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Interview with Peter Paule

Toufik Mansour

Peter Paule obtained a Ph.D. from the University of Vienna in 1982 under the supervision of Johann Cigler. In 1996 he earned a habilitation from Johannes Kepler University. Since 1990 he has held a faculty position at the Research Institute for Symbolic Computation (RISC) of the Johannes Kepler University of Linz and is currently head of the Institute. Places he has held visiting positions include Pennsylvania State University, the University of Waterloo, the University of Witwatersrand, and the Université Marne-la-Vallée, etc. His main research interests include symbolic computation and its connections to enumerative combinatorics, number theory, and special functions. He is an editorial board member of the Journal of Symbolic Computation and The Ramanujan Journal, and was a managing editor for the Annals of Combinatorics for nearly twenty years. The Academy of Europe elected him as a member in 2011 and shortly after he was elected as a Fellow of the American Mathematical Society.

Mansour: Professor Paule, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Paule: The meanings of the words "combinatorics" or "combinatorial" have a tremendously wide reach. For example, in 1666 Gottfried Willhelm Leibniz published his "Dissertatio de arte combinatoria"[1] as an extended version of his doctoral thesis in philosophy. As summarized in Wikipedia, the main idea behind the text is that of an alphabet of human thought, just as words are combinations of letters. In this context, Leibniz discusses combinatorial notions like permutations ("variationes ordinis") or combinations of elements (e.g., "combinations" and "complexions"). Among other things, one finds Pascal's triangle together with its recursive explanation[1].

Despite being also interested in such philosophical aspects, my combinatorial horizon is mainly restricted to enumerative combinatorics. My standard description of this field is problem-oriented: for a certain class of objects, which can be specified in many different ways, determine the number of objects belonging to this class.[2] To solve such "counting problems", general methods (for instance, from algebraic combinatorics) are used, and in many cases, answers are expressed in explicit terms. Often these explicit terms come in the form of complicated expressions (e.g., nested sums) which one would like to represent in a more accessible form. Despite being interested also in general "counting methods", much of my research was devoted to this second part of combinatorial work; for example, the development of symbolic summation methods to simplify expressions.

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[1] G. W. Leibniz, *Dissertatio de Arte Combinatoria*, Lipsiae: apud Joh. Simon. Fickium et Joh. Polycarp. Seuboldum in Platea Nicolaea, literis Spörelianis, 1666. Available at https://archive.org/details/ita-bnc-mag-00000844-001.
[2] A variation of this theme is the problem of constructing such objects or configurations.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Paule: In his review[3] of the book "The Unity of Combinatorics" by Ezra Brown and Richard Guy, Igor Pak addresses the evolution of combinatorics from a "bag of isolated tricks"[4] to a reputation expressed by Richard Stanley[5] in a recent interview in this journal, "There has been fantastic development since I started doing combinatorics in the 1960s. Algebraic combinatorics by definition involves the relationship between combinatorics and algebra. It is now a major subarea of combinatorics ... Of course, areas of combinatorics besides algebraic combinatorics also have a deep relationship with other parts of mathematics ... All these connections are great examples of the unity of mathematics."

Concerning areas besides algebraic combinatorics, I feel that algorithmic combinatorics[6] is taking more and more ground. This is also reflected by increased funding, at least in Austria. For example, on the basis of an international peer-review, for the period 2013–2021 the Austrian Science Funds (FWF) has granted a major research project with the title "Algorithmic and Enumerative Combinatorics" involving computer algebraists from the Johannes Kepler University Linz and combinatorialists from Vienna around Michael Drmota (Vienna University of Technology) and Christian Krattenthaler (University of Vienna).

Mansour: What have been some of the main goals of your research?

Paule: A general driving force was the desire to understand. Concerning mathematics, there is a saying that in order to really understand things, you need to program them. It took me many years to develop a really reflected appreciation of the importance of algorithmic thinking (and programming!) in mathematical research. Nevertheless, influenced by George Andrews, Bruno Buchberger, and Volker Strehl, since the end of the 1980s, an overall goal of my research was to combine computer algebra with my other interests, combinatorics, number theory, and special functions.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Paule: In the early years of my childhood, mathematics did not play any remarkable role, except that I always found doing calculations easy and fun, also according to the memories of my mother. The first serious influence came at the age of fourteen or fifteen when I discovered the writings of Bertrand Russell[7]. As a consequence, I was not sure what to study: mathematics or philosophy, so I enrolled at the Vienna University in both. But after attending courses in both subjects, very soon the definitive decision for mathematics was easy to make. Maybe, I should mention that my father actually did not agree to either choice. He began his professional life as a rope maker in a small-scale shop of his uncle. Being the designated successor, he took this business over and succeeded to extend it to a much bigger scale which also included the selling of goods like carpets and curtains. My father was quite proud of this development. And, as a consequence, he would have strongly preferred if I would have followed him in his business.

Mansour: Were there specific problems that made you first interested in combinatorics? What was the reason you chose the University of Vienna for your Ph.D. and your advisor Johann Cigler?

Paule: I can answer both questions "in one stroke". A study outside of Austria was no option at that time; as a consequence, the University of Vienna was the most natural choice. Namely, all people I asked, like my mathematics teacher at the Gymnasium (high school), confirmed that this was the best decision, also in view of the remarkable tradition of the mathematics department there. Another natural choice was Johann Cigler as a Ph.D.

[3] I. Pak, *Book review: "Unity of Combinatorics" by Ezra Brown and Richard K. Guy*, Notices of the Amer. Math. Soc. 69 (2022), 108–111.

[4] R. K. Guy, *The unity of combinatorics*, pp. 129–159 in: Combinatorics Advances, C.J. Colbourn and E.S. Mahmoodian eds., Kluwer, Dordrecht, 1995.

[5] ECA, *Interview with Richard P. Stanley*, Enumer. Combin. Appl. 1:1 (2021), Interview #S3I1. Available at `http://ecajournal.haifa.ac.il/Volume2021/ECA2021_S3I1.pdf`.

[6] Michael F. Singer is using the term "symbolic combinatorics".

[7] See `https://plato.stanford.edu/entries/russell`.

advisor. I found his lectures carefully prepared and most inspiring. In particular, I had the good luck of being in the right place at the right time: I attended the first-course Cigler ever gave on combinatorics. A truly formative experience!

Mansour: What was the problem you worked on in your thesis?

Paule: There was a variety of themes I found attractive. Cigler was very much interested in umbral calculus which in these days was revitalized by Gian-Carlo Rota et al. More concretely, Cigler did interesting work in establishing analogous operator methods for q-identities (q-binomial identities, q-difference equations, etc.). Around the same time, Cigler was studying the first Russian edition of Egorychev's book "Integral Representation and the Computation of Combinatorial sums"[8]; in particular, he organized an unforgettable and highly inspiring seminar on this topical area. Nevertheless, my main attraction came from another side, namely, the entry on the Rogers-Ramanujan identities in the book by Hardy and Wright[9]. Without any further knowledge of the context, these identities loudly spoke to me. I was deeply impressed by their structure and, most important when reading Hardy's statement about the various proofs, I got intrigued by the sentence, "No proof is really easy (and it would perhaps be unreasonable to expect an easy proof)." As a consequence, I decided to work on this problem in my thesis.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Paule: There is no definite pattern on this. However, there might be at least some tendency towards working on questions of a more general nature such as Hardy's statement (mentioned above) on possible easy proofs of the Rogers-Ramanujan identities. Another example is my attempt to understand Gosper's algorithm, which in turn led to the concept of greatest factorial factorization[10] and its q-analogue[11]. A recent example is my paper[12] which originated in a project to explain Gauss' classical work on contiguous relations between hypergeometric series in the light of algorithmic developments like Zeilberger's algorithm and parameterized creative telescoping.

Mansour[a]: When you are working on a problem, do you feel that something is true even before you have the proof?

Paule: Definitely so. However, such feelings are varying. For example, when working on a crucial problem of my thesis, at a sudden moment the full solution came to my mind in a crystal clear manner. When I wrote it down (two days later), I was able to reproduce every detail in a one-to-one fashion. But such experiences are rare, unfortunately. Usually, it is so that I have a concrete feeling that things should work or fit together in a certain way, but then working out the details, often is a hard painstaking job. It also happens that my feelings are wrong, needless to say, ...

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Paule: Instead of three results, let me mention three papers [13,14,15]. All these articles circle around Zeilberger's holonomic systems approach to special functions identities which influenced the development of the whole field and of neighboring areas in substantial ways.

Mansour: What are the top three open questions in your list?

Paule: A major research direction concerns connections of classical q-series (q-hypergeometric functions, theory of partitions, etc.) with the theory of modular functions and forms. My particular focus is on the development of computer algebra algorithms. Of spe-

[8] G. P. Egorychev, *Integral Representation and the Computation of Combinatorial Sums*, Amer. Math. Soc., Providence, 1984.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford University Press, Oxford, 1979.

[10] P. Paule, *Greatest Factorial Factorization and Symbolic Summation*, J. Symbolic Computation 20 (1995), 235–268.

[11] P. Paule and A. Riese, *A Mathematica q-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping*, Pages 179–210 in: *Special Functions, q-Series and Related Topics* (Toronto, ON, 1995), Fields Inst. Commun. 14, Amer. Math. Soc., Providence, RI, 1997.

[12] P. Paule, *Contiguous Relations and Creative Telescoping*, Pages 335-394 in: Blümlein J. and Schneider C. (eds). *Anti-Differentiation and the Calculation of Feynman Amplitudes*, Texts & Monographs in Symbolic Computation, Springer, 2021.

[13] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. 80 (1990), 207–211.

[14] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. 32 (1990), 321–368.

[15] D. Zeilberger and H. S. Wilf, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral*, Invent. Math. 108 (1992), 575–633.

[16] F. Klein, *Gesammelte Mathematische Abhandlungen* (3 vols.), R. Fricke and A. Ostrowski (eds.) Berlin, Springer, 1921. Available at `https://gdz.sub.uni-goettingen.de/id/PPN237839962`.

cial interest in this regard is the work of Felix Klein[16].

Mansour: What kind of mathematics would you like to see in the next ten to twenty years as the continuation of your work?

Paule: A major driving force for my research was the development of computer algebra algorithms to assist work in classical areas of mathematics such as combinatorics, number theory, and special functions. Despite the fact that similar efforts have been made also in many other mathematical fields, I consider this kind of algorithmization still in its early stages. In some cases, these algorithmic tools will influence the way research in these areas is done.

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Paule: As in many aspects of life, it is a natural phenomenon to find various mainstream developments. To me, valid statements about importance essentially can be made only in the hindsight. Nevertheless, one can observe certain tendencies, in particular, in connection with the use of computers in mathematics. After the blossoming of numerical analysis, I feel/hope that symbolic computation will continue to catch up even more visibly. This is also related to the dichotomy "continuous vs. discrete," a pairing that played a prominent role already in the work of Leibniz[17]. From my personal experience, I can say that methods combining the best of the two worlds have the potential to be particularly fruitful. In other words, I expect future developments in these directions which include also a push towards discrete mathematics.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Paule: To me, the distinction between pure and applied mathematics does not make sense intrinsically (although it might be useful for bureaucratic reasons, e.g., for funding agencies). My impression is that this way one artificially erects fences where there should not be any. I find such tags restrictive in many contexts, for instance, when designing mathematics curricula. To those who are developing algorithms as tools to tackle mathematical problems, it is quite irrelevant whether the problems come from pure or applied mathematics. To give a concrete example, our mathematics department in Linz took an effort to tear down such fences: numerical analysts (PDEs, direct and inverse problems) and symbolic computation people from RISC set up a joint doctoral program "Computational Mathematics" to combine these two different worlds. It has been granted by the Austrian Science Funds FWF for the period 2008 to 2022; for further details see `https://www.dk-compmath.jku.at`.

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Paule: Without any thought, my first and main advice is a motto I found as a student in a book by my advisor, Johann Cigler: "Stick honestly to your own understanding."

A piece of standard advice I give to my Ph.D. students: Try to be flexible with regard to your mathematical interests; you cannot expect that in your professional future, regardless of whether in academia or industry, you will work on topics closely related to your Ph.D. thesis.

Another piece of advice: Learn programing.

Mansour: Would you tell us about your interests besides mathematics?

Paule: Mathematics occupies most of my time, so there is not much to tell: hiking and reading.

Mansour: Since 2009, you are the head of the Research Institute for Symbolic Computation of the Johannes Kepler University of Linz. Would you tell us about some of the projects related to enumerative combinatorics?

Paule: As mentioned above, a major research enterprise was from 2013 to 2021 the FWF sponsored project "Algorithmic and Enumerative Combinatorics," a joint action of computer algebraists from the Johannes Kepler University Linz and combinatorialists around Michael Drmota (Vienna University of Technology) and Christian Krattenthaler (University of Vienna). Another partner in Linz was the Johann Radon Institute for Computational and Applied Mathematics (RICAM).

[17] J. Jost, *Leibniz und die moderne Naturwissenschaft*, Springer, 2019.

Topical areas treated in the subprojects were: Shape Characteristics of Planar Maps and Planar Graphs (Michael Drmota), Combinatorics of Tree-Like Structures and Enriched Trees (Bernhard Gittenberger), Algorithmic Lattice Path Counting Using the Kernel Method (Manuel Kauers), Determinantal and Recursive Methods in Enumeration (Christian Krattenthaler), Partition Analysis (Peter Paule), Computer Algebra and Combinatorial Inequalities (Veronika Pillwein), Combinatorial and Algorithmic Aspects of Elliptic Hypergeometric Series (Michael Schlosser), Computer Algebra for Nested Sums and Products (Carsten Schneider), Alternating Sign Arrays of Triangular Shapes (Ilse Fischer), and Certificate-Free Summation and Integration (Christoph Koutschan). See `https://www.sfb050.risc.jku.at` for project descriptions, publications, and further details.

As also mentioned above, from 2008 to 2022 RISC participated in the doctoral program "Computational Mathematics", another excellence program granted by the Austrian Science Funds FWF. The general theme of my subproject was "Computer Algebra Tools for Special Functions"; the titles of the particular Ph.D. theses were: "Definite integration in differential fields" (Clemens G. Raab), "Complex analysis based computer algebra algorithms for proving Jacobi theta function identities" (Liangjie Ye), "Computer algebra with the fifth operation: applications of modular functions to partition congruences" (Nicolas A. Smoot), "New inequalities for special functions and sequences" (tentative title; Koustav Banerjee, ongoing). The Ph.D. theses can be found at the project page `https://www.dk-compmath.jku.at`.

Mansour: What are the main breakthroughs in symbolic computation since 2000?

Paule: Allow me to restrict to two success stories I am particularly familiar with, and which are of relevance to combinatorics and number theory

In the 1980s Michael Karr[18,19] developed an indefinite summation analog to the symbolic indefinite integration algorithm by Robert Risch[20]. Beginning with his thesis in 2001, Carsten Schneider started a major project with the goal to extend and streamline the work of Karr. A second goal was to implement all of his theoretical achievements within the framework of his summation package Sigma[21,22,23,24], written in the Mathematica system. Within difference fields and rings, Schneider's summation theories enable one to simplify definite nested multi-sums to representations in terms of indefinite nested sums and products. Special emphasis is put on representations that are optimal, e.g., concerning their nested depth.

Among other features, Schneider's package can be used to: (a) compute recurrences (based on the paradigm of Zeilberger's creative telescoping) for definite sums with summands given in terms of indefinite nested sums and products; (b) solve recurrences in terms of all solutions that are expressible in terms of indefinite nested sums and products (d'Alembertian solutions); (c) eliminate all algebraic relations among the summation objects and simplify them to representations with optimal nesting depth.

Besides numerous applications in enumerative combinatorics, the power of Sigma has been demonstrated impressively in quantum field applications studied jointly in a collaboration of Schneider's RISC group with the group at DESY (Deutsches Elektronen Synchrotron) of Johannes Blümlein in Berlin-Zeuthen. For further information on quantum field applications see, for instance, the recent monograph [25]; for combinatorics related applications see the publications on Schneider's web page at `https://www3.risc.jku.at/people/cschneid`.

The second success story, I want to de-

[18] M. Karr, *Summation in finite terms*, J. of the ACM 28 (1981), 305–350.
[19] M. Karr, *Theory of summation in finite terms*, J. Symbolic Comput. 1 (1985), 303–315.
[20] R. H. Risch, *The problem of integration in finite terms*, Trans. Amer. Math. Soc. 139 (1969), 167–189.
[21] C. Schneider, *Symbolic summation assists combinatorics*, Sém. Lothar. Combin. 56 (2007), 1–36.
[22] C. Schneider, *Parameterized telescoping proves algebraic independence of sums*, Ann. Comb. 14 (2010), 533–552.
[23] C. Schneider, *Fast algorithms for refined parameterized telescoping in difference fields*, Pages 157–191 in: Computer Algebra and Polynomials, LNCS 8942, Springer, 2015.
[24] C. Schneider, *A difference ring theory for symbolic summation*, J. Symbolic Comput. 72 (2016), 82–127.
[25] J. Blümlein and C. Schneider (eds.), *Anti-Differentiation and the Calculation of Feynman Amplitudes*, Texts & Monographs in Symbolic Computation (A Series of the Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria), Springer, 2021.

scribe briefly is Radu's Ramanujan-Kolberg algorithm[26] for discovering and proving witness identities for partition congruences.

Such congruences were first observed by Ramanujan: for example, 5 divides $p(5n+4)$ for all non-negative integers n; here $p(n)$ denotes the number of partitions of n. Witness identities present compact proofs of such facts. For example, Ramanujan found

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty}\frac{(1-q^{5j})^5}{(1-q^j)^6}, \qquad \text{(R)}$$

which, by comparing coefficients of powers of q on both sides, immediately implies the Ramanujan's divisibility observation.

Hardy in his obituary notice for Ramanujan said, "It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities, but here Ramanujan must take second place to Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting (R)." Now, equipped with Radu's Ramanujan-Kolberg algorithm one can discover and (at the same time) prove such identities automatically! Bill Chen together with his young collaborators Julia Du and Jack Zhao[27] extended Radu's algorithm to an important class of modular functions. For a Mathematica implementation of Radu's algorithm, I refer to Nicolas Smoot's implementation [28].

Mansour: Together with your group at RISC, you authored and co-authored combinatorial software. Would you describe *fastZeil*? Would you list some recent nontrivial usages of this package?

Paule: At the web page of the RISC Algorithmic Combinatorics Group, `https://www3.risc.jku.at/research/combinat/software` one finds a variety of software packages; fastZeil is one of them. For researchers and non-commercial users the software is free; just send a password request to me. Since the mid-1990s we have received on average two access requests a week. Most of the packages are written in the Mathematica system.

The packages fastZeil and qZeil are implementations of Zeilberger's algorithm[13] and its q-analogue[11]; in [29] and [11] one finds descriptions of how to use them. Concerning non-trivial usages, the number of access requests gives a rough measure that there are plenty of them. I should mention that fastZeil comes with various extra features; e.g., one of these allows parameterized telescoping[12]. As a concrete recent non-trivial application, by invoking the corresponding option "Parameterized" a non-terminating generalization of one of James Wilson's famous relations between hypergeometric ${}_4F_3$-series was derived[12].

Mansour: What do you think about computer-assisted proofs? Would you give some examples of enumerative results obtained thanks to a computer assistant? *Quantamagazine*[30] published an article about intentions to build an artificial intelligence system that can win a gold medal at the International Mathematical Olympiads. How promising do you see these efforts?

Paule: As already mentioned, I think that the usage of symbolic algorithms and of corresponding computer algebra packages is going to play a more and more substantial role in many areas of mathematical research. Such developments will be relevant to exploration and discovery (experimental mathematics) as well as proving.

Concerning examples of enumerative results where the use of the computer played a decisive role: let me select one in which I have been involved.

In 1995, John Stembridge proved that the number of totally symmetric plane partitions (TSPP[31]) with the largest part at most n (i.e., those whose 3D Ferrers diagram is contained in the cube $[0,n]^3$), is given by an elegant product formula that was conjectured by Ian Macdonald. Stembridge's proof[31] combines a variety of masterful steps involving the combinatorics of

[26] C.-S. Radu, *An algorithmic approach to Ramanujan-Kolberg identities*, J. Symbolic Comput. 68 (2015), 225–253.

[27] W. Y. C. Chen, J. Q. D. Du, and J. C. D. Zhao, *Finding modular functions for Ramanujan-type identities*, Ann. Comb. 23 (2019), 613–657.

[28] N. A. Smoot, *On the Computation of Identities Relating Partition Numbers in Arithmetic Progressions with Eta Quotients: An Implementation of Radu's Algorithm*, J. Symbolic Comput. 104 (2021), 276–311.

[29] P. Paule and M. Schorn, *A Mathematica Version of Zeilberger's Algorithm for Proving Binomial Coefficient Identities*, J. Symbolic Comput. 20 (1995), 673–698.

[30] See `https://www.quantamagazine.org/at-the-international-mathematical-olympiad-artificial-intelligence-prepares-to-go-for-the-gold-20200921`.

[31] J. R. Stembridge, *The Enumeration of Totally Symmetric Plane Partitions*, Adv. in Math. 111 (1995), 227—243.

Pfaffians and reduction of such to known determinant representations from which the product formula follows.

Ten years later, we[32] came up with a computer-assisted proof of the TSPP product formula. To this end, we used an ingenious matrix LU-decomposition supplied by George Andrews together with Schneider's Sigma package to prove the resulting hypergeometric multiple-sum identities.

It should be stressed that this problem transformation still required human insight as a preprocessing step. Christoph Koutschan then found a third "human-free" computer proof of Stembridge's theorem which was algorithmic and did not require any substantial human insight into the problem. In a project with Manuel Kauers and Doron Zeilberger, this approach was carried over to the q-case, resulting in the first proof of George Andrews's and David Robbins's q-TSPP conjecture[33].

Concerning your question about intentions to build an artificial intelligence system that can win a gold medal at the International Mathematical Olympiads. No doubt, I find such projects interesting. I was a chess enthusiast in my youth, so in 1997 I was really surprised by Deep(er) Blue's victory against Garry Kasparov, at that time the reigning world champion in chess. Nevertheless, I am following developments in AI only loosely. As I already pointed out, there is still lots to do regarding the development of computer algebra algorithms to assist mathematical discovery and proving. I want to stress the aspect of "assistance" because it is still the human mathematician who needs to ask the right questions — and to interpret and understand the computer output.

Mansour: Besides research in Combinatorics and Computer Algebra your work is much focused on Modular Functions. How would you describe the intersection of combinatorics to modular functions?

Paule: Actually, combinatorics was the first of two major reasons for extending my research interests to modular functions and forms. Namely, in the course of the partition analysis project with George Andrews we were led to questions concerning the combinatorial construction of modular forms. With the help of MacMahon's method and the accompanying Omega package, in [34] we derived an example of such a construction: broken partition diamonds. The related generating functions are infinite products, actually forming an infinite family of modular forms. These, in turn, led to arithmetic theorems and conjectures for the related partition functions.

The second major reason for considering modular functions more closely was Silviu Radu. Silviu did an outstanding computational Master's thesis[35] (new bounds for inverse problems concerning turns of the Rubik cube) with Gert Almkvist at Lund University and Gert had the (excellent!) idea to send him to my group for doing a Ph.D. thesis. Inspired by Andrews's excitement about the connection of combinatorics with modular forms, I suggested to Silviu to work on this topic. In turn, the success of Silviu's Ph.D. project encouraged me to open the door towards modular functions more widely.

Concerning the intersection of combinatorics to modular functions, besides the combinatorial construction of modular forms, there is another major aspect I started to work on only recently. Holonomic (also called: D-finite) functions (formal power series, or functions being analytic at zero) satisfy linear differential equations with polynomial coefficients. Holonomic (also called: P-recursive) sequences satisfy linear recurrences with polynomial coefficients. The generating function of a holonomic sequence is holonomic. Vice versa, the Taylor coefficients of holonomic functions form a holonomic sequence. In his survey on the algorithmic holonomic

[32] G. E. Andrews, P. Paule, and C. Schneider, *Plane Partitions VI: Stembridge's TSPP theorem*, Adv. in Appl. Math. 34 (2005), 709—739.

[33] C. Koutschan, M. Kauers, and D. Zeilberger, *Proof of George Andrews's and David Robbins's q-TSPP conjectur*, Proc. Natl. Acad. Sci. USA 108 (2011), 2196–2199.

[34] G. E. Andrews and P. Paule, *MacMahon's partition analysis XI: Broken diamonds and modular forms*, Acta Arithm. 126 (2007), 281–294.

[35] C.-S. Radu, *A new upper bound on Rubik's cube group*, RISC Technical report no. 07-08, 2007. Available at `http://www.kociemba.org/math/papers/rubik27.pdf`.

[36] M. Kauers, *The Holonomic Toolkit*, Pages 119–144 in: Schneider C., Blümlein J. (eds.), *Computer Algebra in Quantum Field Theory*, Texts & Monographs in Symbolic Computation (A Series of the Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria). Springer, 2013.

toolkit[36] Kauers states that "according to Bruno Salvy ... more than 60% of the entries of Abramowitz/Stegun's table of mathematical functions are holonomic, as well as some 25% of the entries of Sloane's online encyclopedia of integer sequences (OEIS)."

Modular forms are substantially non-holonomic: they satisfy algebraic differential equations of order three with constant coefficients. Nevertheless, there is an important bridge to the holonomic universe. Namely, by local expansion, any pair of a modular form and a modular function give rise to a uniquely defined *holonomic* sequence! In recent work with Silviu Radu, [37] and [38], and with particular attention to algorithmic aspects, we have studied various connections between these two worlds. Applications concern partition congruences, Fricke–Klein relations, irrationality proofs á la Beukers, or approximations to π studied by Ramanujan and the Borweins. As a major ingredient to a "first guess, then prove" strategy, we developed a new algorithm "ModFormDE" for proving (guessed) differential equations for modular forms.

Mansour: Let us talk about one of your recent papers, co-authored with Carsten Schneider, *Towards a symbolic summation theory for unspecified sequences*. The paper addresses the problem of whether indefinite double sums involving a generic sequence can be simplified in terms of indefinite single sums. What are the main motivations for this paper? Would you tell us about the essential ideas behind the proofs?

Paule: The original motivation for this paper was the problem to set up proper q-analogues of an identity due to Neil Calkin[39]. In the course of this project, we were led to consider questions about the "generic" structure of double (also multiple) sums. For example, when simplifying double sums "by hand", one often applies operations like interchanging the order of summation. Obviously, this action is independent of the nature of the summands. So we asked the question when and for which summands such kinds of manipulations have a chance to succeed. To model things as general as possible, we use generic sequences. In order to determine the requirements for simplification, we use the summation algorithms within the framework of the Karr/Schneider[40] theory.

Depending on the structure of the given double sum, the proposed summation machinery may provide a simplification without any exception. But if this fails, the algorithmic mechanism may suggest a "more advanced simplification" by introducing, in addition, a single sum where the summand has to satisfy a particular constraint. In other words, the algorithmic setting automatically delivers additional conditions under which simplification is possible. All this is implemented in Schneider's Sigma package.

Mansour: You have published a series of papers on *MacMahon's partition analysis*. Would you elaborate further on these works and point out some related future research directions?

Paule: In 1997 George Andrews suggested to me as a joint algorithmic project to work on MacMahon's method of partition analysis. For more than 80 years, despite being described in detail in his pioneering book "Combinatory Analysis", MacMahon's ideas have not received due attention with the exception of one shining moment when Richard Stanley successfully utilized partition analysis in his monumental treatment of magic labelings of graphs.

The decisive moment for Andrews to dig out this forgotten method was the beautiful lecture hall partition theorem by Mireille Bousquet-Mélou and Kimmo Eriksson[41]. Namely, Andrews observed that MacMahon's method is fitting just perfectly to the study of settings of this kind. Although I understood this kind of application, at the beginning I was very much in doubt about the possible *algorithmic* content of partition analysis, in particular, after looking at the corresponding pages in MacMahon's book.

[37] P. Paule and C.-S. Radu, *Holonomic relations for modular functions and forms: First guess, then prove*, Internatl. J. of Number Theory 17 (2021), 713—759.

[38] P. Paule and C.-S. Radu, *An algorithm to prove holonomic differential equations for modular forms*, Pages 367–420 in: Alin Bostan and Kilian Raschel (eds.), *Transcendence in Algebra, Combinatorics, Geometry and Number Theory*, TRANS 2019. Springer Proceedings in Mathematics & Statistics, vol 373. Springer, 2021.

[39] N. J. Calkin, *A curious binomial identity*, Discrete Math. 131 (1994), 335–337.

[40] C. Schneider, *Summation theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects*, J. Symbolic Comput. 80 (2017), 616–664.

[41] M. Bousquet-Mélou and K. Eriksson, *Lecture Hall Partitions*, Ramanujan J. 1 (1997), 101–111.

The situation changed completely when Andrews arrived in spring 1998 in Linz to spend part of his sabbatical at RISC. Very soon after, and with the expert advice of Andrews, Axel Riese has programmed a prototype version of partition analysis in the form of the Omega package, written in Mathematica. A series of papers on partition analysis followed, the most recent one being "Partition Analysis XIII"[42] which already stimulated some further combinatorial work [43,44,45].

A major goal of this series of articles was to demonstrate the power of MacMahon's method and also of the accompanying Omega package. The primary aspect of these applications was the usage of Omega as a tool for discovery, and I still feel that many combinatorial treasures are waiting to be discovered and explored by using this toolbox. Another promising area for further investigation is the already mentioned combinatorial construction of modular forms[34]. Further directions for future research are connections to polyhedral geometry (e.g., Erhart theory); as an example, see the work of Felix Breuer and Zafeirakis Zafeirakopoulos[46].

Mansour: Your book *The Concrete Tetrahedron*[47] encompasses computer algebra algorithms for dealing with four topics: symbolic sums, recurrence equations, generating functions, and asymptotic estimates. Do these four topics correspond to the vertices of the tetrahedron from the book title? Have you ever planned to add a new vertex, say, software?

Paule: Right, the vertices of the tetrahedron correspond to exactly these topics; the edges stand for the mutual connections between them. Another structural aspect of the book is a hierarchical one with respect to algebraic domains: it progresses from formal power series (Ch. 2), to polynomials (Ch. 3), C-finite sequences (Ch. 4), hypergeometric series (Ch. 5), and algebraic functions (Ch. 6) to holonomic sequences and power series (Ch. 7).

The whole book is intended as an algorithmic supplement to the bestselling "Concrete Mathematics" by Ron Graham, Don Knuth, and Oren Patashnik[48]. There is already a subsection on software in the appendix of the Concrete Tetrahedron. In a new edition, in case this will ever happen, this entry certainly should be expanded and typos removed. In addition, we have to incorporate many suggestions of Don Knuth who read the book from cover to cover and who sent us pages of remarks and corrections.

Mansour: One of your interesting results, coauthored with Cristian-Silviu Radu, is *The Andrews–Sellers family of partition congruences*. Therein, you proved Sellers' conjecture for all powers of 5. Would you tell us about the main ideas behind it?

Paule: Let me begin with some historical background. As already mentioned, partition congruences were first observed by Ramanujan, also the fact that some of them form infinite families. The first of these families start as follows: 5 divides $p(5n + 4)$, 5^2 divides $p(5^2n + 24)$, and so on; again $p(n)$ denotes the number of partitions of n.

In 1994, James Sellers[49] conjectured an infinite family of Ramanujan type congruences for 2-colored Frobenius partitions introduced by George Andrews[50]. These congruences also arise modulo powers of 5. In 2002 Dennis Eichhorn and Sellers[51] were able to settle the conjecture for powers up to 4. Until 2012, when our paper[52] was published, no further progress had been made.

[42] G. E. Andrews and P. Paule, *MacMahon's partition analysis XIII: Schmidt type partitions and modular forms*, J. Number Theory, in press. Available at `https://doi.org/10.1016/j.jnt.2021.09.008`.

[43] R. da Silva, M. D. Hirschhorn, and J. A. Sellers, *Elementary proofs of infinitely many congruences for k-elongated partition diamonds*, preprint, 2021. Available at `https://arxiv.org/pdf/2112.06328.pdf`.

[44] K. Q. Ji, *A combinatorial proof of a Schmidt type theorem of Andrews and Paule*, preprint, 2021. Available at `https://arxiv.org/pdf/2111.03367.pdf`.

[45] N. A. Smoot, *A congruence family for 2-elongated plane partitions: An application of the localization method*, preprint, 2021. Available at `https://arxiv.org/pdf/2111.07131.pdf`.

[46] F. Breuer and Z. Zafeirakopoulos, *Polyhedral Omega: a new algorithm for solving linear diophantine systems*, Ann. Combin. 21 (2017), 211–280.

[47] M. Kauers and P. Paule, *The Concrete Tetrahedron*, Symbolic sums, recurrence equations, generating functions, asymptotic estimates. Texts and Monographs in Symbolic Computation. SpringerWienNewYork, Vienna, 2011.

[48] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, 1994.

[49] J. A. Sellers, *Congruences involving F-partition functions*, International Journal of Mathematics and Mathematical Sciences 17 (1994), 187–188.

[50] G. E. Andrews, *Generalized Frobenius Partitions*, Memoirs AMS, 49(301), 1984.

[51] D. Eichhorn and J. A. Sellers, *Computational Proofs of Congruences for 2-Colored Frobenius Partitions*, International Journal of Mathematics and Mathematical Sciences 29 (2002), 333–340.

[52] P. Paule and C.-S. Radu, *The Andrews–Sellers family of partition congruences*, Adv. in Math. 230 (2012), 819—838.

At a first glance, the congruences in question seem to fit the standard pattern of Ramanujan type congruence, and one would expect those standard methods would apply in a straightforward manner. But it turns out that a basic feature of such approaches is missing here, namely, ℓ-adic convergence to zero of sequences formed by the application of U-operators to Atkin basis functions[53]. This, we feel, is the reason why Sellers' conjecture has remained open for more than fifteen years.

We were able to recover ℓ-adic zero convergence by the introduction of a new type of subspace of modular functions which behave well under the action of the U-operators. These subspaces came as a perfect surprise to us. They were found by Radu on the basis of cleverly arranged computer experiments

In other words, we found a reason why the Andrews–Sellers family is significantly different from classical congruences modulo powers of primes. And, despite having proved the conjecture, the underlying mathematics still bears some mysteries - at least to me.

Mansour: You have advised more than 15 Ph.D. students and about the same number of post-doctoral fellows. How important is working with young researchers and passing knowledge to them? Do you have any long-time collaborators among them?

Paule: To work with young people always has been a major source of inspiration to me. Some of my former Ph.D. students stayed in Linz for some additional period after their Ph.D. or became colleagues in the RISC faculty. Over the recent years, Silviu Radu has been my closest collaborator from this group of former Ph.D. students. Concerning former PostDocs, although they are following their own tracks, I am still in touch with most of them.

Mansour: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

Paule: As described above, a major portion of my work deals with aspects of simplification, namely, the simplification of answers given by enumerative techniques. Nevertheless, at the beginning of my career, I was very interested in methods such as counting under group actions. In this regard, I learned a lot from Adalbert Kerber who at the beginning of the 1980s was my Alexander von Humboldt host at the University of Bayreuth. During this time I produced two articles[54,55].

The first article [54] is kind of a survey on various settings for the "involution principle", including the work of Garsia and Milne. Among other things, it contains a combinatorial proof of a q-binomial identity which can be used as a fundamental building block for iteratively proving identities like the Rogers-Ramanujan identities. This setting is a special case of Andrews's general conceptual framework of Bailey chain iteration.

The note [55] describes a graph theoretic interpretation of the celebrated Garsia-Milne involution principle; more precisely, this mechanism can be viewed as an application of the so-called Linkage Lemma of Ingleton and Piff, which provides a general framework for the iterative construction of bijections. Also Basil Gordon's "complementary bijection principle" fits under this umbrella.

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "eureka moment"?

Paule: On general grounds, my answer to your question[a] gives already some picture. Otherwise, I am afraid I cannot present any news here; i.e., I guess the creative process in my case is not much different from others: first come feelings about what could/should be true, when one tries to work out the details. If no improvement/solution is in sight, one thinks of suitable modifications; then one introduces those variations which look reasonable and most promising and goes through the whole process again. Concerning "eureka moments," I feel they are actually there all the time; this means, each time when the different mental building blocks "glue" together to form a new mental entity that turns out to be of help in further reasoning steps. In most cases such "eureka" moments happen silently and unno-

[53] A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Mathematical Journal 8 (1967), 14–32.

[54] P. Paule, *Über das Involutionsprinzip von Garsia-Milne*, Bayreuther Math. Schriften 21 (1986), 295–319.

[55] P. Paule, *A remark on a lemma of Ingleton and Piff and the construction of bijections*, Bayreuther Math. Schriften 25 (1987), 123–127.

ticed; only on a (very) few exciting occasions, one can feel them so strongly as Archimedes did.

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Paule: One example which immediately comes to my mind stems from my collaboration with George Andrews on partition analysis. As already mentioned, MacMahon described this method in his book "Combinatory Analysis". More precisely, he devoted more than one hundred pages to this topic. MacMahon clearly hoped to hone this tool into one that could prove his conjectures on the generating functions of plane partitions. Clearly, the problems can all be set up in the language of his partition analysis, being a special calculus involving what MacMahon called the Omega operator. However, he was unable to develop this machinery adequately. Sadly he sets up the general problem on page 186 in Volume 2 of his book, but at the same time he was forced to conclude: "Our knowledge of the Omega operation is not sufficient to enable us to establish the final form of result."

Consequently, as one major milestone in our partition analysis project, we set up the goal to complete MacMahon's original project. Article [56] shows that we finally succeeded - but only after many discussions, fruitless attempts, and mutual visits. In the course of this project, it turned out that we needed to develop better insight into how the Omega operator works. To this end, the Omega package was extremely helpful. I remember vividly a major breakthrough experience in George's PennState office: when (again) trying out a variety of variations on a theme, at a certain moment the Omega package returned - instead of "Sauerkraut" - a beautiful product expansion! This was the decisive eureka moment towards the complete proof. Nevertheless, setting up the complete induction argument required still quite some (less exciting) way to go.

Mansour: My last question is philosophical: have you figured out why we are here?

Paule: As a dweller of the mathematical universe my vocation is trying to understand and to find good explanations.

Mansour: Professor Peter Paule, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Paule: It was my pleasure! I congratulate you on having founded this new journal and wish you, and all the people involved in the production, all the best for its future!

[56] P. Paule and G. E. Andrews, *MacMahon's partition analysis XII: Plane partitions*, J. London Math. Soc. 76 (2007), 647–666.