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Interview with Helmut Prodinger

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Helmut Prodinger obtained a Ph.D. from the Technical University of Vienna in 1978 under the supervision of Werner Kuich. In 2016 he obtained a Doctor of Science degree from the University of Stellenbosch. He is currently an emeritus professor at the Department of Mathematics of Stellenbosch University. His numerous awards include the prize from the Austrian Mathematical Society (1985), Gold medal from the South African Mathematical Society (2001), Vice-chancellor's research award (2001), South African Mathematical Society award for research distinction (2003), Honorary Professor at the Technical University of Graz (2005). Professor Prodinger has served as a member of the editorial board of several journals, that include Theoretical Computer Science, Journal of Algorithms, Transactions on Algorithms, Discrete Mathematics and Theoretical Computer Science, and Quaestiones Mathematicae.

Mansour: Professor Prodinger, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Prodinger: I read the answers to this question by all the people you interviewed. I cannot add much that was not mentioned already. Personally, I prefer the term "Concrete Mathematics" coined by Donald Knuth. I am a pragmatic person; when I feel that I can add something to a certain subject, then I will do it, whether it is called Combinatorics, Number Theory, Analysis of Algorithms, or whatever.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Prodinger: Since Philippe Flajolet[1] was my major influence, I am particularly impressed by the interplay of combinatorics and the Analysis of Algorithms. Computer Algebra (Zeilberger's algorithm[2], but there are many other algorithms of interest) are of major importance to me. I am mostly a user not a developer of such software. But it extended my range of activities so much that I joked 30 years ago: "I don't need coauthor XXX anymore, now I have Maple."

Mansour: What have been some of the main goals of your research?

Prodinger: I do not have longtime plans; I am used to following the current literature closely and reacting quickly. This is so much easier now than it was when I was young.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Prodinger: My grandfather was a high-school teacher in mathematics, but he died the same year I was born. I was always intrigued by mathematics, but the education in high school ('gymnasium' in German) was extremely poor. I felt underprepared to become a full-time

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[1] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.

[2] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric series identities*, Discrete Math. 80 (1990), 207–211.

mathematician when I started. I thought: when you are going to study the piano, you must be already an excellent piano player, and in mathematics, it will be the same. So I started with computer science, which was a very new subject in 1972, so everybody should have the same chance. Very soon, however, I discovered that my comparison with studying the piano was wrong, and I added the study of (pure) mathematics to my curriculum after one year. During high school, I spent perhaps more time playing the electric guitar and reading German literature. I cannot report any influence from my family (except for some basic talent) and I soon realized that I will be on my own and that I must be creative myself.

Mansour: Were there specific problems that made you first interested in combinatorics?

Prodinger: Since I was employed in a department of theoretical computer science, combinatorics was the closest to mathematics for me. I had some ideas in combinatorics on words, although I did not know that such a discipline really existed.

Mansour: What was the reason you chose the Technical University of Vienna for your Ph.D. and your advisor Werner Kuich?

Prodinger: I got a full-time position at a very young age in the department of Formal Languages. I felt that it is my responsibility to choose the head of the department as a supervisor. But I was really self-trained, and chose subjects for MS.c. and Ph.D. myself. About Kuich's influence, I only remember that he told me in 1975 that he learned from Samuel Eilenberg about the binomial coefficients for words. Somehow I felt that I will have to deal with non-commutativity for the rest of my life. Fortunately, this did not happen. Curiously, automata played a role in my life much later, for instance when studying (redundant) digit expansions. Around 1978/79, I got in contact with Rainer Kemp and Philippe Flajolet, and this opened a new world for me. I had a reasonable background in (analytic) number theory (like many students in Vienna) but I was not trained in asymptotics. Well, in Vienna at the period nobody was. But I read Comtet's book[3] and it had a small section on asymptotics. There was Darboux's method[1] (later overshadowed by singularity analysis of generating functions), and I studied a short paper by van Lint. Later, I discovered that it was really the saddle point method that he used, but it was never mentioned. And once I mastered this, I applied it to problems of my own. I kept this attitude to this day. I do not have the energy to 'just read', I feel that I have to write about what I read.

Mansour: What was the problem you worked on in your thesis?

Prodinger: I used the binomial coefficients for words and formal power series to embed the free monoid into larger and richer structures, among other things. I was a bit naive at the time; I rediscovered something related to the Magnus transform, although I forgot the details. Perhaps I started too early to do research, especially since I was completely on my own, and I should have studied more. But somehow I learned while doing it like a dog learns to swim when thrown into the pool. I regretted that I started with computer science and not immediately with mathematics, but perhaps there was not too much damage to it.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Prodinger: When I feel that I can contribute something, I will do it. I never felt that I want to be a theory builder. Some people tried to do that in combinatorics, but I felt that their best work was done when they were young and solved problems (I avoid mentioning names).

I spend a lot of time on the internet. Some people think that it is excessive. However, I find many things that are worthwhile for further study.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Prodinger: I am certainly equipped with a good portion of intuition, comparable to talent in musical improvisation, although I admire proper virtuosos (in any field).

My most cited paper is called "Fibonacci numbers of Graphs"[4] and was coauthored with Tichy. It was almost designed as a joke after reading an exercise in Comtet's book. When studying skip lists, I saw the larger picture:

[3] L. Comtet, *Advanced combinatorics*, D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[4] H. Prodinger and R. F. Tichy, *Fibonacci numbers of graphs*, Fibonacci Quart. 9 (1982), 16–21.

combinatorics of geometrically distributed random variables. I wrote several papers on the subject, and Arnold Knopfmacher and his team continued the series.

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Prodinger: Zeilberger's algorithm wins this by a mile. Such a wonderful combination of theory and practice. Singularity analysis of generating functions (Flajolet and Odlyzko[5]), is a cornerstone of asymptotic enumeration. (By the way, I never liked the notion of "Analytic Combinatorics". I like the field, but not the name; for me, asymptotic enumeration is good enough.) For the third place, I do not have a clear favorite, but I like hypergeometric functions (something I never learned as a student) and the q-world.

Mansour: What are the top three open questions on your list?

Prodinger: I do not really have any; I live hand to mouth.

Mansour: What kind of mathematics would you like to see in the next ten to twenty years as the continuation of your work?

Prodinger: That is a tough question. Some stuff that I liked, like digital problems (first seen in Delange[6], 1975) does not produce any interesting questions anymore. Some people still try, but it is like squeezing water from a stone. Some authors write their first book again and again; other artists (like Canadian band Rush or German author Arno Schmidt) develop and produce something really new each time. I do not have a 'vision', but somehow I feel that things that I liked are on a decline. Perhaps they are too involved in terms of computations/manipulations, for which young people do not have enough patience anymore. To give you an example: in 1993, Philippe Flajolet, Rainer Kemp, and yours truly started a seminar series 'Analysis of Algorithms'. It still exists, but no algorithms are analyzed, and these seminars are crowded with people from random graphs. I escaped from this more than 10 years ago. Kemp and Flajolet died young, I am still alive, but I want to be true to myself.

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Prodinger: There are certainly mainstream areas that are more popular than others, especially when it was enough to be smart and technical skills are secondary. I do not want to point at certain areas, but when you go to a general conference in combinatorics, only a small fraction of the contribution deals with combinatorics. I cannot report much about core subjects, as I do not 'read'; I remember that somebody told me many years ago that you can either read a lot or do a lot, not both.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Prodinger: It is difficult to give a diplomatic answer. I was never good at that. Zeilberger's algorithm[2] is the ideal incarnation of both. There are just things that I do not like, but it is better not to mention examples here. I have a private opinion about what 'ugly mathematics' means for me. It is better not to mention that here.

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Prodinger: If you really feel the drive to do it – go for it. But it is more than good grades in school and universities. You must feel that you can create. If you need others to permanently tell you what to do next, you will be in the wrong place. I have seen too many frustrated mathematicians who became administrators and mainly teachers and just wait for retirement – even some who were 'very promising' when they were young.

Mansour: Would you tell us about your interests besides mathematics?

Prodinger: I mentioned it already. Music is one; I never became particularly good on any instrument, but I have good ears and can sing from a score (first sight) surprisingly well. So much so that a professional singer was very surprised about it and said that many singers in professional choirs don't have that ability. It is just pattern recognition and an inner ear. Unfortunately, I do not have much of a voice.

[5] P. Flajolet and A. Odlyzko, *Singularity analysis of generating functions*, SIAM J. Discrete Math. 3:2 (1990), 216–240.

[6] H. Delange, *Sur la fonction sommatoire de la fonction "somme des chiffres"*, Enseign. Math. (2) 21:1 (1975), 31–47.

When I was younger, I concentrated on the electric guitar and always tried to treat it in the style of a violin. Many virtuosos do this now; youtube is full of amazing players.

Another interest is literature. The real eye-opener was the German author Arno Schmidt. His approach of working 100 hours per week somehow saved me, as I had a period when I did not work very much (around 1970). In my reading, I followed Schmidt's 'canon', reading half-forgotten authors from the 19th century, like Bulwer-Lytton and the Bronte sisters. German authors, I will not mention here since the international audience of your journal will not connect anything with them.

Mansour: A few years ago, at a workshop in BIRS on Analytic and Probabilistic Combinatorics, you gave a talk entitled as *Forty years of tree enumeration*. Would you tell us about the evolution of those tree enumerations?

Prodinger: This is not a full account of 'forty years'. This is just about forty years of my research life, related to tree enumeration and lattice path enumeration. Of course, I did other things as well.

I just gave many examples of what I have seen during the years. Something I liked, in particular, was my early work with Alois Panholzer where we could grind out many explicit enumerations that most people would not do, because of the complexity of the computations and the lack of *sitzfleisch*.

Mansour: One of your very recent papers, coauthored with Nancy Gu, is *Combinatorics on lattice paths in strips*[7]. Would you tell a little bit more about this work and point out some possible future research directions?

Prodinger: Johann Cigler from Vienna, who is remarkably active, although in his eighties, posted a few observations in a Mathematics Forum. Thomas Prellberg[8] quickly solved a special case, and I also linked a special case to an earlier invention of mine (Elena trees[9]). The general bijection was still a mystery – until I mentioned it to Nancy. She is just very good at that. The proofs of Cigler's[10] observations with generating functions are not hard, but the bijections are certainly tricky. I used to have a list of people whom I could approach for specific problems as I knew they are good at it. Nowadays I consider myself as *the last man standing* and working mostly alone.

Mansour: You, with Benjamin Hackl, have a paper[11] on the *necklace process* published in 2018. You derived some wonderful formulas for some quantities related to the process using singularity analysis and also proved some interesting probabilistic results. Would you tell us more about this process, and your motivation for this work and list some interesting open questions?

Prodinger: With one of the usual search machines, I discovered that Nakata[12] (unknown to me) cited an old (slightly obscure) paper by Brennan and myself[13], which was a surprise. Furthermore, he mentioned Polya urns. From earlier experience, I had a feeling that generating functions are a more powerful tool than Polya urns. I am certainly biased, but I am a big fan and practitioner of generating functions. So I looked at the paper. There was eventually a partial differential equation to be solved. My visitor, a big fan of Sage, tried his tool – no success. But old Maple (after some human interactions) did it! I never felt the desire to switch from Maple to something else.

Mansour: Several of your works are related to the well-known computer science masterpiece *the Art of Computer Programming*. Would you tell us more about such correlations?

Prodinger: Since Knuth has initiated analysis of algorithms, and Philippe Flajolet was the European head of this area for several decades, it is very natural that Knuth's *magnum opus* played a major role also for me. One major item is the *Gamma function method*, which was later identified to be an instance of the *Mellin transform method*[1]. This tool is very useful for the asymptotics of harmonic sums, and I

[7] N. S. S. Gu and H. Prodinger, *Combinatorics on lattice paths in strips*, European J. Combin. 94 (2021), 103310.

[8] T. Prellberg, *Is there a simple bijection between the following sets A_n and B_n which are counted by the Fibonacci numbers?* (Answer) www.researchgate.net, 2015.

[9] H. Prodinger, *Words, Dyck paths, Trees, and Bijections, in: Words, Semigroups, and Transductions*, World Scientific, 2001, 369–379.

[10] J. Cigler, *Is there a simple bijection between the following sets A_n and B_n which are counted by the Fibonacci numbers?* www.researchgate.net, 2015.

[11] B. Hackl and H. Prodinger, *The necklace process: A generating function approach*, Stat. Prob. Lett. 142 (2018), 57–61.

[12] T. Nakata, *Necklace processes via Pólya urns*, J. Appl. Probab. 46:1 (2009), 284–295

[13] C. A. C. Brennan and H. Prodinger, *The pills problem revisited*, Quaest. Math. 26 (2003), 427–439.

was a co-author of a relevant paper[14] (digital sums), a paper that is still cited very often today. Another asymptotic method of continuing interest for me is Rice's method, a contour integration technique that is also useful to derive identities in some instances. The manipulation of harmonic numbers is of course fascinating, and Carsten Schneider from Peter Paule's team designed Computer Algebra packages that are helpful. Tries, patricia trees, and digital search trees were of course fascinating, and we managed to extend some of the results that can be found in Knuth's book. Of course, I detected the kernel method there as well. From volume 2, I just want to mention redundant representations of integers, something that kept my interest (with some coauthors) alive for about a decade. As you probably know, volume 3 has a long chapter on partition analysis, so there is a fruitful interaction between mathematics and computer science to be seen.

Mansour: One of your favorite subjects is the *Kernel Method*. You also have published a nice survey[15] on the subject in which you described the method as *dear to your heart*. Why it is so? What are the three fascinating results obtained by the kernel method so far?

Prodinger: I attended the 50th session of the Séminaire Lotharingien de Combinatoire (SLC). Peter Paule, who spoke after me, told me that he would need a lot of time. So I decided to stop a bit early and helped him to an extended time slot. I had prepared some examples of applications of the kernel method but had no time to present them. When there was a call for papers from SLC, I took the chance and collected them. I never felt that I am a great specialist, but I found it cute, initially surprising, and powerful. Somebody said (not into my face) that my survey 'only deals with quadratic cases'. Such things, unfortunately, I remember the best. During the last 3 years, I had a chance to deal with various *cubic* equations, so even I learned something new. Surprisingly, this innocent collection of examples (Knödel walks, Banach's matchbox problem, toilet paper problem, etc.) is one of my most cited papers!

For another issue of SLC (2022), I collected 57 pages and called it 'A walk through my lattice path garden'[16]. It is dedicated to Neil Peart, who wrote the lyrics of 'The garden'. It has many applications of the kernel method!

Mansour: You have published a series of papers on *the Gaussian q-binomial coefficients*. Would you elaborate on these works?

Prodinger: About 12 years ago, Emrah Kilic from Turkey asked me a question that I was able to solve by translating from Fibonacci coefficients to q-binomial coefficients and applying the q-Rice formula, a tool[17] that I liked and applied frequently. So I became a person to write to for Kilic, and I often reacted. Once, I also answered to Roberto Tauroso, and a similar approach was useful; I don't remember much.

When studying the path length for various data structures, in particular digital search trees, a few years earlier, a strange mixture of ordinary binomial coefficients and q-binomial coefficients appeared; in order to apply Rice's method, a certain sum, only defined for integers n, had to be extended to the complex plane in a meaningful way, and then certain residues had to be computed. That and an alternative analysis of Approximate Counting were the first occasions when I was confronted with q-analysis.

Mansour: One of your interesting results, co-authored with Conrado Martinez and Alois Panholzer, is *Partial match queries in relaxed multidimensional search trees*[18]. Would you tell us about the main ideas behind this result?

Prodinger: Philippe Flajolet (together with Jean-Marc Steyaert) was on a mission to carry the gospel of analysis of algorithms to Barcelona in the early days. I met a young Conrado Martinez in Dagstuhl, and I had a feeling that after finishing his Ph.D., he was looking for new adventures. I tried to step into Flajolet's (giant) footsteps. Since we got a bit of a joint grant, I suggested investigating skiplists (popular at the period) and Hoare's FIND algorithm with median-of-three partition. (I

[14] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy, *Mellin transforms and asymptotics: Digital sums*, Theoret. Comput. Sci. 123 (1994), 291–314.

[15] H. Prodinger, *The kernel method: a collection of examples*, Sém. Lothar. Combin. 50 (2004), Article B50f.

[16] H. Prodinger, *A walk in my lattice path garden*, arXiv:2111.14797.

[17] E. Kilic and H. Prodinger, *A generalized Filbert matrix*, Fibonacci Quart. 48 (2010), 29–33.

[18] C. Martinez, A. Panholzer, and H. Prodinger, *Partial match queries in relaxed multidimensional search trees*, Algorithmica 29 (2001), 181–204.

was later involved in the variance of Hoare's FIND algorithm with a fantastically complex explicit formula, for which a reciprocity theorem for sums involving harmonic numbers had to be developed.) We also did something about binary search trees that Hosam Mahmoud started with Janice Lent. My ambition was, as always, to find *explicit* results, even if they are complicated. Eventually, Alois joined the team, and then we could do much more than before. Some research that I remember was about partial queries in multi-dimensional tries – only certain components/coordinates need to match – and Alois mastered partial differential equations! Yes, classical analysis plays a role in some branches of combinatorics!

Mansour: You have advised several graduates and postdocs. How important is working with Ph.D. and postdocs and passing knowledge to them? Do you keep working with them?

Prodinger: Well, some are great, others not so much. You pass knowledge to them, as you say – and then they forget you. I had a few bad experiences. Alois Panholzer is a wonderful man and did much more than I asked for. Since I left Vienna in 1998, certain cooperations are not so much alive anymore. For bijections, Nancy Gu is a wonderful coauthor, and for hard analytic questions related to asymptotics, I always communicate with Peter Grabner.

Mansour: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

Prodinger: You need a feeling of which tool fits which question, so I would never apply urn models or Riordan arrays. The technique is of course important and in my case 'creative guessing'. Creating a list of values and factoring them and searching for recursions (often with gfun, a Maple programme, due to Bruno Salvy and various coauthors) is something I do all the time. And I do manipulations with TEX, not with pencil and paper, so I can do just minuscule steps and always see the result. For a publication, I would typically delete every second line. Like anybody else, I have a preferred toolkit. Some people use Lindström-Gessel-Viennot[19,20] as often as possible, others the Brownian motion, '*was dem einen sin uhl ist dem andern sin nachtigall*'.[21]

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "eureka moment"?

Prodinger: Sometimes I have a feeling that something *must* be possible. Like the short paper about Kirkman's identity: I *knew* that the Lagrange inversion formula would do it – and then Alois Panholzer and I did the details[22]. About the eureka moment I can mention Hoare's FIND algorithm with median-of-three partition. It was just the beginning of my guessing career, and I wasn't too good with Maple around 1992. When I identified certain numbers to be expressible with harmonic numbers, I really thought that I achieved something. Although, in 2022, I would say that it was relatively easy.

I remember that Edmund Hlawka from Vienna advised that if you don't find a proof/result within 3 weeks, leave it, and come back to it later. I follow that advice.

Mansour: On your web page, it is written that "the most striking aspect of his mathematical practice is his extraordinary skill at calculations." Is there something you have tried hard to calculate for a long time but have not achieved your goal yet?

Prodinger: This is a citation I found about Georg Frobenius. But it is very close to my own approach. Of course, I don't compare myself to Frobenius. Another phrase that very much describes my interests is by Richard Askey: "And when an explicit formula can be found there is nothing to beat it. In the rush to abstraction and generalization, we often forget this."

Mansour: You have eight pictures of Pocahontas on your web page. Why is she special to you? Would you tell us about her story?

Prodinger: I was always interested in the American colonization. I read almost the

[19] I. Gessel and X. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. 58 (1985), 300–321
[20] B. Lindström, *On the vector representations of induced matroids*, Bull. London Math. Soc. 5 (1973), 85–90.
[21] One man's meat is another man's poison.
[22] A. Panholzer and H. Prodinger, *Kirkman's hypothesis revisited*, Integers 1 (2001), Article A05.
[23] G. Dekker, *James Fenimore Cooper The Novelist*, 1st ed., Routledge, 1967.

full set of novels of James Fenimore Cooper, also the less known ones, like The Littlepage-Trilogy[23]. There is a song 'Fever' with the line 'Captain Smith and Pocahontas . . . '. Austrian people were notoriously ignorant about this. Perhaps the Disney movie from 1995 has changed this a bit; when I started my web page, this movie was new. I never distinguish between doing mathematics, music, reading, or whatever: it is always just me, and things are notoriously interwoven in my brain and personality.

Mansour: In many of your research papers, the word Fibonacci appears even in the title. Can you elaborate on this?

Prodinger: The paper[4] (with Tichy) about Fibonacci numbers of graphs, which was just a little fun project, became my most cited paper; it was also rediscovered in combinatorial chemistry under a different name. Since this was written (in 1979, I believe), I often superimpose a condition that two neighbors cannot occur together on various combinatorial structures and use the epitheton 'Fibonacci'. As I explained earlier, Fibonacci and q are not too far away for me.

Mansour: The works of Ramanujan appear fairly often in your work. Can you elaborate on this?

Prodinger: Ramanujan's identity for $\zeta(2n+1)$[24] could be used to prove that a certain quantity that popped up in the computation of the variance of a parameter of a certain data structure is actually identical to zero. This had some significance, as the order of the variance was much smaller than originally thought. Similar phenomena appeared more often, not always labeled by the name Ramanujan, but various approaches (Mellin transform, residue calculus) were collected in a survey paper[25] for the Iranian Mathematical Society. Then sharp asymptotic bounds for Ramanujan's Q-function were derived, settling a conjecture of Ramanujan.

Garrett, Ismail, and Stanton[26] found an 'm-version' of the celebrated Rogers-Ramanujan identities. Very early on, I had a feeling that some relatively simple operations should be enough to show that. And I was a fan of Drew Sills' list of Slaters' identities, with finite versions included. I convinced Nancy Gu[27] to work with me on m-versions of many of the identities in the long list, as well as (later) on various continued fraction expansions. So, Sills' list was very inspiring for me.

A few smaller projects are also linked to Ramanujan's name. A long and exciting project was never finished: I invited somebody to work with me and, after saying yes, not much happened, and I was trapped. Similar things happened more often to me. Now I work mostly alone. I had almost 100 coauthors; most of them have disappeared in one way or another.

Mansour: My last question is philosophical: have you figured out why we are here?

Prodinger: No, but currently I am intrigued by a series called 'Urknall, Weltall, Leben',[28] with close to 100 episodes on youtube. Of course, Giordano Bruno, who was burnt in 1600, was mentioned, as well as Galileo Galilei; an official apology was only provided in 1992. There is still so much to learn for me. My physics education was even poorer than my mathematics education. I always had the feeling that I have to catch up, so I somehow overcompensated for the last 50 years.

Mansour: Professor Helmut Prodinger, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Prodinger: It was an honor for me and I wish you good luck with your journal; I provided a historical paper[29] about the early combinatorial efforts of my hero Philippe Flajolet and hope that your readers enjoyed it.

[24] S. Ramanujan, *Notebooks*, Volume 2, Tata Institute of Fundamental Research, Bombay, 1957; 2nd ed., 2012, Page 173.
[25] H. Prodinger, *Periodic oscillations in the analysis of algorithms and their cancellations*, JIRSS 3:2 (2004), 251–270.
[26] T. Garrett, M. Ismail, and D. Stanton, *Variants of the Rogers-Ramanujan identities*, Adv. Appl. Math. 23 (1999), 274–299.
[27] N. S. S. Gu and H. Prodinger, *One-parameter generalizations of Rogers-Ramanujan type identities*, Adv. in Appl. Math. 45 (2010), 149-196.
[28] Big bang, the universe, life.
[29] H. Prodinger, *Philippe Flajolet's early work in combinatorics*, ECA 2:1 (2022), Article S1H1.