n_3 = 1$ bead of color Y with co-period that divides $v = 2$ is

$$B_2(1, 1, 1) = \frac{1}{2}(M_2(1, 1, 1) + R_2(1, 1, 1)) = \frac{2 + 0}{2} = 1.$$

Indeed, the equivalence class of the single bracelet is

$$D = \{BWY, WYB, YBW, YWB, BYW, WBY\}.$$

It has period 3 and co-period 1.

Finally, from the above discussion, we see that $S^{\mathrm{I}}(1, 1, 1) = 0 = S^{\mathrm{II}}(1, 1, 1)$.

Example 2.3. *Suppose $q = 3$ and $\mathbf{n} = (n_1, n_2, n_3) = (6, 12, 6)$. In Table 1, we list the values of $M_v(\mathbf{n})$, $R_v(\mathbf{n})$, and $B_v(\mathbf{n})$ for $v = 0, 1, \ldots, 5$. Because of Eqs. (6), (11), and (13), and the fact that*

$$\gcd(\mathbf{n}, 6m + r) = \gcd(6, 12, 6, 6m + r) = \gcd(6, r) \quad \textit{for } m \in \mathbb{Z}_{\geq 0} \textit{ and } r \in \{0, 1, \ldots, 5\},$$

the three enumerative quantities are periodic with period 6.

3. Proof of Theorem 1.2

Let $q \in \mathbb{Z}_{>0}$, $v \in \mathbb{Z}_{\geq 0}$, and $\mathbf{n} = (n_1, \ldots, n_q) \in (\mathbb{Z}_{\geq 0})^q$ with $|\mathbf{n}| = \sum_{i=1}^q n_i > 0$. In this section, we prove Theorem 1.2. We divide the proof of the theorem into three steps:

- In Step 1, we prove Eq. (9) about $R_0(\mathbf{n})$.
- In Step 2, we prove
  - (i) Eq. (10) that expresses the quantity $R_1(\mathbf{n})$ in terms of the quantity $R_0(\mathbf{m})$ and
  - (ii) Eq. (11) that expresses the quantity $R_v(\mathbf{n})$ in terms of the quantity $R_1(\mathbf{m})$ (with $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^q$).
- In Step 3, we prove Eqs. (12) and (13) regarding $B_v(\mathbf{n})$.

Step 1. Recall that $R_0(\mathbf{n})$ is the number of symmetric q-ary necklaces of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$. Here we prove Eq. (9), which is the formula about $R_0(\mathbf{n})$ conjectured in Reis [12].

It follows from the general results † in Sections 1 and 2 of Hadjicostas and Zhang [7] that $R_0(\mathbf{n})$ is the average of the number of Type I and Type II palindromic q-ary strings of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$. As before, denote these numbers by $S^{\mathrm{I}}(\mathbf{n})$ and $S^{\mathrm{II}}(\mathbf{n})$, respectively.

(i) Assume first $|\mathbf{n}|$ is even. In a linear palindromic q-ary string of Type I of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$, all the n_i's must be even. In such a case,

$$S^{\mathrm{I}}(\mathbf{n}) = \binom{\sum_{i=1}^q \frac{n_i}{2}}{\frac{n_1}{2}, \ldots, \frac{n_q}{2}}.$$

In all the other cases, when at least one n_i is odd, no such linear q-ary string of Type I of length $|\mathbf{n}|$ exists (i.e., $S^{\mathrm{I}}(\mathbf{n}) = 0$).

† As stated in Hadjicostas and Zhang [7], even though that paper deals with cyclic compositions of positive integers, the results in Lemmas 2.2–2.4 in that paper apply to general palindromic strings (of both types).

In a linear palindromic q-ary string $(b_1, b_2, \ldots, b_{|\mathbf{n}|})$ of Type II of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$, we consider two cases: (A) beads b_1 and $b_{(|\mathbf{n}|/2)+1}$ are of the same color, and (B) beads b_1 and $b_{(|\mathbf{n}|/2)+1}$ are of a different color.

In case (A), we may move bead b_1 between beads $b_{|\mathbf{n}|/2}$ and $b_{(|\mathbf{n}|/2)+1}$ and get a palindromic string of Type I (and this process can be reversed). In such a case, all the n_i's are even, and

$$S^{\mathrm{II}}(\mathbf{n}) = \binom{\sum_{i=1}^{q} \frac{n_i}{2}}{\frac{n_1}{2}, \ldots, \frac{n_q}{2}}.$$

Hence, in the case all the n_i's are even, we have $S^{\mathrm{I}}(\mathbf{n}) = S^{\mathrm{II}}(\mathbf{n})$.

In case (B), there are exactly two of the n_i's, say n_r and n_s with $1 \leq r < s \leq q$, that are odd, and the rest are even. Since n_r and n_s are odd, only the colors a_r and a_s can appear in beads b_1 and $b_{(|\mathbf{n}|/2)+1}$, respectively, or vice versa. Removing those two beads, we get a palindromic string of Type I of length $|\mathbf{n}| - 2$, where color a_i (with $i \in \{1, \ldots, q\} - \{r, s\}$) appears n_i times, while colors a_r and a_s appear $n_r - 1$ and $n_s - 1$ times, respectively. Since

$$\left\lfloor \frac{m}{2} \right\rfloor = \frac{m-1}{2} \quad \text{for } m \in \{n_r, n_s\},$$

in case (B), we get

$$S^{\mathrm{II}}(\mathbf{n}) = 2\binom{\sum_{i=1}^{q} \lfloor \frac{n_i}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor, \ldots, \lfloor \frac{n_q}{2} \rfloor}.$$

In the case of exactly two of the n_i's being odd, we have $S^{\mathrm{I}}(\mathbf{n}) = 0$.

In the above two situations (when all the n_i's are even or when exactly two n_i's are odd), we have $J(\mathbf{n}) = 1$ and

$$R_0(\mathbf{n}) = \frac{S^{\mathrm{I}}(\mathbf{n}) + S^{\mathrm{II}}(\mathbf{n})}{2} = \binom{\sum_{i=1}^{q} \lfloor \frac{n_i}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor, \ldots, \lfloor \frac{n_q}{2} \rfloor}. \tag{14}$$

In all the other cases of part (i), we have $S^{\mathrm{I}}(\mathbf{n}) = S^{\mathrm{II}}(\mathbf{n}) = 0$, $R_0(\mathbf{n}) = (0+0)/2$, and $J(\mathbf{n}) = 0$.

Putting together all of the above situations, we see that we proved Eq. (9) when $|\mathbf{n}|$ is even.

(ii) Next, assume $|\mathbf{n}|$ is odd. In a linear palindromic q-ary string of either Type I or Type II of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$, exactly one n_i is odd while the rest are even. In such a case, $J(\mathbf{n}) = 1$ and Eqs. (14) hold again because

$$S^{\mathrm{I}}(\mathbf{n}) = \binom{\sum_{i=1}^{q} \lfloor \frac{n_i}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor, \ldots, \lfloor \frac{n_q}{2} \rfloor} = S^{\mathrm{II}}(\mathbf{n}).$$

In all the other cases of part (ii), we have $S^{\mathrm{I}}(\mathbf{n}) = S^{\mathrm{II}}(\mathbf{n}) = 0$, $R_0(\mathbf{n}) = (0+0)/2$, and $J(\mathbf{n}) = 0$.

Putting together all of the above situations, we see that we proved Eq. (9) when $|\mathbf{n}|$ is odd.

Step 2. Recall that $R_v(\mathbf{n})$ is the number of symmetric q-ary necklaces of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$ and co-periods dividing v. Here we prove Eqs. (10) and (11).

According to Lemma 2.2 in Hadjicostas and Zhang [7], if d is the period of the (linear) string $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)$ (where $d, K \in \mathbb{Z}_{>0}$ with $d|K$) and $\boldsymbol{\lambda}$ is palindromic of either Type I or Type II, then the (linear) string $(\lambda_1, \ldots, \lambda_d)$ is also palindromic of the same type. In addition, Lemma 2.4 in Hadjicostas and Zhang [7], which is quite general and deals with cyclic shifts of a string, implies that the equivalence class of a symmetric necklace contains exactly two palindromic linear strings of either type when the period is $d > 1$. (If $d = 1$, then the equivalence class of the symmetric necklace has only one linear string that is trivially palindromic of both types.) We use both of these lemmas below.

Denote by $R_0(\mathbf{n}; h)$ the number of symmetric q-ary necklaces of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$ with co-period h. In such a case, h must divide each n_i, and each of the palindromic representatives in the equivalence class of such a (symmetric) necklace can be obtained by repeating h times a representative in the equivalence class of a symmetric aperiodic necklace of length $|\mathbf{n}|/h$ with n_i/h beads of color a_i for $i = 1, \ldots, q$. It can be easily proved that the above process can be reversed, and thus

$$R_0(\mathbf{n}; h) = R_0\left(\frac{\mathbf{n}}{h}; 1\right) = R_1\left(\frac{\mathbf{n}}{h}\right) \quad \text{for each} \quad h|\gcd(\mathbf{n}). \tag{15}$$

We then have

$$R_0(\mathbf{n}) = \sum_{h|\gcd(\mathbf{n})} R_0(\mathbf{n}; h) = \sum_{h|\gcd(\mathbf{n})} R_1\left(\frac{\mathbf{n}}{h}\right).$$

By Möbius inversion, we get Eq. (10).

Using the definitions of $R_0(\mathbf{n}; h)$ and $R_v(\mathbf{n})$ and Eqs. (15), we obtain

$$R_v(\mathbf{n}) = \sum_{h \mid \gcd(\mathbf{n}, v)} R_0(\mathbf{n}; h) = \sum_{h \mid \gcd(\mathbf{n}, v)} R_1\left(\frac{\mathbf{n}}{h}\right),$$

which is Eq. (11).

Step 3. Here we prove Eqs. (12) and (13)) regarding $B_v(\mathbf{n})$, which counts q-ary bracelets of length $|\mathbf{n}|$ over the color set $\{a_1, \ldots, a_q\}$ with n_i beads of color a_i for $i = 1, \ldots, q$ and co-periods dividing v.

Bracelets are divided into symmetric and non-symmetric ones. Each non-symmetric bracelet corresponds to two different necklaces, each of which can be obtained from the other by changing direction (clockwise to counterclockwise, or vice versa). Thus,

$$B_v(\mathbf{n}) = R_v(\mathbf{n}) + \frac{M_v(\mathbf{n}) - R_v(\mathbf{n})}{2} = \frac{M_v(\mathbf{n}) + R_v(\mathbf{n})}{2}.$$

We thus proved Eq. (12).

Eq. (13) follows immediately from Eqs. (6), (11), and (12).

Acknowledgements

We would like to thank an anonymous referee whose comments and suggestions help us improve the presentation of the paper.

References

[1] T. Brysiewicz, *Necklaces count polynomial parametric osculants*, J. Symbolic Comput. 103 (2021), 95–107.

[2] G. A. T. F. da Costa and G. A. Zimmermann, *An analogue of the Witt identity*, Pacific J. Math. 263(2) (2013), 475–494.

[3] A. Elashvili, M. Jibladze, and D. Pataraia, *Combinatorics of necklaces and "Hermite reciprocity"*, J. Algebraic Combin. 10(2) (1999), 173–188.

[4] M. L. Fredman, *A symmetry relationship for a class of partitions*, J. Combin. Theory Ser. A. 18 (1975), 199–202.

[5] H. Gupta, *Enumeration of incongruent cyclic k-gons*, Indian J. Pure Appl. Math. 10(8) (1979), 964–999.

[6] P. Hadjicostas, *On the number of necklaces whose co-periods divide a given integer*, to appear in Ramanujan J. (2023). Available at `https://doi.org/10.1007/s11139-023-00723-3`.

[7] P. Hadjicostas and L. Zhang, *Sommerville's symmetrical cyclic compositions of a positive integer with parts avoiding multiples of an integer*, Fibonacci Quart. 55(1) (2017), 54–73.

[8] J. C. Kluyver, *Some formulae concerning the integers less than n and prime to n*, in: KNAW, Proceedings, 9 I, 1906, Amsterdam, pp. 408–414, 1906.

[9] P. Moree, *The formal series Witt transform*, Discrete Math. 295(1–3) (2005), 143–160.

[10] D. I. Panyushev, *Fredman's reciprocity, invariants of abelian groups, and the permanent of the Cayley table*, J. Algebraic Combin. 33(1) (2011), 111–125.

[11] S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Trans. Camb. Phil. Soc. 22(13) (1918), 259–276.

[12] R. H. Reis, *A formula for $C(T)$ in Gupta's paper*, Indian J. Pure Appl. Math. 10(8) (1979), 1000–1001.

[13] C. Reutenauer, *Free Lie Algebras*, London Mathematical Society Monographs, New Series 7, Oxford University Press, New York, 1993.

[14] S. Sherman, *Combinatorial aspects of the Ising model for ferromagnetism. II. An analogue of the Witt identity*, Bull. Amer. Math. Soc. 68 (1962), 225–229.

[15] V. S. Shevelev, *Necklaces and convex k-gons*, Indian J. Pure Appl. Math. 35(5) (2004), 629–638.

[16] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available electronically at `http://oeis.org/`.

[17] D. M. Y. Sommerville, *On certain periodic properties of cyclic compositions of numbers*, Proc. London Math. Soc. S2-7 (1909), 263–313.

[18] A. Wintner, *On a statistics of the Ramanujan sums*, Amer. J. Math. 64 (1942), 106–114.

[19] E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. 177 (1937), 152–160.

[20] N. Zagaglia Salvi, *Ordered partitions and colourings of cycles and necklaces*, Bull. Inst. Combin. Appl. 27 (1999), 37–40.