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The Characteristic Quasi-Polynomials of Hyperplane Arrangements over Residually Finite Dedekind Domains

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Abstract: Kamiya, Takemura, and Terao initiated the theory of the characteristic quasi-polynomial of an integral arrangement, which is a function counting the elements in the complement of the arrangement modulo positive integers.

They gave a period of the characteristic quasi-polynomial, called the LCM-period, and showed that the first constituent of the characteristic quasi-polynomial coincides with the characteristic polynomial of the corresponding hyperplane arrangement.

Recently, Liu, Tran, and Yoshinaga showed that the last constituent of the characteristic quasi-polynomial coincides with the characteristic polynomial of the corresponding toric arrangement.

In addition, by using the theory of toric arrangements, Higashitani, Tran, and Yoshinaga proved that the LCM-period is the minimum period of the characteristic quasi-polynomial.

In this paper, we study an arrangements over a Dedekind domain such that every residue ring with a nonzero ideal is finite and give algebraic generalizations of the above results.

Keywords: Characteristic quasi-polynomial; Dedekind domain; Hyperplane arrangement; Root system
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1. Introduction

For a positive integer ℓ, let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathbb{Z}^\ell$ be a finite subset consisting of nonzero integral column vectors. Roughly speaking, an *arrangement* means a finite collection of subspaces of codimension 1 of a space. We can define three kinds of arrangements from $\mathcal{A}$ as follows:

The first is the *hyperplane arrangement* $\mathcal{A}(\mathbb{R}) = \{H_1, \dots, H_n\}$ in the vector space $\mathbb{R}^\ell$ consisting of hyperplanes

$$H_j := \left\{ \boldsymbol{x} = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell \;\middle|\; \boldsymbol{x} c_j = 0 \right\} \quad (j \in [n] := \{1, \dots, n\}).$$

The second is the *q-reduced arrangement* $\mathcal{A}(\mathbb{Z}/q\mathbb{Z}) = \{H_{1,q}, \dots, H_{n,q}\}$ in $(\mathbb{Z}/q\mathbb{Z})^\ell$ for any positive integer q, where

$$H_{j,q} := \left\{ [\boldsymbol{x}]_q \in (\mathbb{Z}/q\mathbb{Z})^\ell \;\middle|\; \boldsymbol{x} c_j \equiv 0 \pmod q \right\} \quad (j \in [n])$$

and $[\boldsymbol{x}]_q$ denotes the equivalence class of $\boldsymbol{x}$.

The third is the *toric arrangement* $\mathcal{A}(\mathbb{C}^\times) = \{T_1, \dots, T_n\}$ in the algebraic torus $(\mathbb{C}^\times)^\ell$, where

$$T_j := \left\{ (t_1, \dots, t_n) \in (\mathbb{C}^\times)^\ell \;\middle|\; \prod_{i=1}^{\ell} t_i^{c_{ij}} = 1 \right\} \quad (j \in [n])$$

and c_{ij} denotes the i-th coefficient of the column vector c_j.

Combinatorics of $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{C}^\times)$ is described as the posets of the intersections defined as follows:

$$L(\mathcal{A}(\mathbb{R})) := \{ H_J \mid J \subseteq [n] \}, \qquad L(\mathcal{A}(\mathbb{C}^\times)) := \{ Z \mid J \subseteq [n],\ Z \text{ is a connected component of } T_J \},$$

where $H_J := \bigcap_{j\in J} H_j$ and $T_J := \bigcap_{j\in J} T_j$, and the partial order defined by reverse inclusion for each poset. Note that $H_\varnothing = \mathbb{R}^\ell$ and $T_\varnothing = (\mathbb{C}^\times)^\ell$ are the minimal elements.

Main concerns for study of arrangements are relations among combinatrics, algebra, and geometry of arrangements. The characteristic polynomials of the posets play important rolls, which are defined by using the Möbius functions on posets. Let P be a finite poset with unique minimal element $\hat{0}$. The *Möbius function* μ on P is defined recursively by

$$\mu(\hat{0}) := 1 \qquad \text{and} \qquad \mu(Z) := -\sum_{Y<Z} \mu(Y) \text{ for } Z \neq \hat{0}.$$

The *characteristic polynomials* $\chi_{\mathcal{A}(\mathbb{R})}(t)$ and $\chi_{\mathcal{A}(\mathbb{C}^\times)}$ are defined by

$$\chi_{\mathcal{A}(\mathbb{R})}(t) := \sum_{Z\in L(\mathcal{A}(\mathbb{R}))} \mu(Z)t^{\dim Z}, \qquad \chi_{\mathcal{A}(\mathbb{C}^\times)}(t) := \sum_{Z\in L(\mathcal{A}(\mathbb{C}^\times))} \mu(Z)t^{\dim Z}.$$

The *complement* of an arrangement is the complement of the union of the members of the arrangement in the ambient space. Each connected component of the complement of $\mathcal{A}(\mathbb{R})$ is called a chamber. Zaslavsky [18] proved that the numbers of chambers and bounded chambers coincide with $|\chi_{\mathcal{A}(\mathbb{R})}(-1)|$ and $|\chi_{\mathcal{A}(\mathbb{R})}(1)|$. Orlik and Solomon [11] proved that $\chi_{\mathcal{A}(\mathbb{R})}(t)$ is equivalent to the Poincaré polynomial of the complement of the complexification of $\mathcal{A}(\mathbb{R})$. Moci [10, Corollary 5.12] showed that $\chi_{\mathcal{A}(\mathbb{C}^\times)}(t)$ is equivalent to the Poincaré polynomial of the complement of $\mathcal{A}(\mathbb{C}^\times)$. These facts present a strong association between combinatorics and the geometry of arrangements.

Athanasiadis [1, Theorem 2.2] provided a method to compute the characteristic polynomial of an integral arrangement by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/p\mathbb{Z})$ for large enough prime numbers p. Athanasiadis [2, Theorem 2.1] also proved that the characteristic polynomial can be computed by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for large enough integers q relatively prime a constant which depends only on $\mathcal{A}$.

Kamiya, Takemura, and Terao developed Athanasiadis' method by considering the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for all positive integers q as follows. For a nonempty subset $J = \{j_1, \dots, j_k\} \subseteq [n]$, suppose that the matrix $C_J := (c_{j_1} \ \cdots \ c_{j_k})$ has the Smith normal form

$$\begin{pmatrix}
d_{J,1} & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & d_{J,2} & & \vdots & & & \vdots \\
\vdots & & \ddots & 0 & & & \\
0 & \cdots & 0 & d_{J,r(J)} & & & \\
\vdots & & & & 0 & & \\
\vdots & & & & & \ddots & \vdots \\
0 & \cdots & & & & \cdots & 0
\end{pmatrix},$$

where $d_{J,i}$ is a positive integer such that $d_{J,i}$ divides $d_{J,i+1}$. Define $\rho_{\mathcal{A}} \in \mathbb{Z}_{>0}$ by

$$\rho_{\mathcal{A}} := \operatorname{lcm}\left\{\, d_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \,\right\}.$$

Theorem 1.1 (Kamiya-Takemura-Terao [6, Theorem 2.4])**.** *Let $M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z})) := (\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{J\subseteq[n]} H_{J,q}$ denote the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$. Then the function $|M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ is a monic integral quasi-polynomial in $q \in \mathbb{Z}_{>0}$ with a period $\rho_{\mathcal{A}}$. Namely, there exist monic polynomials $f^k_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ $(1 \leq k \leq \rho_{\mathcal{A}})$ such that $f^k_{\mathcal{A}}(q) = |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ if $q \equiv k \pmod{\rho_{\mathcal{A}}}$. Furthermore, the quasi-polynomial has the GCD-property, that is, $f^k_{\mathcal{A}}(t) = f^{k'}_{\mathcal{A}}(t)$ when $\gcd(k, \rho_{\mathcal{A}}) = \gcd(k', \rho_{\mathcal{A}})$.*

Definition 1.1. *We call the quasi-polynomial*

$$\chi^{\mathrm{quasi}}_{\mathcal{A}}(q) := |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$$

the characteristic quasi-polynomial of $\mathcal{A}$. The period $\rho_{\mathcal{A}}$ is called the LCM-period. The polynomial $f^k_{\mathcal{A}}(t)$ is said to be the k-constituent of $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$.

Interestingly enough, each constituent of the characteristic quasi-polynomial has a combinatorial interpretation.

Theorem 1.2 (Kamiya-Takemura-Terao [6, Theorem 2.5])**.** *The 1-constituent of the characteristic quasi-polynomial of $\mathcal{A}$ is the characteristic polynomial of the hyperplane arrangement $\mathcal{A}(\mathbb{R})$. Namely, $f^1_{\mathcal{A}}(t) = \chi_{\mathcal{A}(\mathbb{R})}(t)$.*

Theorem 1.3 (Liu–Tran–Yoshinaga [9, Corollary 5.6]). *The $\rho_{\mathcal{A}}$-constituent of the characteristic quasi-polynomial of $\mathcal{A}$ is the characteristic polynomial of the toric arrangement $\mathcal{A}(\mathbb{C}^{\times})$. Namely, $f_{\mathcal{A}}^{\rho_{\mathcal{A}}}(t) = \chi_{\mathcal{A}(\mathbb{C}^{\times})}(t)$.*

Tran and Yoshinaga [15] gave combinatorial interpretation for the other constituents and unified Theorem 1.2 and Theorem 1.3. For a positive integer k, define

$$\begin{aligned} L(\mathcal{A}(\mathbb{C}^{\times}))[k] &:= \left\{ Z \in L(\mathcal{A}(\mathbb{C}^{\times})) \mid Z \text{ contains a } k\text{-torsion element} \right\}, \\ \chi^{k}_{\mathcal{A}(\mathbb{C}^{\times})}(t) &:= \sum_{Z \in L(\mathcal{A}(\mathbb{C}^{\times}))[k]} \mu(Z) t^{\dim Z}. \end{aligned}$$

Note that $\chi^{1}_{\mathcal{A}(\mathbb{C}^{\times})}(t) = \chi_{\mathcal{A}(\mathbb{R})}(t)$ and $\chi^{\rho_{\mathcal{A}}}_{\mathcal{A}(\mathbb{C}^{\times})}(t) = \chi_{\mathcal{A}(\mathbb{C}^{\times})}(t)$.

Theorem 1.4 (Tran-Yoshinaga [15, Corollary 4.8]). *Let $1 \leq k \leq \rho_{\mathcal{A}}$. Then $f^{k}_{\mathcal{A}}(t) = \chi^{k}_{\mathcal{A}(\mathbb{C}^{\times})}(t)$.*

For a decade, it was an open problem whether the LCM-period is minimum or not. Recently, using Theorem 1.4, Higashitani, Tran, and Yoshinaga gave an affirmative answer for central arrangements.

Theorem 1.5 (Higashitani-Tran-Yoshinaga [4, Theorem 1.2]). *The LCM-period $\rho_{\mathcal{A}}$ is the minimum period of the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$.*

Remark 1.1. *The characteristic quasi-polynomial and its LCM-period can be considered for non-central arrangements [8]. Higashitani, Tran, and Yoshinaga [4] also studied non-central arrangements such that the LCM-periods are not minimum.*

The properties of the characteristic quasi-polynomial highly depend on ring-theoretic properties of $\mathbb{Z}$. In particular, the theory of Smith normal forms and their invariant factors are essential. It is a natural question to ask what occurs if we replace the ring $\mathbb{Z}$ with another ring $\mathcal{O}$. Let $\mathcal{A}$ be a finite subset of $\mathcal{O}^{\ell}$ and we will investigate the function $|M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ in nonzero ideals of $\mathfrak{a}$. In order to do this, $|M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ must be finite for every $\mathfrak{a}$. Moreover, the theory of finitely generated modules of Dedekind domains is considered to be a direct generalization of the theory of Smith normal forms.

For the reasons above, in this article, we will consider the function $|M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ under the following convention.

- $\mathcal{O}$ is a Dedekind domain, that is, an integral domain in which every nonzero proper ideal has a unique factorization into prime ideals.
- $\mathcal{O}/\mathfrak{a}$ is a finite ring for every nonzero ideal $\mathfrak{a}$.

Such a ring $\mathcal{O}$ is called a *residually finite Dedekind domain* or a *Dedekind domain with the finite norm property.* The ring $\mathbb{Z}$ is a typical example of a residually finite Dedekind domain. More generally, it is well known that the ring of integers of an algebraic field is a residually finite Dedekind domain. The polynomial ring $\mathbb{F}_q[t]$ over a finite field $\mathbb{F}_q$ and the ring of p-adic integers $\mathbb{Z}_p$ are also examples of residually finite Dedekind domains.

The purpose of this paper is to give algebraic generalizations of Theorem 1.1, 1.2, 1.3, 1.4, and 1.5 for a finite subset $\mathcal{A} = \{c_1, \dots, c_n\}$ of $\mathcal{O}^{\ell}$. Let $K := \operatorname{Frac}\mathcal{O}$ denote the fraction filed of $\mathcal{O}$. We will consider the following three kinds of arrangements:

The first is the *hyperplane arrangement* $\mathcal{A}(K) = \{H_1, \dots, H_n\}$, where

$$H_j := \left\{ \boldsymbol{x} \in K^{\ell} \mid \boldsymbol{x}c_j = 0 \right\}.$$

The second is the $\mathfrak{a}$*-reduced arrangement* $\mathcal{A}(\mathcal{O}/\mathfrak{a}) = \{H_{1,\mathfrak{a}}, \dots, H_{n,\mathfrak{a}}\}$ for every nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}$, where

$$H_{j,\mathfrak{a}} := \left\{ [\boldsymbol{x}]_{\mathfrak{a}} \in (\mathcal{O}/\mathfrak{a})^{\ell} \mid \boldsymbol{x}c_j \equiv 0 \pmod{\mathfrak{a}} \right\}$$

and $[\boldsymbol{x}]_{\mathfrak{a}}$ denotes the equivalent class of $\boldsymbol{x}$.

The third is the *torsion arrangement* $\mathcal{A}(K/\mathcal{O}) = \{T_1, \dots, T_n\}$, where

$$T_j := \left\{ \pi(\boldsymbol{x}) \in (K/\mathcal{O})^{\ell} \mid \boldsymbol{x}c_j \equiv 0 \pmod{\mathcal{O}} \right\}$$

and $\pi\colon K \to K/\mathcal{O}$ denotes the canonical projection. Note that $\mathbb{Q} = \operatorname{Frac}\mathbb{Z}$ and if $\mathcal{A} \subseteq \mathbb{Z}^{\ell}$, then $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{C}^{\times})$ are combinatorially equivalent to $\mathcal{A}(\mathbb{Q})$ and $\mathcal{A}(\mathbb{Q}/\mathbb{Z})$, respectively. The torsion arrangement $\mathcal{A}(K/\mathcal{O})$ will work as a substitute of the toric arrangement.

Let $I(\mathcal{O})$ be the set consisting of nonzero ideals of $\mathcal{O}$. For every $\mathfrak{a} \in I(\mathcal{O})$, define the *complement* $M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))$ by

$$M(\mathcal{A}(\mathcal{O}/\mathfrak{a})) := (\mathcal{O}/\mathfrak{a})^{\ell} \setminus \bigcup_{j=1}^{n} H_{j,\mathfrak{a}}.$$

Definition 1.2. *The function $\chi^{\text{quasi}}_{\mathcal{A}}\colon I(\mathcal{O}) \to \mathbb{Z}$ determined by $\chi^{\text{quasi}}_{\mathcal{A}}(\mathfrak{a}) := |M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ is called the characteristic quasi-polynomial of $\mathcal{A}$.*

The organization of this article is as follows: In Section 2, we introduce and study an algebraic generalization of a quasi-polynomial with GCD-property (Definition 2.1), which is a function on nonzero proper ideals of $\mathcal{O}$ represented by using "periodically" finitely many polynomials, called constituents. A "period" of this generalization is a nonzero ideal and every constituent corresponds with a factor of a period.

In Section 3, we will show that the characteristic quasi-polynomial $\chi^{\text{quasi}}_{\mathcal{A}}$ is a quasi-polynomial in the sense of the previous section with the LCM-period $\rho_{\mathcal{A}}$ (Theorem 3.1), which generalize Theorem 1.1. Moreover, we will give a generalization of Theorem 1.2 which states that $\langle 1\rangle$-constituent of $\chi^{\text{quasi}}_{\mathcal{A}}(\mathfrak{a})$ coincides with the characteristic polynomial $\chi_{\mathcal{A}(K)}(t)$ of the hyperplane arrangement $\mathcal{A}(K)$ (Theorem 3.2), where $\langle 1\rangle = \mathcal{O}$ stands for the unit ideal.

In Section 4, we will study the torsion arrangement $\mathcal{A}(K/\mathcal{O})$ and the poset $L(\mathcal{A}(K/\mathcal{O}))$ defined by the information of the intersections of $\mathcal{A}(K/\mathcal{O})$, called the poset of layers. We will introduce the κ-torsion subposet $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$ and its characteristic polynomial $\chi^{\kappa}_{\mathcal{A}(K/\mathcal{O})}(t)$ for each factor κ of the LCM-period $\rho_{\mathcal{A}}$ and prove that $\chi^{\kappa}_{\mathcal{A}(K/\mathcal{O})}(t)$ coincides with the κ-constituent of $\chi^{\text{quasi}}_{\mathcal{A}}$ (Theorem 4.3). This result is a generalization of Theorem 1.4 including Theorem 1.3.

In Section 5, with the similar technique as the proof of Theorem 1.5, we will prove that the LCM-period $\rho_{\mathcal{A}}$ is the minimum period of $\chi^{\text{quasi}}_{\mathcal{A}}$ (Theorem 5.1).

In Section 6, we will study the behavior of the characteristic quasi-polynomials and the posets of layers under taking localization of the base rings.

In Section 7, we will discuss general properties of the characteristic quasi-polynomials, which are generalizations of results in [8] and are very useful for computing.

Kamiya, Takemura, and Terao [8] studied the characteristic quasi-polynomial of the irreducible crystallographic root systems. Thanks to the results in this article, we can consider the characteristic quasi-polynomials of non-crystallographic root systems. In Section 8, we will give the characteristic quasi-polynomials explicitly for non-crystallographic root systems of types H_2, H_3, and H_4.

2. Quasi-polynomials on the set of nonzero ideals

Let $\mathcal{O}$ be a residually finite Dedekind domain. Let $I(\mathcal{O})$ denote the set consisting of nonzero ideals of $\mathcal{O}$ and $N(\mathfrak{a})$ the *absolute norm* of $\mathfrak{a} \in I(\mathcal{O})$, that is, the cardinality of the quotient ring $\mathcal{O}/\mathfrak{a}$.

Note that every ideal in $I(\mathcal{O})$ factors into a product of prime ideals uniquely. We write $\mathfrak{a} \mid \mathfrak{b}$ if $\mathfrak{a}$ is a factor of $\mathfrak{b}$, or equivalently $\mathfrak{a} \supseteq \mathfrak{b}$. The sum $\mathfrak{a}+\mathfrak{b}$ coincides with the greatest common divisor $\gcd(\mathfrak{a}, \mathfrak{b})$ and the intersection $\mathfrak{a} \cap \mathfrak{b}$ coincides the least common multiple $\operatorname{lcm}(\mathfrak{a}, \mathfrak{b})$.

Definition 2.1. *A function $\phi\colon I(\mathcal{O}) \to \mathbb{Z}$ is called a quasi-polynomial on $I(\mathcal{O})$ if there exists an ideal $\rho \in I(\mathcal{O})$ and polynomials $f^{\kappa}(t) \in \mathbb{Z}[t]$ for each $\kappa \mid \rho$ such that for any $\mathfrak{a} \in I(\mathcal{O})$ with $\mathfrak{a} + \rho = \kappa$,*

$$\phi(\mathfrak{a}) = f^{\kappa}(N(\mathfrak{a})).$$

The ideal ρ is called a period and the polynomial $f^{\kappa}(t)$ is called the κ-constituent of the quasi-polynomial ϕ associated with ρ. A period that divides every period is called the minimum period of the quasi-polynomial ϕ.

Remark 2.1. *A quasi-polynomial on $I(\mathbb{Z})$ is a quasi-polynomial* with the GCD-property. *Our generalization of quasi-polynomial* does not *contain quasi-polynomials without the GCD-property.*

For any nonzero prime ideal $\mathfrak{p} \in I(\mathcal{O})$, we define $\operatorname{ord}_{\mathfrak{p}}\colon I(\mathcal{O}) \to \mathbb{Z}_{\geq 0}$ by $\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) := \max\{\, a \in \mathbb{Z}_{\geq 0} \mid \mathfrak{p}^a \mid \mathfrak{a} \,\}$.

Proposition 2.1. *Let ϕ be a quasi-polynomial on $I(\mathcal{O})$. The following statements hold:*

(i) For any periods ρ_1 and ρ_2 of ϕ, the sum $\rho_1 + \rho_2$ is a period of ϕ.

(ii) The minimum period of ϕ always exists.

Proof. (i) For $i \in \{1, 2\}$, let $f^{\kappa}_i(t) \in \mathbb{Z}[t]$ denote the κ-constituent of ϕ associated with the period ρ_i for $\kappa \mid \rho_i$. Let $\rho := \rho_1 + \rho_2$ and fix a divisor κ of ρ. It is sufficient to show that there exists a polynomial $f^{\kappa}(t) \in \mathbb{Z}[t]$ such that $\mathfrak{a} + \rho = \kappa$ implies $\phi(\mathfrak{a}) = f^{\kappa}(N(\mathfrak{a}))$.

First suppose that $\operatorname{ord}_{\mathfrak{p}}(\kappa) < \operatorname{ord}_{\mathfrak{p}}(\rho)$ for any nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$. Note that this happens only when the number of prime ideals of $\mathcal{O}$ is finite. If $\mathfrak{a} + \rho = \kappa$, then $\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) = \operatorname{ord}_{\mathfrak{p}}(\kappa)$ for all nonzero prime ideals $\mathfrak{p}$ in $\mathcal{O}$ and hence $\mathfrak{a} = \kappa$. Since $\mathfrak{a} + \rho_1 = \kappa + \rho_1 = \kappa + \rho = \kappa$, we have $\phi(\mathfrak{a}) = f^{\kappa}_1(N(\mathfrak{a}))$. Thus the polynomial $f^{\kappa}(t) := f^{\kappa}_1(t)$ has the desired property.

Now, we consider the case there exists a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ such that $\mathrm{ord}_{\mathfrak{p}}(\kappa) = \mathrm{ord}_{\mathfrak{p}}(\rho)$. Without loss of generality, we can assume that $\mathrm{ord}_{\mathfrak{p}}(\rho_1) \leq \mathrm{ord}_{\mathfrak{p}}(\rho_2)$. Suppose that κ' is an ideal such that $\kappa \mid \kappa'$ and $\kappa' \mid \rho_1$. Note that $\mathrm{ord}_{\mathfrak{p}}(\kappa) = \mathrm{ord}_{\mathfrak{p}}(\kappa') = \mathrm{ord}_{\mathfrak{p}}(\rho_1)$.

Put $\nu := \mathrm{ord}_{\mathfrak{p}}(\rho_2) - \mathrm{ord}_{\mathfrak{p}}(\rho_1) \geq 0$. Then for any integer $m \geq \nu$, we have $\mathfrak{p}^m\kappa' + \rho_1 = \kappa'$ and $\mathfrak{p}^m\kappa + \rho_2 = \mathfrak{p}^\nu\kappa$. Therefore $f_1^{\kappa'}(N(\mathfrak{p}^m\kappa)) = \phi(\mathfrak{p}^m\kappa) = f_2^{\mathfrak{p}^\nu\kappa}(N(\mathfrak{p}^m\kappa))$ for any $m \geq \nu$. Thus $f_1^{\kappa'}(t) = f_2^{\mathfrak{p}^\nu\kappa}(t)$. In particular, $f_1^{\kappa'}(t) = f_1^{\kappa}(t)$ for any ideal κ' such that $\kappa \mid \kappa'$ and $\kappa' \mid \rho_1$.

Suppose that $\mathfrak{a}+\rho = \kappa$. Then $\kappa' := \mathfrak{a}+\rho_1$ satisfies $\kappa \mid \kappa'$ and $\kappa' \mid \rho_1$. Therefore $\phi(\mathfrak{a}) = f_1^{\kappa'}(N(\mathfrak{a})) = f_1^{\kappa}(N(\mathfrak{a}))$. Hence $f^\kappa(t) := f_1^\kappa(t)$ satisfies the desired condition.

(ii) Let ρ_0 be the greatest common divisor of all periods of ϕ. It suffices to show that ρ_0 is a period of ϕ. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the prime factors of ρ_0. Then there exists a period ρ_i of ϕ such that $\mathrm{ord}_{\mathfrak{p}_i}(\rho_i) = \mathrm{ord}_{\mathfrak{p}_i}(\rho_0)$ for each $i \in \{1, \dots, s\}$. Since

$$\rho_0 = \prod_{i=1}^{s} \mathfrak{p}_i^{\mathrm{ord}_{\mathfrak{p}_i}(\rho_0)} = \rho_1 + \cdots + \rho_s.$$

we can conclude that ρ_0 is a period of ϕ by (i). $\square$

Proposition 2.2. *Suppose that ϕ_1 and ϕ_2 are quasi-polynomials on $I(\mathcal{O})$ with period ρ_1 and ρ_2, respectively. Then the function $(\phi_1+\phi_2)(\mathfrak{a}) := \phi_1(\mathfrak{a}) + \phi_2(\mathfrak{a})$ is a quasi-polynomial of $I(\mathcal{O})$ with period* $\mathrm{lcm}(\rho_1, \rho_2)$.

Proof. For $i \in \{1,2\}$, let $f_i^\kappa(t) \in \mathbb{Z}[t]$ denote the κ-constituent of ϕ_i associated with the period ρ_i. For any divisor κ of $\mathrm{lcm}(\rho_1,\rho_2)$, define $f^\kappa(t) \in \mathbb{Z}[t]$ by $f^\kappa(t) := f_1^{\kappa+\rho_1}(t) + f_2^{\kappa+\rho_2}(t)$. Assume that $\mathfrak{a} + \mathrm{lcm}(\rho_1,\rho_2) = \kappa$. Then $\mathfrak{a}+\rho_i = \kappa+\rho_i$ for each $i \in \{1,2\}$. Therefore $f^\kappa(N(\mathfrak{a})) = f_1^{\kappa+\rho_1}(N(\mathfrak{a})) + f_2^{\kappa+\rho_2}(N(\mathfrak{a})) = \phi_1(\mathfrak{a}) + \phi_2(\mathfrak{a}) = (\phi_1+\phi_2)(\mathfrak{a})$. $\square$

3. The characteristic quasi-polynomial of $\mathcal{A}$

Let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathcal{O}^\ell$ and $\mathfrak{a} \in I(\mathcal{O})$. Recall that the $\mathfrak{a}$-*reduced arrangement* $\mathcal{A}(\mathcal{O}/\mathfrak{a}) := \{\, H_{j,\mathfrak{a}} \mid j \in [n] \,\}$ consists of hyperplanes

$$H_{j,\mathfrak{a}} := \left\{ [\boldsymbol{x}]_{\mathfrak{a}} \in (\mathcal{O}/\mathfrak{a})^\ell \;\middle|\; \boldsymbol{x}c_j \equiv 0 \pmod{\mathfrak{a}} \right\}.$$

Let $M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))$ denote the complement of $\mathcal{A}(\mathcal{O}/\mathfrak{a})$. Namely

$$M(\mathcal{A}(\mathcal{O}/\mathfrak{a})) := (\mathcal{O}/\mathfrak{a})^\ell \setminus \bigcup_{j=1}^{n} H_{j,\mathfrak{a}}.$$

The main purpose of this section is to prove that the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a}) = |M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ is a quasi-polynomial discussed in Section 2. The proof will proceed in a similar way with the proof of Theorem 1.1. However, since in general $\mathcal{O}$ is not a principal ideal domain, we cannot use the Smith normal forms and we will use the structure theorem of finitely generated modules over a Dedekind domain instead of them.

For a nonempty subset $J = \{j_1, \dots, j_k\} \subseteq [n]$, we define

$$H_{J,\mathfrak{a}} := \bigcap_{j \in J} H_{j,\mathfrak{a}} = H_{j_1,\mathfrak{a}} \cap \cdots \cap H_{j_k,\mathfrak{a}}.$$

By the inclusion-exclusion principle,

$$\begin{aligned}\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a}) = |M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))| &= \left|(\mathcal{O}/\mathfrak{a})^\ell\right| - \left|\bigcup_{j=1}^{n} H_{j,\mathfrak{a}}\right| = N(\mathfrak{a})^\ell - \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subseteq [n] \\ |J|=k}} \left|\bigcap_{j\in J} H_{j,\mathfrak{a}}\right| \\ &= N(\mathfrak{a})^\ell + \sum_{\varnothing \neq J \subseteq [n]} (-1)^{|J|} \left|H_{J,\mathfrak{a}}\right|.\end{aligned}$$

In order to prove that $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a})$ is a quasi-polynomial, it is sufficient to show that the cardinality $|H_{J,\mathfrak{a}}|$ is a quasi-polynomial for each J by Proposition 2.2.

Let $C_J := (c_{j_1}, \dots c_{j_k}) \in \mathrm{Mat}_{\ell\times k}(\mathcal{O})$. We define an $\mathcal{O}$-homomorphism $\phi_J \colon \mathcal{O}^\ell \to \mathcal{O}^{|J|}$ by $\phi_J(\boldsymbol{x}) := \boldsymbol{x}C_J$. It induces an $\mathcal{O}$-homomorphism $\phi_{J,\mathfrak{a}} \colon (\mathcal{O}/\mathfrak{a})^\ell \to (\mathcal{O}/\mathfrak{a})^{|J|}$ defined by $\phi_{J,\mathfrak{a}}([\boldsymbol{x}]_{\mathfrak{a}}) := [\phi_J(\boldsymbol{x})]_{\mathfrak{a}}$. Note that $|H_{J,\mathfrak{a}}| = |\ker \phi_{J,\mathfrak{a}}|$.

Since $\mathrm{coker}\,\phi_J = \mathcal{O}^k / \mathrm{Im}\,\phi_J$ is a finitely generated $\mathcal{O}$-module, by the structure theorem for finitely generated modules over a Dedekind domain (See for example [3, Theorem 5.1]), we obtain the following key lemma.

Lemma 3.1. *There exist nonzero ideals $\mathfrak{d}_J, \mathfrak{d}_{J,1}, \mathfrak{d}_{J,2}, \dots, \mathfrak{d}_{J,r(J)}$ such that*

$$\operatorname{coker}\phi_J \simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/\mathfrak{d}_{J,i} \oplus \mathfrak{d}_J \oplus \mathcal{O}^{|J|-r(J)-1} \qquad (\mathfrak{d}_{J,i} \mid \mathfrak{d}_{J,i+1}, \quad \text{for } 1 \leq i \leq r(J)-1),$$

where $r(J) := \operatorname{rank}(\operatorname{Im}\phi_J)$ and if $r(J) = |J|$, then we regard $\mathfrak{d}_J \oplus \mathcal{O}^{|J|-r(J)-1}$ as $\{0\}$.

Lemma 3.2 (See also [6, Lemma 2.1])**.**

$$|H_{J,\mathfrak{a}}| = |\ker\phi_{J,\mathfrak{a}}| = \left(\prod_{i=1}^{r(J)} N(\mathfrak{a}+\mathfrak{d}_{J,i})\right) N(\mathfrak{a})^{\ell-r(J)}.$$

Moreover, $|H_{J,\mathfrak{a}}|$ is a quasi-monomial with GCD-property of degree $\ell - r(J)$ and its minimum period is $\mathfrak{d}_{J,r(J)}$.

Proof. Since $(\mathcal{O}/\mathfrak{a})^\ell / \ker\phi_{J,\mathfrak{a}} \simeq \operatorname{Im}\phi_{J,\mathfrak{a}}$ and $\operatorname{coker}\phi_{J,\mathfrak{a}} = (\mathcal{O}/\mathfrak{a})^k / \operatorname{Im}\phi_{J,\mathfrak{a}}$, we have $|\operatorname{Im}\phi_{J,\mathfrak{a}}| = N(\mathfrak{a})^\ell / |\ker\phi_{J,\mathfrak{a}}|$ and $|\operatorname{coker}\phi_{J,\mathfrak{a}}| = N(\mathfrak{a})^k / |\operatorname{Im}\phi_{J,\mathfrak{a}}|$. Hence we obtain

$$|\ker\phi_{J,\mathfrak{a}}| = \frac{N(\mathfrak{a})^\ell}{|\operatorname{Im}\phi_{J,\mathfrak{a}}|} = \frac{N(\mathfrak{a})^\ell\,|\operatorname{coker}\phi_{J,\mathfrak{a}}|}{N(\mathfrak{a})^k} = N(\mathfrak{a})^{\ell-k}\,|\operatorname{coker}\phi_{J,\mathfrak{a}}|\,.$$

Consider the exact sequence

$$\mathcal{O}^\ell \xrightarrow{\phi_J} \mathcal{O}^k \longrightarrow \operatorname{coker}\phi_J \longrightarrow 0.$$

Since tensoring with $\mathcal{O}/\mathfrak{a}$ is a right exact functor, we obtain the exact sequence

$$\mathcal{O}^\ell \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} \xrightarrow{\phi_J\otimes\mathrm{id}} \mathcal{O}^k \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} \longrightarrow (\operatorname{coker}\phi_J) \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} \longrightarrow 0.$$

Also, we have the following commutative diagram of $\mathcal{O}$-modules, where the vertical arrows show isomorphisms:

$$\begin{array}{ccc} \mathcal{O}^\ell \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} & \xrightarrow{\phi_J\otimes\mathrm{id}} & \mathcal{O}^k \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} \\ \downarrow & & \downarrow \\ (\mathcal{O}/\mathfrak{a})^\ell & \xrightarrow{\phi_{J,\mathfrak{a}}} & (\mathcal{O}/\mathfrak{a})^k \end{array}$$

Then we obtain

$$\begin{aligned} \operatorname{coker}\phi_{J,\mathfrak{a}} &\simeq (\operatorname{coker}\phi_J) \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a} \\ &\simeq \bigoplus_{i=1}^{r(J)} (\mathcal{O}/\mathfrak{d}_{J,i} \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a}) \oplus (\mathfrak{d}_J \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a}) \oplus \left(\mathcal{O}^{k-r(J)-1} \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{a}\right) \\ &\simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/\left(\mathfrak{a}+\mathfrak{d}_{J,i}\right) \oplus \mathfrak{d}_J/\mathfrak{a}\mathfrak{d}_J \oplus (\mathcal{O}/\mathfrak{a})^{k-r(J)-1}. \end{aligned}$$

Since $\mathcal{O}$ is a Dedekind domain, $\mathfrak{d}_J/\mathfrak{a}\mathfrak{d}_J \simeq \mathcal{O}/\mathfrak{a}$. Hence

$$\operatorname{coker}\phi_{J,\mathfrak{a}} \simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/\left(\mathfrak{a}+\mathfrak{d}_{J,i}\right) \oplus (\mathcal{O}/\mathfrak{a})^{k-r(J)}.$$

Therefore we obtain

$$\begin{aligned} |H_{J,\mathfrak{a}}| = |\ker\phi_{J,\mathfrak{a}}| &= N(\mathfrak{a})^{\ell-k}\,|\operatorname{coker}\phi_{J,\mathfrak{a}}| = N(\mathfrak{a})^{\ell-k} \cdot \left(\prod_{i=1}^{r(J)} N\left(\mathfrak{a}+\mathfrak{d}_{J,i}\right)\right) \cdot N(\mathfrak{a})^{k-r(J)} \\ &= \left(\prod_{i=1}^{r(J)} N\left(\mathfrak{a}+\mathfrak{d}_{J,i}\right)\right) N(\mathfrak{a})^{\ell-r(J)}. \end{aligned}$$

Next, we will show that $|H_{J,\mathfrak{a}}|$ is a quasi-monomial with period $\mathfrak{d}_{J,r(J)}$. For each $\kappa \mid \mathfrak{d}_{J,r(J)}$, define the monomial $f^{\kappa}(t) \in \mathbb{Z}[t]$ by

$$f^{\kappa}(t) := \left(\prod_{i=1}^{r(J)} N\left(\kappa + \mathfrak{d}_{J,i}\right)\right) t^{\ell - r(J)}.$$

Suppose that $\mathfrak{a} + \mathfrak{d}_{J,r(J)} = \kappa$. Then $\mathfrak{a} + \mathfrak{d}_{J,i} = \kappa + \mathfrak{d}_{J,i}$ for each $i \in \{1, \dots, r(J)\}$ since $\mathfrak{d}_{J,i} \mid \mathfrak{d}_{J,r(J)}$. Hence $f^{\kappa}(N(\mathfrak{a})) = |H_{J,\mathfrak{a}}|$.

Finally, we will show that $\mathfrak{d}_{J,r(J)}$ is the minimum period of $|H_{J,\mathfrak{a}}|$. Let ρ_0 be the minimum period. For any $\kappa \mid \rho_0$ there exists $g^{\kappa}(t) \in \mathbb{Z}[t]$ such that $\mathfrak{a} + \rho_0 = \kappa$ implies $g^{\kappa}(N(\mathfrak{a})) = |H_{J,\mathfrak{a}}|$. Suppose that $\mathfrak{a}$ is a multiple of $\mathfrak{d}_{J,r(J)}$. Then $\mathfrak{a} + \mathfrak{d}_{J,r(J)} = \mathfrak{d}_{J,r(J)}$ and $\mathfrak{a} + \rho_0 = \rho_0$. Therefore $f^{\mathfrak{d}_{J,r(J)}}(N(\mathfrak{a})) = |H_{J,\mathfrak{a}}| = g^{\rho_0}(N(\mathfrak{a}))$. Since there are infinitely many such ideals $\mathfrak{a}$, we have $f^{\mathfrak{d}_{J,r(J)}}(t) = g^{\rho_0}(t)$. Hence

$$f^{\mathfrak{d}_{J,r(J)}}(N(\rho_0)) = g^{\rho_0}(N(\rho_0)) = |H_{J,\rho_0}| = f^{\rho_0}(N(\rho_0)).$$

Therefore

$$\prod_{i=1}^{r(J)} N(\mathfrak{d}_{J,i}) = \prod_{i=1}^{r(J)} N(\mathfrak{d}_{J,r(J)} + \mathfrak{d}_{J,i}) = \prod_{i=1}^{r(J)} N(\rho_0 + \mathfrak{d}_{J,i}).$$

Since $N(\mathfrak{d}_{J,i}) \geq N(\mathfrak{d}_{J,i} + \rho_0)$ for each $i \in \{1, \dots, r(J)\}$, we have $N(\mathfrak{d}_{J,i}) = N(\mathfrak{d}_{J,i} + \rho_0)$ for every $i \in \{1, \dots, r(J)\}$. In particular $N(\mathfrak{d}_{J,r(J)}) = N(\mathfrak{d}_{J,r(J)} + \rho_0) = N(\rho_0)$. Since $\rho_0 \mid \mathfrak{d}_{J,r(J)}$, we have $\rho_0 = \mathfrak{d}_{J,r(J)}$. ☐

From the discussion so far, we have

$$\begin{aligned}
\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a}) &= N(\mathfrak{a})^{\ell} + \sum_{\varnothing \neq J \subseteq [n]} (-1)^{|J|} |H_{J,\mathfrak{a}}| \\
&= N(\mathfrak{a})^{\ell} + \sum_{\varnothing \neq J \subseteq [n]} (-1)^{|J|} \left(\prod_{i=1}^{r(J)} N\left(\mathfrak{a} + \mathfrak{d}_{J,i}\right)\right) N(\mathfrak{a})^{\ell - r(J)}.
\end{aligned}$$

Since $\mathfrak{d}_{J,r(J)}$ is a period of $|H_{J,\mathfrak{a}}|$,

$$\rho_{\mathcal{A}} := \operatorname{lcm}\left\{ \mathfrak{d}_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \right\}$$

is a period of $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a})$ by Proposition 2.2. We call $\rho_{\mathcal{A}}$ the *LCM-period*. In summary, we obtain the following results:

Theorem 3.1 (Generalization of Theorem 1.1)**.** *The characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a})$ is a monic quasi-polynomial on $I(\mathcal{O})$ of degree ℓ with period $\rho_{\mathcal{A}}$ and for each $\kappa \mid \rho_{\mathcal{A}}$ the κ-constituent is given by*

$$f_{\mathcal{A}}^{\kappa}(t) = \sum_{J \subseteq [n]} (-1)^{|J|} m(J, \kappa) t^{\ell - r(J)},$$

where we put $m(J, \mathfrak{a}) := |\mathrm{tors}(\mathrm{coker}(\phi_{J,\mathfrak{a}}))| = \prod_{i=1}^{r(J)} N\left(\mathfrak{a} + \mathfrak{d}_{J,i}\right)$ for nonempty $J \subseteq [n]$ and $\mathfrak{a} \in I(\mathcal{O})$. When $J = \varnothing$, we define $r(\varnothing) := 0$ and $m(\varnothing, \kappa) := 1$.

Remark 3.1. *The LCM-period $\rho_{\mathcal{A}}$ is actually the minimum period of $\chi_{\mathcal{A}}^{\mathrm{quasi}}$ (See Theorem 5.1).*

Theorem 3.2 (Generalization of Theorem 1.2)**.** *The $\langle 1 \rangle$-constituent of the characteristic quasi-polynomial of $\mathcal{A}$ coincides with the characteristic polynomial of the hyperplane arrangement $\mathcal{A}(K)$, that is, $f_{\mathcal{A}}^{\langle 1 \rangle}(t) = \chi_{\mathcal{A}(K)}(t)$.*

Proof. By Theorem 3.1,

$$f_{\mathcal{A}}^{\langle 1 \rangle}(t) = \sum_{J \subseteq [n]} (-1)^{|J|} t^{\ell - r(J)}.$$

Using Whitney's theorem (See for example [12, Theorem 2.4]), we obtain $\chi_{\mathcal{A}(K)}(t) = f_{\mathcal{A}}^{\langle 1 \rangle}(t)$. ☐

Example 3.1. *Let $\mathcal{O} = \mathbb{Z}[\sqrt{-1}]$ and $\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix} \right\}$. Since $\mathcal{O}$ is a principal ideal domain, we can determine the LCM-period by computing Smith normal forms. The LCM-period is $\langle 2 \rangle = \mathfrak{p}^2$, where $\mathfrak{p} = \langle 1 + \sqrt{-1} \rangle$. The constituents of $\chi_{\mathcal{A}}^{\mathrm{quasi}}$ are given by*

$$f_{\mathcal{A}}^{\langle 1 \rangle} = t^2 - 4t + 3, \quad f_{\mathcal{A}}^{\mathfrak{p}} = t^2 - 4t + 6, \quad f_{\mathcal{A}}^{\langle 2 \rangle} = t^2 - 4t + 10.$$

Example 3.2. *Let K be a quadratic field $\mathbb{Q}(\sqrt{-5})$. Then its ring of integers is $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$. Note that $\mathcal{O}$ is not a principal ideal domain. For example, prime ideals $\mathfrak{p} := \langle 2, 1-\sqrt{-5}\rangle$ and $\mathfrak{q} := \langle 3, 1+\sqrt{-5}\rangle$ are not principal.*

Let

$$\mathcal{A} := \{c_1, c_2\}, \quad \textit{where } c_1 := \begin{pmatrix} 2 \\ 1-\sqrt{-5} \end{pmatrix}, \ c_2 := \begin{pmatrix} 1+\sqrt{-5} \\ 3 \end{pmatrix}.$$

Then we have

$$\operatorname{Im}\phi_{\{1\}} = \mathfrak{p}, \quad \operatorname{Im}\phi_{\{2\}} = \mathfrak{q}, \quad \operatorname{Im}\phi_{\{1,2\}} = \left\langle (2, 1+\sqrt{-5}),\ (1-\sqrt{-5}, 3) \right\rangle.$$

Define a map $g\colon \mathcal{O}^2 \to \mathcal{O}$ by $g(a,b) := (1+\sqrt{-5})a - 2b$. Then it holds that $\ker g = \operatorname{Im}\phi_{\{1,2\}}$ and $\operatorname{Im} g = \mathfrak{p}$. We will give a proof only for $\ker g \subseteq \operatorname{Im}\phi_{\{1,2\}}$ since the rest is clear. Taking $(a,b) \in \ker g$, we have $2b = (1+\sqrt{-5})a \in \langle 1+\sqrt{-5}\rangle = \mathfrak{pq} \subseteq \mathfrak{q}$. Since $2 \notin \mathfrak{q}$, we have $b \in \mathfrak{q}$. Therefore there exist $c, d \in \mathcal{O}$ such that $b = 3c + (1+\sqrt{-5})d$. Then $(1+\sqrt{-5})a = 2b = 6c + 2(1+\sqrt{-5})d$ and hence $a = (1-\sqrt{-5})c + 2d$. Hence $(a,b) = ((1-\sqrt{-5}c + 2d, 3c + (1+\sqrt{-5})d)) = (1-\sqrt{-5}, 3)c + (2, 1+\sqrt{-5})d \in \operatorname{Im}\phi_{\{1,2\}}$. Thus, $\operatorname{Im}\phi_{\{1,2\}} \supseteq \ker g$.

We obtain

$$\operatorname{coker}\phi_{\{1\}} = \mathcal{O}/\mathfrak{p}, \quad \operatorname{coker}\phi_{\{2\}} = \mathcal{O}/\mathfrak{q}, \quad \operatorname{coker}\phi_{\{1,2\}} = \operatorname{Im} g = \mathfrak{p}.$$

Therefore $\rho_{\mathcal{A}} = \operatorname{lcm}\{\mathfrak{p}, \mathfrak{q}\} = \mathfrak{pq} = \langle 1+\sqrt{-5}\rangle$ and the constituents of $\chi_{\mathcal{A}}^{\mathrm{quasi}}$ are given by

$$f_{\mathcal{A}}^{\langle 1\rangle}(t) = t^2 - t, \quad f_{\mathcal{A}}^{\mathfrak{p}}(t) = t^2 - 2t, \quad f_{\mathcal{A}}^{\mathfrak{q}}(t) = t^2 - 3t, \quad f_{\mathcal{A}}^{\mathfrak{pq}}(t) = t^2 - 4t.$$

Moreover, the hyperplanes $H_1 = \left\{ \boldsymbol{x} \in K^2 \,\middle|\, \boldsymbol{x}c_1 = 0 \right\}$ and $H_2 = \left\{ \boldsymbol{x} \in K^2 \,\middle|\, \boldsymbol{x}c_2 = 0 \right\}$ are equal since $(1+\sqrt{-5})c_1 = 2c_2$. Therefore $\mathcal{A}(K) = \{H_1\}$ and we obtain $\chi_{\mathcal{A}(K)}(t) = t^2 - t = f_{\mathcal{A}}^{\langle 1\rangle}(t)$ directly.

4. The torsion arrangement and its poset of layers

Recall that the hyperplane arrangement $\mathcal{A}(K)$ is defined by $\mathcal{A}(K) = \{H_1, \dots, H_n\}$, where

$$H_j = \left\{ \boldsymbol{x} \in K^\ell \,\middle|\, \boldsymbol{x}c_j = 0 \right\}$$

and the torsion arrangement $\mathcal{A}(K/\mathcal{O})$ is defined by $\mathcal{A}(K/\mathcal{O}) = \{ T_1, \dots, T_n \}$, where

$$T_j = \left\{ \pi(\boldsymbol{x}) \in (K/\mathcal{O})^\ell \,\middle|\, \boldsymbol{x}c_j \equiv 0 \pmod{\mathcal{O}} \right\}$$

and $\pi\colon K^\ell \to (K/\mathcal{O})^\ell$ denotes the natural projection.

For each $\varnothing \neq J \subseteq [n]$, let $H_J := \bigcap_{j\in J} H_j$ and $T_J := \bigcap_{j\in J} T_j$. Moreover, let $H_\varnothing := K^\ell$ and $T_\varnothing := (K/\mathcal{O})^\ell$. Note that for each $\varnothing \neq J \subseteq [n]$,

$$\pi(H_J) = \left\{ \pi(\boldsymbol{x}) \in (K/\mathcal{O})^\ell \,\middle|\, \boldsymbol{x}C_J = 0 \right\}, \qquad T_J = \left\{ \pi(\boldsymbol{x}) \in (K/\mathcal{O})^\ell \,\middle|\, \boldsymbol{x}C_J \in \mathcal{O}^{|J|} \right\}$$

and $\pi(H_J)$ is an $\mathcal{O}$-submodule of T_J.

The intersection poset of $\mathcal{A}(K)$ is defined by

$$L(\mathcal{A}(K)) := \left\{ H_J \subseteq K^\ell \,\middle|\, J \subseteq [n] \right\}$$

equipped with the order defined by the reverse inclusion. It is well known that $L(\mathcal{A})$ is a finite geometric lattice.

To construct the intersection poset for toric arrangements, one uses connected components of the intersections of tori. However, the field K may not have reasonable topology to define connected components of the intersection T_J. To define the poset for the torsion arrangement $\mathcal{A}(K/\mathcal{O})$, we use the quotient module $T_J/\pi(H_J)$ (See also [15, The paragraph before Lemma 4.5]).

Definition 4.1. *Define the poset of layers $L(\mathcal{A}(K/\mathcal{O}))$ by*

$$L(\mathcal{A}(K/\mathcal{O})) := \left\{ Z \subseteq (K/\mathcal{O})^\ell \,\middle|\, J \subseteq [n],\ Z \in T_J/\pi(H_J) \right\} = \{ \pi(\boldsymbol{x}) + \pi(H_J) \mid J \subseteq [n], \boldsymbol{x} \in T_J \}.$$

equipped with the order defined by the reverse inclusion. The poset $L(\mathcal{A}(K/\mathcal{O}))$ has a unique minimal element $T_\varnothing = (K/\mathcal{O})^\ell$. We call an element of $L(\mathcal{A}(K/\mathcal{O}))$ a layer.

Remark 4.1. *Every T_J can be represented as the disjoint union of layers: $T_J = \bigsqcup_{Z\in T_J/\pi(H_J)} Z$. Thus a layer behaves like a "connected component" of the intersection T_J. For each J, $\pi(H_J)$ is a unique component of T_J containing the unit element $\pi(\boldsymbol{0})$.*

4.1 Finiteness of the poset of layers

Lemma 4.1. *For each $\varnothing \neq J \subseteq [n]$,*

$$T_J/\pi(H_J) \simeq \operatorname{tors}(\operatorname{coker}\phi_J) \simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/\mathfrak{d}_{J,i}.$$

as $\mathcal{O}$-modules, where $\phi_J\colon \mathcal{O}^\ell \to \mathcal{O}^{|J|}$ is defined by $\phi_J(\boldsymbol{x}) = \boldsymbol{x}C_J$ and $\mathfrak{d}_{J,1}, \ldots, \mathfrak{d}_{J,r(J)}$ are the ideals determined in Lemma 3.1.

Proof. The latter isomorphism follows by Lemma 3.1. We verify the former isomorphism.

Define an $\mathcal{O}$-homomorphism $f_J\colon T_J/\pi(H_J) \to \operatorname{tors}(\operatorname{coker}\phi_J)$ by $f_J(\boldsymbol{x} + \pi(H_J)) := \boldsymbol{x}C_J + \operatorname{Im}\phi_J$. Note that when $\boldsymbol{x} \in T_J$ we have $\boldsymbol{x}C_J \in \mathcal{O}^J$ and, since every element in $(K/\mathcal{O})^\ell$ is a torsion element, we have $\boldsymbol{x}C_J + \operatorname{Im}\phi_J \in \operatorname{tors}(\operatorname{coker}\phi_J)$.

First, we prove that f_J is well-defined. Suppose that $\pi(\boldsymbol{x}) + \pi(H_J) = \pi(\boldsymbol{y}) + \pi(H_J) \in T_J/\pi(H_J)$. Then $\pi(\boldsymbol{x}) - \pi(\boldsymbol{y}) \in \pi(H_J)$. Hence there exists $\boldsymbol{v} \in H_J$ such that $\pi(\boldsymbol{x}) - \pi(\boldsymbol{y}) = \pi(\boldsymbol{v})$, and hence there exists $\boldsymbol{a} \in \mathcal{O}^\ell$ such that $\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{v} = \boldsymbol{a}$. Since $\boldsymbol{v} \in H_J$ and $\boldsymbol{a} \in \mathcal{O}^\ell$, $\boldsymbol{x}C_J - \boldsymbol{y}C_J = \boldsymbol{v}C_J + \boldsymbol{a}C_J = \boldsymbol{a}C_J \in \operatorname{Im}\phi_J$. Thus $\boldsymbol{x}C_J + \operatorname{Im}\phi_J = \boldsymbol{y}C_J + \operatorname{Im}\phi_J$ and hence f_J is well-defined.

Second, we prove that f_J is injective. Assume that $\boldsymbol{x}C_J + \operatorname{Im}\phi_J = \boldsymbol{y}C_J + \operatorname{Im}\phi_J$, where $\boldsymbol{x}, \boldsymbol{y} \in K^\ell$. Then $\boldsymbol{x}C_J - \boldsymbol{y}C_J \in \operatorname{Im}\phi_J$ and hence there exists $\boldsymbol{a} \in \mathcal{O}^\ell$ such that $\boldsymbol{x}C_J - \boldsymbol{y}C_J = \phi_J(\boldsymbol{a}) = \boldsymbol{a}C_J$. Therefore $(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{a})C_J = 0$ and $\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{a} \in H_J$. Thus $\pi(\boldsymbol{x}) - \pi(\boldsymbol{y}) = \pi(\boldsymbol{x}) - \pi(\boldsymbol{y}) - \pi(\boldsymbol{a}) = \pi(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{a}) \in \pi(H_J)$ and hence f_J is injective.

Finally, we prove that f is surjective. For any element $\boldsymbol{y} + \operatorname{Im}\phi_J \in \operatorname{tors}(\operatorname{coker}\phi_J)$, there exists $b \in \mathcal{O} \setminus \{0\}$ such that $b\,(\boldsymbol{y} + \operatorname{Im}\phi_J) = \operatorname{Im}\phi_J$, that is, $b\boldsymbol{y} \in \operatorname{Im}\phi_J$. Hence there exists $\boldsymbol{a} \in \mathcal{O}^\ell$ such that $b\boldsymbol{y} = \phi_J(\boldsymbol{a}) = \boldsymbol{a}C_J$. Let $\boldsymbol{x} = (1/b)\boldsymbol{a} \in K^\ell$. Then $\boldsymbol{x}C_J = (1/b)\boldsymbol{a}C_J = \boldsymbol{y} \in \mathcal{O}^{|J|}$, and hence $\pi(\boldsymbol{x}) \in T_J$. Therefore $f_J\,(\pi(\boldsymbol{x}) + \pi(H_J)) = \boldsymbol{x}C_J + \operatorname{Im}\phi_J = \boldsymbol{y} + \operatorname{Im}\phi_J$. Hence f_J is surjective.

Thus f_J is isomorphic and we obtain the desired isomorphism $T_J/\pi(H_J) \simeq \operatorname{tors}(\operatorname{coker}\phi_J)$ as $\mathcal{O}$-modules. □

Theorem 4.1. *The poset of layers $L(\mathcal{A}(K/\mathcal{O}))$ is finite.*

Proof. It follows immediately from Lemma 4.1 and the assumption that $\mathcal{O}$ is residually finite. Also note that $T_\varnothing/\pi(H_\varnothing) = \{T_\varnothing\}$. □

4.2 Relations between $L(\mathcal{A}(K/\mathcal{O}))$ and $L(\mathcal{A}(K))$

First of all, we will show a crucial lemma for describing relations between $L(\mathcal{A}(K/\mathcal{O}))$ and $L(\mathcal{A}(K)$. We need the topology on K^ℓ defined by a fixed nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $\mathcal{O}_\mathfrak{p}$ be the local ring of $\mathcal{O}$ at $\mathfrak{p}$. Then $\mathcal{O}_\mathfrak{p}$ is a discrete valuation ring with the uniformizer p. Since $K = \operatorname{Frac}(\mathcal{O}_\mathfrak{p})$, any element $x \in K \setminus \{0\}$ is given by the form $x = up^m$ uniquely, where $u \in \mathcal{O}_\mathfrak{p}^\times$ and $m \in \mathbb{Z}$. We denote the exponent m by $\operatorname{ord}_p(x)$. Putting $\operatorname{ord}_p(0) := \infty$, we obtain a discrete valuation $\operatorname{ord}_p\colon K \to \mathbb{Z} \cup \{\infty\}$ and then K is a topological space with an open base consisting of $B_m(a) := \{\, x \in K \mid \operatorname{ord}_p(x - a) > m \,\}$, where $a \in K$ and $m \in \mathbb{Z}$.

We can consider that K^ℓ is a topological space defined by the product topology. Note that any vector subspace of K^ℓ is closed and K^ℓ has an open base consisting of

$$B_m(\boldsymbol{a}) := \left\{\, \boldsymbol{x} \in K^\ell \;\middle|\; \operatorname{ord}_p(\boldsymbol{x} - \boldsymbol{a}) > m \,\right\},$$

where $\boldsymbol{x} = (x_1, \ldots, x_\ell)$, $\boldsymbol{a} = (a_1, \ldots, a_\ell) \in K^\ell$, $m \in \mathbb{Z}$, and $\operatorname{ord}_p(\boldsymbol{x} - \boldsymbol{a}) := \min_{1 \le i \le \ell} \operatorname{ord}_p(x_i - a_i)$.

Lemma 4.2. *Let U and V be vector subspaces of K^ℓ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in K^\ell$. Suppose that $\pi(U) \subseteq \bigcup_{i=1}^r (\pi(\boldsymbol{x}_i) + \pi(V))$. Then $U \subseteq V$.*

Proof. Let $\boldsymbol{u} \in U$. We will show that $\boldsymbol{u}$ is an accumulation point of V, that is, for any $m \in \mathbb{Z}, B_m(\boldsymbol{u}) \cap V \neq \varnothing$. Take an element $a \in \mathcal{O} \setminus \{0\}$ such that $a\boldsymbol{x}_i \in \mathcal{O}^\ell$ for each $i \in \{1, \ldots, r\}$ and consider the vector $a^{-1}p^{-m-1}\boldsymbol{u} \in U$.

From the assumption, we have $\pi(a^{-1}p^{-m-1}\boldsymbol{u}) = \pi(\boldsymbol{x}_i) + \pi(\boldsymbol{v})$ for some $\boldsymbol{x}_i$ and $\boldsymbol{v} \in V$. Then $\pi(p^{-m-1}\boldsymbol{u}) = \pi(a\boldsymbol{x}_i) + \pi(a\boldsymbol{v}) = \pi(\boldsymbol{0}) + \pi(a\boldsymbol{v}) = \pi(a\boldsymbol{v})$. Therefore there exists $\boldsymbol{b} \in \mathcal{O}^\ell$ such that $p^{-m-1}\boldsymbol{u} = a\boldsymbol{v} + \boldsymbol{b}$. Then $\operatorname{ord}_p(ap^{m+1}\boldsymbol{v} - \boldsymbol{u}) = \operatorname{ord}_p(p^{m+1}\boldsymbol{b}) > m$. Thus $ap^{m+1}\boldsymbol{v} \in B_m(\boldsymbol{u}) \cap V$ and hence $\boldsymbol{u}$ is an accumulation point of V.

Since the vector subspace V is closed, we have $\boldsymbol{u} \in V$. Hence $U \subseteq V$. □

Proposition 4.1. *Define a map $\pi\colon L(\mathcal{A}(K)) \to L(\mathcal{A}(K/\mathcal{O}))$ by $H_J \mapsto \pi(H_J)$. Then π is an order-preserving injection.*

Proof. It is clear that π preserves the order. Injectivity follows from Lemma 4.2. □

Proposition 4.2. *Let $Z \in T_{J_1}/\pi(H_{J_1})$ be a layer. Suppose that $Z \subseteq T_{J_2}$. Then there exists a layer $W \in T_{J_2}/\pi(H_{J_2})$ such that $Z \subseteq W$.*

Proof. Let $\pi(\boldsymbol{x}) \in Z$. Then $Z = \pi(\boldsymbol{x}) + \pi(H_{J_1})$. Since $T_{J_2}/\pi(H_{J_2})$ is finite, there exist $\pi(\boldsymbol{x}_1), \dots, \pi(\boldsymbol{x}_r) \in T_{J_2}$ such that

$$\pi(\boldsymbol{x}) + \pi(H_{J_1}) = Z \subseteq T_{J_2} = \bigsqcup_{i=1}^{r} (\pi(\boldsymbol{x}_i) + \pi(H_{J_2})).$$

Then $\pi(H_{J_1}) \subseteq \bigsqcup_{i=1}^{r} (\pi(\boldsymbol{x}_i - \boldsymbol{x}) + \pi(H_{J_2}))$. Using Lemma 4.2, we have $H_{J_1} \subseteq H_{J_2}$. Therefore the layer $W := \pi(\boldsymbol{x}) + \pi(H_{J_2})$ is a desired layer. □

Lemma 4.3. *Given a layer $Z \in L(\mathcal{A}(K/\mathcal{O}))$, there exists a unique $H_Z \in L(\mathcal{A})$ such that $Z = \pi(\boldsymbol{x}) + \pi(H_Z)$ for any $\pi(\boldsymbol{x}) \in Z$.*

Proof. Since $Z \in T_J/\pi(H_J)$ for some $J \subseteq [n]$, the existence holds. To show the uniqueness, suppose that $Z = \pi(\boldsymbol{x}) + \pi(H_{J_1}) = \pi(\boldsymbol{x}) + \pi(H_{J_2})$, where $\boldsymbol{x} \in Z$ and $J_i \subseteq [n]$ for each $i \in \{1, 2\}$. Then $\pi(H_{J_1}) = \pi(H_{J_2})$. By Lemma 4.2, we obtain $H_{J_1} = H_{J_2}$. □

Definition 4.2. *Define a map $\psi\colon L(\mathcal{A}(K/\mathcal{O})) \to L(\mathcal{A}(K))$ by $\psi(Z) := H_Z$. Also, define the dimension of Z by $\dim Z := \dim H_Z$.*

Definition 4.3. *Let $Z \in L(\mathcal{A}(K/\mathcal{O}))$. Define*

$$J_Z := \{\, j \in [n] \mid T_j \supseteq Z \,\}, \qquad \mathcal{A}_Z := \{\, c_j \in \mathcal{A} \mid T_j \supseteq Z \,\} = \{\, c_j \in \mathcal{A} \mid j \in J_Z \,\}.$$

We call $\mathcal{A}_Z$ the localization of $\mathcal{A}$ at Z.

Lemma 4.4. *The following statements hold:*

(i) The map ψ is order-preserving. Namely, if $Z \subseteq W$, then $H_Z \subseteq H_W$.

(ii) The map ψ induces an isomorphism from the interval $[T_\varnothing, Z]$ to $L(\mathcal{A}_Z(K))$, where

$$[T_\varnothing, Z] := \{\, W \in L(\mathcal{A}(K/\mathcal{O})) \mid T_\varnothing \leq W \leq Z \,\}.$$

(iii) The Möbius function on $L(\mathcal{A}(K/\mathcal{O}))$ strictly alternates in sign.

Proof. (i) Let $\pi(\boldsymbol{x}) \in Z$. Then $Z = \pi(\boldsymbol{x}) + \pi(H_Z)$ and $W = \pi(\boldsymbol{x}) + \pi(H_W)$ since $Z \subseteq W$. Then we have $\pi(H_Z) \subseteq \pi(H_W)$. By Lemma 4.2, $H_Z \subseteq H_W$.

(ii) We show the injectivity of the restriction $\psi|_{[T_\varnothing, Z]}$. Let $W_1, W_2 \in [T_\varnothing, Z]$. Then $Z \subseteq W_1 \cap W_2$. Letting $\pi(\boldsymbol{x}) \in Z$, we have $W_1 = \pi(\boldsymbol{x}) + \pi(H_{W_1})$ and $W_2 = \pi(\boldsymbol{x}) + \pi(H_{W_2})$. Suppose that $\psi(W_1) = \psi(W_2)$. Then $H_{W_1} = H_{W_2}$ and hence $W_1 = W_2$. Thus the restriction $\psi|_{[T_\varnothing, Z]}$ is injective.

Now, we show the surjectivity. Let $H_J \in L(\mathcal{A}_Z(K))$, where $J \subseteq J_Z$. Then $T_J \supseteq Z$ and hence there exists a layer $W \in T_J/\pi(H_J)$ such that $W \supseteq Z$ by Proposition 4.2. Then $\psi(W) = H_J$ and hence $\psi|_{[T_\varnothing, Z]}$ is surjective.

(iii) Note that the restriction of the Möbius function μ on $L(\mathcal{A}(K/\mathcal{O}))$ to the interval $[T_\varnothing, Z]$ coincides with the Möbius function on $[T_\varnothing, Z]$. By (ii), the interval $[T_\varnothing, Z]$ is a geometric lattice. Therefore the assertion holds. □

4.3 The κ-torsion subposet

Let M be a $\mathcal{O}$-module and let $\kappa \in I(\mathcal{O})$. An element $m \in M$ is called a *κ-torsion element* if $\kappa \subseteq \operatorname{Ann} m := \{\, a \in \mathcal{O} \mid am = 0 \in M \,\}$.

A layer $\pi(\boldsymbol{x}) + \pi(H_J) \in T_J/\pi(H_J)$ is κ-torsion if and only if for any $a \in \kappa$, $\pi(a\boldsymbol{x}) \in \pi(H_J)$. In particular whether a layer $Z \in L(\mathcal{A}(K/\mathcal{O}))$ is κ-torsion is independent of the choice of J such that $Z \in T_J/\pi(H_J)$ by Lemma 4.3. Namely, Z is κ-torsion if and only if

$$\kappa \subseteq \operatorname{Ann} Z := \{\, a \in \mathcal{O} \mid aZ \subseteq \pi(H_Z) \,\},$$

where $aZ := \left\{\, \pi(a\boldsymbol{x}) \in (K/\mathcal{O})^\ell \;\middle|\; \boldsymbol{x} \in Z \,\right\}$.

Definition 4.4. *For any $\kappa \in I(\mathcal{O})$, the κ-torsion subposet of $L(\mathcal{A}(K/\mathcal{O}))$ is defined by*

$$L(\mathcal{A}(K/\mathcal{O}))[\kappa] := \{\, Z \in L(\mathcal{A}(K/\mathcal{O})) \mid Z \text{ is } \kappa\text{-torsion} \,\}.$$

Remark 4.2. *In the original definition [15, Definition 4.4(1)], which can be considered as the case $\mathcal{O} = \mathbb{Z}$, the poset consists of layers containing a κ-torsion element. At least when κ is a principal ideal, one can prove easily that a layer Z contains a κ-torsion element if and only if Z itself is a κ-torsion element (See also the formula (4.3) in the proof of Lemma 4.5 in [15]).*

Proposition 4.3. *The κ-torsion subposet $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$ is an order ideal of $L(\mathcal{A}(K/\mathcal{O}))$. Namely if $Z \leq W$ in $L(\mathcal{A}(K/\mathcal{O}))$, or equivalently $Z \supseteq W$, and $W \in L(\mathcal{A}(K/\mathcal{O}))[\kappa]$, then $Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa]$. In particular, the restriction of the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))$ to $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$ coincides with the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$.*

Proof. Let $\pi(\boldsymbol{x}) \in W$. Then $W = \pi(\boldsymbol{x}) + \pi(H_W)$ and $Z = \pi(\boldsymbol{x}) + \pi(H_Z)$ since $Z \supseteq W$. Let $a \in \kappa$. Since W is κ-torsion, $\pi(a\boldsymbol{x}) \in \pi(H_W) \subseteq \pi(H_Z)$. Therefore Z is κ-torsion. $\square$

Given an $\mathcal{O}$-module M and $\kappa \in I(\mathcal{O})$, let $M[\kappa]$ denote the submodule consisting of κ-torsion elements.

Lemma 4.5. *For any κ, $\mathfrak{a} \in I(\mathcal{O})$, $(\mathcal{O}/\mathfrak{a})\,[\kappa] = \mathcal{O}/(\kappa + \mathfrak{a})$.*

Proof. Since $\mathcal{O}$ is a Dedekind domain, $\mathcal{O}/(\kappa+\mathfrak{a}) \simeq \mathfrak{a}(\kappa+\mathfrak{a})^{-1}/\mathfrak{a}$. Moreover, since

$$(\mathcal{O}/\mathfrak{a})\,[\kappa] = \{\, [\omega]_{\mathfrak{a}} \in \mathcal{O}/\mathfrak{a} \mid a\omega \in \mathfrak{a} \text{ for all } a \in \kappa \,\},$$

it suffices to show that

$$(\mathfrak{a} \colon \kappa) := \{\, \omega \in \mathcal{O} \mid a\omega \in \mathfrak{a} \text{ for all } a \in \kappa \,\} = \mathfrak{a}(\kappa+\mathfrak{a})^{-1}.$$

Let $\omega = \sum_i x_i y_i \in \mathfrak{a}(\kappa+\mathfrak{a})^{-1}$ $(x_i \in \mathfrak{a}, y_i \in (\kappa+\mathfrak{a})^{-1})$. Since $(\kappa+\mathfrak{a})^{-1} \subseteq \mathfrak{a}^{-1}$, we have $\omega \in \mathfrak{a}(\kappa+\mathfrak{a})^{-1} \subseteq \mathfrak{a}\mathfrak{a}^{-1} = \mathcal{O}$. Since $(\kappa+\mathfrak{a})^{-1} \subseteq \kappa^{-1}$, for any $a \in \kappa$,

$$a\left(\sum_i x_i y_i\right) = \sum_i x_i(ay_i) \in \mathfrak{a}\mathcal{O} = \mathfrak{a}.$$

Therefore $\omega \in (\mathfrak{a} \colon \kappa)$ and hence $(\mathfrak{a} \colon \kappa) \supseteq \mathfrak{a}(\kappa+\mathfrak{a})^{-1}$.

Let $\omega \in (\mathfrak{a} \colon \kappa)$. Since $\mathcal{O} = (\kappa+\mathfrak{a})(\kappa+\mathfrak{a})^{-1}$, there exist $x_i \in (\kappa+\mathfrak{a})$, $y_i \in (\kappa+\mathfrak{a})^{-1}$ such that $1 = \sum_i x_i y_i$. Since $\omega x_i \in \omega(\kappa+\mathfrak{a}) \subseteq \omega\kappa + \mathfrak{a} \subseteq \mathfrak{a}$, we have

$$\omega = \omega\left(\sum_i x_i y_i\right) = \sum_i (\omega x_i) y_i \in \mathfrak{a}(\kappa+\mathfrak{a})^{-1}.$$

Hence $(\mathfrak{a} \colon \kappa) \subseteq \mathfrak{a}(\kappa+\mathfrak{a})^{-1}$. Therefore $(\mathfrak{a} \colon \kappa) = \mathfrak{a}(\kappa+\mathfrak{a})^{-1}$. $\square$

Theorem 4.2. *Let $\varnothing \neq J \subseteq [n]$. For any $\kappa \in I(\mathcal{O})$,*

$$(T_J/\pi(H_J))\,[\kappa] \simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/\left(\kappa + \mathfrak{d}_{J,i}\right).$$

In particular, $|(T_J/\pi(H_J))\,[\kappa]| = \prod_{i=1}^{r(J)} N\left(\kappa + \mathfrak{d}_{J,i}\right) = m(J,\kappa)$, where $\mathfrak{d}_{J,1}, \ldots, \mathfrak{d}_{J,r(J)}$ are the ideals determined in Lemma 3.1.

Proof. By Lemma 4.1 and 4.5, we obtain

$$(T_J/\pi(H_J))\,[\kappa] \simeq \bigoplus_{i=1}^{r(J)} (\mathcal{O}/\mathfrak{d}_{J,i})\,[\kappa] \simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}/(\kappa + \mathfrak{d}_{J,i}).$$

$\square$

Corollary 4.1. *For any $\kappa \in I(\mathcal{O})$,*

$$L(\mathcal{A}(K/\mathcal{O}))[\kappa] = L(\mathcal{A}(K/\mathcal{O}))[\kappa + \rho_{\mathcal{A}}],$$

where $\rho_{\mathcal{A}} = \operatorname{lcm}\left\{\, \mathfrak{d}_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \,\right\}$ denotes the LCM-period of $\mathcal{A}$.

Proof. Since $\kappa + \rho_{\mathcal{A}} + \mathfrak{d}_{J,i} = \kappa + \mathfrak{d}_{J,i}$ for each i, the assertion holds. $\square$

Corollary 4.2. $L(\mathcal{A}(K/\mathcal{O}))[\langle 1\rangle] \simeq L(\mathcal{A}(K))$ *and* $L(\mathcal{A}(K/\mathcal{O}))[\rho_{\mathcal{A}}] = L(\mathcal{A}(K/\mathcal{O}))$.

Proof. A layer $Z \in L(\mathcal{A}(K/\mathcal{O}))$ is $\langle 1\rangle$-torsion if and only if $Z = \pi(H_Z)$. Therefore $L(\mathcal{A}(K/\mathcal{O}))[\langle 1\rangle] \simeq L(\mathcal{A}(K))$ by Proposition 4.1.

By Theorem 4.2, we have $T_J/\pi(H_J)[\rho_{\mathcal{A}}] = T_J/\pi(H_J)$. Therefore $L(\mathcal{A}(K/\mathcal{O}))[\rho_{\mathcal{A}}] = L(\mathcal{A}(K/\mathcal{O}))$. $\square$

4.4 The constituents and the characteristic polynomials

The characteristic polynomial $\chi_{\mathcal{A}(K)}(t)$ is defined by

$$\chi_{\mathcal{A}(K)}(t) := \sum_{X \in L(\mathcal{A}(K))} \mu(H_{\varnothing}, X) t^{\dim X},$$

where μ denotes the Möbius function on $L(\mathcal{A}(K))$.

The characteristic polynomial $\chi_{\mathcal{A}(K/\mathcal{O})}(t)$ is defined by

$$\chi_{\mathcal{A}(K/\mathcal{O})}(t) := \sum_{Z \in L(\mathcal{A}(K/\mathcal{O}))} \mu(T_{\varnothing}, Z) t^{\dim Z},$$

where μ denotes the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))$ and $\dim Z = \dim H_Z$.

Definition 4.5. *For each $\kappa \mid \rho_{\mathcal{A}}$, the κ-characteristic polynomial of $\mathcal{A}(K/\mathcal{O})$ is defined by*

$$\chi^{\kappa}_{\mathcal{A}(K/\mathcal{O})}(t) := \sum_{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa]} \mu(T_{\varnothing}, Z) t^{\dim Z},$$

where μ denotes the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))$.

By Proposition 4.3, the restriction of the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))$ to $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$ coincides with the Möbius function on $L(\mathcal{A}(K/\mathcal{O}))[\kappa]$. Also, it follows that $\chi^{\langle 1\rangle}_{\mathcal{A}(K/\mathcal{O})}(t) = \chi_{\mathcal{A}(K)}(t)$ and $\chi^{\rho_{\mathcal{A}}}_{\mathcal{A}(K/\mathcal{O})}(t) = \chi_{\mathcal{A}(K/\mathcal{O})}(t)$ by Corollary 4.2.

Example 4.1. *Let $\mathcal{O} = \mathbb{Z}[\sqrt{-1}], K = \mathbb{Q}(\sqrt{-1})$, and*

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix} \right\}$$

as in Example 3.1. Figure 1 shows the Hasse diagram of $L(\mathcal{A}(K/\mathcal{O}))$.

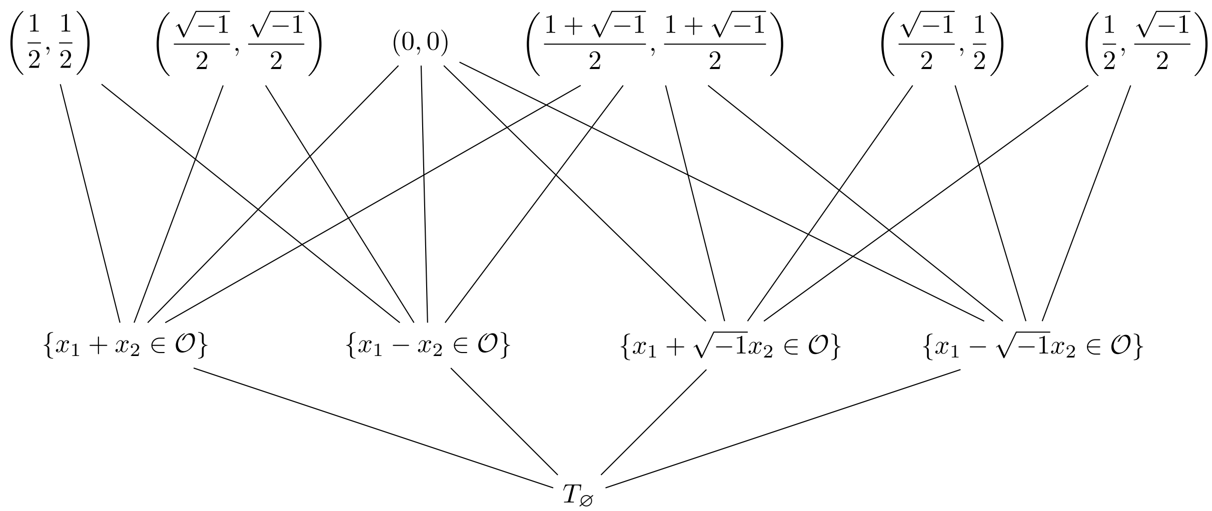

Figure 1: The Hasse diagram of $L(\mathcal{A}(K/\mathcal{O}))$ in Example 4.1

Recall that the LCM-period of $\mathcal{A}$ is $\rho_{\mathcal{A}} = \langle 2\rangle = \mathfrak{p}^2$, where $\mathfrak{p} = \langle 1 + \sqrt{-1}\rangle$. The subposets of $L(\mathcal{A}(K\mathcal{O}))$ are as follows:

$$\begin{aligned} L(\mathcal{A}(K/\mathcal{O}))[\langle 1\rangle] &= [T_{\varnothing}, (0,0)] \\ L(\mathcal{A}(K/\mathcal{O}))[\mathfrak{p}] &= [T_{\varnothing}, (0,0)] \cup \left[T_{\varnothing}, \left(\frac{\sqrt{-1}}{2}, \frac{\sqrt{-1}}{2}\right)\right] \\ L(\mathcal{A}(K/\mathcal{O}))[\mathfrak{p}^2] &= L(\mathcal{A}(K/\mathcal{O})). \end{aligned}$$

Therefore we obtain

$$\chi^{\langle 1\rangle}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 4t + 3, \quad \chi^{\mathfrak{p}}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 4t + 6, \quad \chi^{\mathfrak{p}^2}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 4t + 10.$$

In particular, by Example 3.1, we have $f^{\kappa}_{\mathcal{A}}(t) = \chi^{\kappa}_{\mathcal{A}(K/\mathcal{O})}(t)$ for each ideal $\kappa \mid \rho_{\mathcal{A}} = \mathfrak{p}^2$.

Example 4.2. *Let $\mathcal{O} = \mathbb{Z}[\sqrt{-5}], K = \mathbb{Q}(\sqrt{-5})$, and*

$$\mathcal{A} = \{c_1, c_2\}, \quad \text{where } c_1 = \begin{pmatrix} 2 \\ 1-\sqrt{-5} \end{pmatrix},\ c_2 = \begin{pmatrix} 1+\sqrt{-5} \\ 3 \end{pmatrix}$$

as in Example 3.2. Then $H := H_{\{1\}} = H_{\{2\}} = H_{\{1,2\}} = \langle(-3, 1+\sqrt{-5})\rangle_K \subseteq K^2$ and we have

$$T_J/\pi(H_J) = \begin{cases} \{T_\varnothing\} & \text{if } J = \varnothing, \\ \left\{\overline{(0,0)},\ \overline{(1/2,0)}\right\} & \text{if } J = \{1\}, \\ \left\{\overline{(0,0)},\ \overline{(0,1/3)},\ \overline{(0,2/3)}\right\} & \text{if } J = \{2\}, \\ \left\{\overline{(0,0)}\right\} & \text{if } J = \{1,2\}, \end{cases}$$

where $\overline{(a,b)}$ denotes $\pi(a,b) + \pi(H)$. Figure 2 shows the Hasse diagram of the poset of layers $L(\mathcal{A}(K/\mathcal{O}))$.

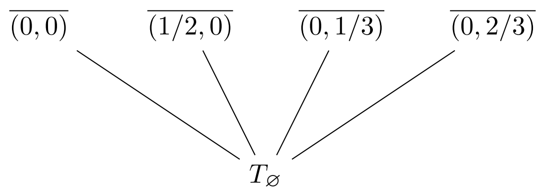

Figure 2: The Hasse diagram of $L(\mathcal{A}(K/\mathcal{O}))$ in Example 4.2

Recall the LCM-period of $\mathcal{A}$ is $\rho_\mathcal{A} = \mathfrak{pq} = \langle 1+\sqrt{-5}\rangle$, where $\mathfrak{p} = \langle 2, 1-\sqrt{-5}\rangle, \mathfrak{q} = \langle 3, 1+\sqrt{-5}\rangle$. The torsion subposets of $L(\mathcal{A}(K/\mathcal{O}))$ are as follows:

$$L(\mathcal{A}(K/\mathcal{O}))[\langle 1\rangle] = \{T_\varnothing, \overline{(0,0)}\}, \quad L(\mathcal{A}(K/\mathcal{O}))[\mathfrak{p}] = \{T_\varnothing, \overline{(0,0)}, \overline{(1/2,0)}\},$$
$$L(\mathcal{A}(K/\mathcal{O}))[\mathfrak{q}] = \{T_\varnothing, \overline{(0,0)}, \overline{(0,1/3)}, \overline{(0,2/3)}\}, \quad L(\mathcal{A}(K/\mathcal{O}))[\mathfrak{pq}] = L(\mathcal{A}(K/\mathcal{O})).$$

Since $\dim T_\varnothing = 2$ *and* $\dim Z = \dim H = 1$ *for each $Z \in L(\mathcal{A}(K/\mathcal{O})) \setminus \{T_\varnothing\}$, we obtain*

$$\chi^{\langle 1\rangle}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - t, \qquad \chi^{\mathfrak{p}}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 2t,$$
$$\chi^{\mathfrak{q}}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 3t, \qquad \chi^{\mathfrak{pq}}_{\mathcal{A}(K/\mathcal{O})}(t) = t^2 - 4t.$$

In particular, by Example 3.2, we have $f^\kappa_\mathcal{A}(t) = \chi^\kappa_{\mathcal{A}(K/\mathcal{O})}(t)$ for each ideal $\kappa \mid \rho_\mathcal{A} = \mathfrak{pq}$.

In the rest of this subsection, we will prove that for each ideal $\kappa \mid \rho_\mathcal{A}$, the κ-constituent $f^\kappa_\mathcal{A}(t)$ of the characteristic quasi-polynomial $\chi^{\text{quasi}}_\mathcal{A}$ coincides with the κ-characteristic polynomial $\chi^\kappa_{\mathcal{A}(K/\mathcal{O})}(t)$ of $\mathcal{A}(K/\mathcal{O})$.

Definition 4.6. *Given a layer $Z \in L(\mathcal{A}(K/\mathcal{O}))$, define $D(Z)$ by*

$$D(Z) := \{\ J \subseteq [n] \mid Z \in T_J/\pi(H_J)\ \}.$$

Lemma 4.6. *Let $Z \in L(\mathcal{A}(K/\mathcal{O}))$. Then the following hold.*

(i) If $J \in D(Z)$, then $H_J = H_Z$.

(ii) $J_Z = \{\ j \in [n] \mid T_j \supseteq Z\ \}$ is a unique maximal element of $D(Z)$.

(iii) $$\bigsqcup_{\substack{W \in L(\mathcal{A}(K/\mathcal{O})) \\ W \supseteq Z}} D(W) = \{\ J \subseteq [n] \mid J \subseteq J_Z\ \}.$$

Proof. (i) It follows from Lemma 4.3.

(ii) Let $J \in D(Z)$. Then $T_J \supseteq Z$ and hence $T_j \supseteq Z$ for any $j \in J$. Therefore $J \subseteq J_Z$ and hence the maximality holds. Also, we have $H_Z = H_J \supseteq H_{J_Z}$.

For each $j \in J_Z$, we have $T_j \supseteq Z$ and hence $T_{J_Z} \supseteq Z$. By Proposition 4.2, there exists a layer $W \in T_{J_Z}/\pi(H_{J_Z})$ such that $W \supseteq Z$. Therefore we have $H_{J_Z} \supseteq H_Z$ by Lemma 4.4(i). Thus $H_{J_Z} = H_Z$ and $J_Z \in D(Z)$.

(iii) First, we prove that the union is disjoint. Suppose that $W_1 \cap W_2 \supseteq Z$ and $W_1 \neq W_2$. Assume that there exists $J \in D(W_1) \cap D(W_2)$. Then $H_{W_1} = H_J = H_{W_2}$. By Lemma 4.4(ii), we have $W_1 = W_2$, a contradiction. Thus $D(W_1) \cap D(W_2) = \varnothing$.

Second, let $J \in D(W)$ such that $W \supseteq Z$. We will prove $J \subseteq J_Z$. By (i) and Lemma 4.4(i), $H_{J_Z} = H_Z \subseteq H_W = H_J$. Thus $H_{J \cup J_Z} = H_J \cap H_{J_Z} = H_{J_Z}$. Furthermore, since $Z \subseteq T_{J_Z}$ and $Z \subseteq W \subseteq T_J$, we have $Z \subseteq T_{J_Z} \cap T_J = T_{J \cup J_Z}$. Therefore Z can be written as $Z = \pi(\boldsymbol{x}) + \pi(H_{J \cup J_Z})$ with $\pi(\boldsymbol{x}) \in T_{J \cup J_Z}$ and hence $J \cup J_Z \in D(Z)$. By (ii), we have $J \cup J_Z \subseteq J_Z$. Thus $J \subseteq J_Z$.

Finally, suppose that $J \subseteq J_Z$. Then we have $Z \subseteq T_{J_Z} \subseteq T_J$ and $H_{J_Z} \subseteq H_J$. Let $\pi(\boldsymbol{x}) \in Z$ and $W := \pi(\boldsymbol{x}) + \pi(H_J)$. Then $Z = \pi(\boldsymbol{x}) + \pi(H_{J_Z}) \subseteq \pi(\boldsymbol{x}) + \pi(H_J) = W$ and $W = \pi(\boldsymbol{x}) + \pi(H_J) \in T_J/\pi(H_J)$, that is, $J \in D(W)$. ☐

Lemma 4.7 (See also [10, Lemma 5.5])**.** *For any $Z \in L(\mathcal{A}(K/\mathcal{O}))$,*

$$\mu(T_\varnothing, Z) = \sum_{J \in D(Z)} (-1)^{|J|}.$$

Proof. We prove it by induction on the codimension of Z. If $\operatorname{codim} Z = 0$ then $Z = T_\varnothing$ and $D(T_\varnothing) = \{\varnothing\}$. Hence the statement holds. We assume that $\operatorname{codim} Z \geq 1$. By the definition of the Möbius function, the induction hypothesis, and Lemma 4.6(iii), we have

$$\begin{aligned}
\mu(T_\varnothing, Z) &= -\sum_{W \supsetneq Z} \mu(T_\varnothing, W) \\
&= -\sum_{W \supsetneq Z} \sum_{J \in D(W)} (-1)^{|J|} \\
&= \sum_{J \in D(Z)} (-1)^{|J|} - \sum_{W \supseteq Z} \sum_{J \in D(W)} (-1)^{|J|} \\
&= \sum_{J \in D(Z)} (-1)^{|J|} - \sum_{J \subseteq J_Z} (-1)^{|J|} \\
&= \sum_{J \in D(Z)} (-1)^{|J|}.
\end{aligned}$$

☐

Theorem 4.3 (Generalization of Theorem 1.4)**.** *Let $\kappa \mid \rho_{\mathcal{A}}$. Then*

$$f_{\mathcal{A}}^{\kappa}(t) = \chi_{\mathcal{A}(K/\mathcal{O})}^{\kappa}(t).$$

Proof. By Theorem 3.1, the κ-constituent $f_{\mathcal{A}}^{\kappa}(t)$ of the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}$ is given by

$$f_{\mathcal{A}}^{\kappa}(t) = \sum_{J \subseteq [n]} (-1)^{|J|} m(J, \kappa) t^{\ell - r(J)} = t^{\ell} + \sum_{r=0}^{\ell-1} \left(\sum_{\substack{J \subseteq [n] \\ \ell - r(J) = r}} (-1)^{|J|} m(J, \kappa) \right) t^r.$$

The κ-characteristic polynomial $\chi_{\mathcal{A}(K/\mathcal{O})}^{\kappa}(t)$ of $\mathcal{A}(K/\mathcal{O})$ is given by

$$\chi_{\mathcal{A}(K/\mathcal{O})}^{\kappa}(t) = \sum_{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa]} \mu(T_\varnothing, Z) t^{\dim Z} = t^{\ell} + \sum_{r=0}^{\ell-1} \left(\sum_{\substack{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa] \\ \dim Z = r}} \mu(T_\varnothing, Z) \right) t^r.$$

Since if $J \in D(Z)$ then $\dim Z = \dim H_J = \ell - r(J)$, by Theorem 4.2 and Lemma 4.7, for each $0 \leq r \leq \ell - 1$,

$$\begin{aligned}
\sum_{\substack{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa] \\ \dim Z = r}} \mu(T_\varnothing, Z) &= \sum_{\substack{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa] \\ \dim Z = r}} \sum_{J \in D(Z)} (-1)^{|J|} \\
&= \sum_{\substack{J \subseteq [n] \\ \ell - r(J) = r}} \sum_{\substack{Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa] \\ J \in D(Z)}} (-1)^{|J|} \\
&= \sum_{\substack{J \subseteq [n] \\ \ell - r(J) = r}} (-1)^{|J|} m(J, \kappa).
\end{aligned}$$

Therefore we obtain $f_{\mathcal{A}}^{\kappa}(t) = \chi_{\mathcal{A}(K/\mathcal{O})}^{\kappa}(t)$. ☐

5. Minimality of the LCM-period

For a layer $Z \in L(\mathcal{A}(K/\mathcal{O}))$, put

$$\tau(Z) := \operatorname{Ann} Z = \{ \, a \in \mathcal{O} \mid aZ \subseteq \pi(H_Z) \, \} \, .$$

Note that $Z \in L(\mathcal{A}(K/\mathcal{O}))[\kappa]$ if and only if $\tau(Z) \mid \kappa$.

Lemma 5.1 (See also [4, Lemma 6.2])**.** *Let $\varnothing \neq J \subseteq [n]$.*

(i) $\tau(Z) \mid \mathfrak{d}_{J,r(J)}$ for any $Z \in T_J/\pi(H_J)$.

(ii) $\tau(Z) = \mathfrak{d}_{J,r(J)}$ for some $Z \in T_J/\pi(H_J)$.

Proof. It follows immediately from Lemma 4.1. □

For $r \in \{0, \ldots, \ell-1\}$ we define

$$L^r(\mathcal{A}(K/\mathcal{O})) := \{ \, Z \in L(\mathcal{A}(K/\mathcal{O})) \mid \dim Z = r \, \}$$

and

$$d_r(\mathfrak{a}) := \sum_{\substack{\varnothing \neq J \subseteq [n] \\ r(J) = \ell - r}} (-1)^{|J|} m(J, \mathfrak{a}).$$

By the proof of Theorem 4.3, we have

$$f_{\mathcal{A}}^{\kappa}(t) = t^{\ell} + \sum_{r=0}^{\ell-1} d_r(\kappa) t^r \quad \text{and} \quad d_r(\kappa) = \sum_{\substack{Z \in L^r(\mathcal{A}(K/\mathcal{O})) \\ \tau(Z) \mid \kappa}} \mu(T_{\varnothing}, Z),$$

for each $\kappa \mid \rho_{\mathcal{A}}$.

Lemma 5.2 (See also [4, Claim 6.3])**.** *For each $r \in \{0, \ldots, \ell-1\}$, $d_r(\mathfrak{a})$ is a quasi-constant (quasi-monomial of degree 0) with the minimum period*

$$\rho_r := \operatorname{lcm} \{ \, \tau(Z) \mid Z \in L^r(\mathcal{A}(K/\mathcal{O})) \, \} \, .$$

Proof. We first prove that ρ_r is a period of $d_r(\mathfrak{a})$. By Lemma 3.2, $m(J, \mathfrak{a})$ has the minimum period $\mathfrak{d}_{J,r(J)}$. Hence $d_r(\mathfrak{a})$ has a period

$$\operatorname{lcm} \left\{ \, \mathfrak{d}_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n], \; r(J) = \ell - r \, \right\} .$$

By Lemma 5.1, this period equals $\operatorname{lcm} \{ \, \tau(Z) \mid Z \in L^r(\mathcal{A}(K/\mathcal{O})) \, \} = \rho_r$.

Next, we show the minimality of ρ_r. Let ρ be the minimum period of $d_r(\mathfrak{a})$. Then $\rho \mid \rho_r$ and $d_r(\rho) = d_r(\rho_r)$. By the definition of ρ_r, we have $\tau(Z) \mid \rho_r$ for any $Z \in L^r(\mathcal{A}(K/\mathcal{O}))$. Hence we obtain

$$\begin{aligned}
\sum_{Z \in L^r(\mathcal{A}(K/\mathcal{O}))} (-1)^{\ell-r} \mu(T_{\varnothing}, Z) &= \sum_{\substack{Z \in L^r(\mathcal{A}(K/\mathcal{O})) \\ \tau(Z) \mid \rho_r}} (-1)^{\ell-r} \mu(T_{\varnothing}, Z) \\
&= (-1)^{\ell-r} d_r(\rho_r) = (-1)^{\ell-r} d_r(\rho) \\
&= \sum_{\substack{Z \in L^r(\mathcal{A}(K/\mathcal{O})) \\ \tau(Z) \mid \rho}} (-1)^{\ell-r} \mu(T_{\varnothing}, Z).
\end{aligned}$$

Since $(-1)^{\ell-r} \mu(T_{\varnothing}, Z) > 0$ by Lemma 4.4 (iii), we have

$$L^r(\mathcal{A}(K/\mathcal{O})) = \{ \, Z \in L^r(\mathcal{A}(K/\mathcal{O})) \;\mid\; \tau(Z) \mid \rho \, \} \, .$$

Therefore $\rho_r \mid \rho$ and hence $\rho_r = \rho$. □

Theorem 5.1 (Generalization of Theorem 1.5)**.** *The LCM-period $\rho_{\mathcal{A}}$ is the minimum period of the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}$.*

Proof. The characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}$ is given by

$$\chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{a}) = N(\mathfrak{a})^{\ell} + \sum_{r=0}^{\ell-1} d_r(\mathfrak{a}) N(\mathfrak{a})^r.$$

By Lemma 5.2, each coefficient $d_r(\mathfrak{a})$ has the minimum period ρ_r. Hence the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}$ has the minimum period $\operatorname{lcm}\{\, \rho_r \mid 0 \le r \le \ell - 1 \,\}$. By Lemma 5.1, we obtain

$$\rho_{\mathcal{A}} = \operatorname{lcm}\left\{\, \mathfrak{d}_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \,\right\} = \operatorname{lcm}\{\, \tau(Z) \mid Z \in L(\mathcal{A}(K/\mathcal{O})) \setminus \{T_{\varnothing}\} \,\} = \operatorname{lcm}\{\, \rho_r \mid 0 \le r \le \ell - 1 \,\}.$$

□

6. Localizations of the base rings

Let S be a multiplicative subset of $\mathcal{O}$ and $\mathcal{O}_S$ denotes the localization of $\mathcal{O}$ with respect to S. Since $\mathcal{O}$ is a Dedekind domain, so is $\mathcal{O}_S$.

Let $I_S(\mathcal{O}) := \{\, \mathfrak{A} \cap \mathcal{O} \mid \mathfrak{A} \in I(\mathcal{O}_S) \,\}$, where $I(\mathcal{O}_S)$ denotes the set of nonzero ideals of $\mathcal{O}_S$. By standard ring theory, we have a one-to-one correspondence between ideals in $I_S(\mathcal{O})$ and $I(\mathcal{O}_S)$ as follows:

$$\begin{array}{rcl} I_S(\mathcal{O}) & \longleftrightarrow & I(\mathcal{O}) \\ \mathfrak{a} & \longmapsto & \mathfrak{a}\mathcal{O}_S \\ \mathfrak{A} \cap \mathcal{O} & \longleftarrow & \mathfrak{A}. \end{array}$$

Note that this correspondence preserves prime ideals and

$$I_S(\mathcal{O}) \cap \operatorname{Spec}(\mathcal{O}) = \{\, \mathfrak{p} \in I(\mathcal{O}) \cap \operatorname{Spec}(\mathcal{O}) \mid \mathfrak{p} \cap S = \varnothing \,\},$$

where $\operatorname{Spec}(\mathcal{O})$ denotes the set of prime ideals of $\mathcal{O}$.

Lemma 6.1. *The following statements hold:*

(i) For any $s \in S$ and $\mathfrak{a} \in I_S(\mathcal{O})$, $\mathfrak{a} + \langle s \rangle = \langle 1 \rangle$.

(ii) For any ideal $\mathfrak{a} \in I_S(\mathcal{O})$, we have $\mathcal{O}_S/\mathfrak{a}\mathcal{O}_S \simeq \mathcal{O}/\mathfrak{a}$. In particular, $N(\mathfrak{a}\mathcal{O}_S) = N(\mathfrak{a})$. Moreover, $\mathcal{O}_S/\mathfrak{A} \simeq \mathcal{O}/(\mathfrak{A} \cap \mathcal{O})$ and $N(\mathfrak{A}) = N(\mathfrak{A} \cap \mathcal{O})$ for each $\mathfrak{A} \in I(\mathcal{O}_S)$.

(iii) $\mathcal{O}_S$ is residually finite.

Proof. (i) Since $\mathcal{O}_S$ is a Dedekind domain, the ideal $\mathfrak{a}\mathcal{O}_S$ is decomposed into a product of prime ideals as $\mathfrak{a}\mathcal{O}_S = (\mathfrak{p}_1\mathcal{O}_S) \cdots (\mathfrak{p}_r\mathcal{O}_S)$, where $\mathfrak{p}_i \in I_S(\mathcal{O}) \cap \operatorname{Spec}(\mathcal{O})$. Then $\mathfrak{a}\mathcal{O}_S = (\mathfrak{p}_1\mathcal{O}_S) \cdots (\mathfrak{p}_r\mathcal{O}_S) = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)\mathcal{O}_S$ and hence $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$. Since $\mathcal{O}$ is a Dedekind domain and $\mathfrak{p}_i$ is a prime ideal, $\mathfrak{p}_i + \langle s \rangle$ equals $\mathfrak{p}_i$ or $\langle 1 \rangle$. If $\mathfrak{p}_i + \langle s \rangle = \mathfrak{p}_i$, then $s \in \mathfrak{p}_i$, which contradicts to $\mathfrak{p}_i \cap S = \varnothing$. Therefore $\mathfrak{p}_i + \langle s \rangle = \langle 1 \rangle$. Thus $\mathfrak{a} + \langle s \rangle = \langle 1 \rangle$.

(ii) By (i), every element of S is a unit of $\mathcal{O}/\mathfrak{a}$. Therefore $\mathcal{O}_S/\mathfrak{a}\mathcal{O}_S \simeq (\mathcal{O}/\mathfrak{a}) \otimes_{\mathcal{O}} \mathcal{O}_S \simeq \mathcal{O}/\mathfrak{a}$.

(iii) This follows immediately from (ii). □

Let $\mathcal{A}_S$ denote the finite set $\mathcal{A}$ considered as a subset of $\mathcal{O}_S$ via the natural inclusion $\mathcal{O} \subseteq \mathcal{O}_S$. Since $\mathcal{O}_S$ is a residually finite Dedekind domain by Lemma 6.1(iii), we can consider the characteristic quasi-polynomial and the torsion arrangement of $\mathcal{A}_S$.

6.1 The characteristic quasi-polynomial of $\mathcal{A}_S$

The characteristic quasi-polynomial $\chi_{\mathcal{A}_S}^{\text{quasi}}$ of $\mathcal{A}_S$ is a quasi-polynomial on $I(\mathcal{O}_S)$ defined by $\chi_{\mathcal{A}_S}^{\text{quasi}}(\mathfrak{A}) = |M(\mathcal{A}_S(\mathcal{O}_S/\mathfrak{A}))|$. The following proposition shows that $\chi_{\mathcal{A}_S}^{\text{quasi}}$ is equivalent to the restriction of $\chi_{\mathcal{A}}^{\text{quasi}}$ to $I_S(\mathcal{O})$ via the correspondence described above.

Theorem 6.1. *The following statements hold:*

(i) The LCM-period of $\mathcal{A}_S$ is $\rho_{\mathcal{A}_S} = \rho_{\mathcal{A}}\mathcal{O}_S$.

(ii) For any $\mathfrak{A} \in I(\mathcal{O}_S)$, $\chi_{\mathcal{A}_S}^{\text{quasi}}(\mathfrak{A}) = \chi_{\mathcal{A}}^{\text{quasi}}(\mathfrak{A} \cap \mathcal{O})$.

(iii) For any $\delta \mid \rho_{\mathcal{A}_S}$, $f_{\mathcal{A}_S}^{\delta}(t) = f_{\mathcal{A}}^{\delta \cap \mathcal{O}}(t)$.

Proof. (i) For each nonempty subset $J \subseteq [n]$, we define $\phi_{J,S}\colon \mathcal{O}_S^\ell \to \mathcal{O}_S^{|J|}$ by $\boldsymbol{x} \mapsto \boldsymbol{x}C_J$. Then we have $\operatorname{coker}\phi_{J,S} = (\operatorname{coker}\phi_J) \otimes_{\mathcal{O}} \mathcal{O}_S$. Hence by Lemma 3.1, we obtain

$$\begin{aligned}\operatorname{coker}\phi_{J,S} = (\operatorname{coker}\phi_J) \otimes_{\mathcal{O}} \mathcal{O}_S &\simeq \left(\bigoplus_{i=1}^{r(J)} \mathcal{O}/\mathfrak{d}_{J,i} \oplus \mathfrak{d} \oplus \mathcal{O}^{|J|-r(J)-1}\right) \otimes_{\mathcal{O}} \mathcal{O}_S \\ &\simeq \bigoplus_{i=1}^{r(J)} \mathcal{O}_S/\mathfrak{d}_{J,i}\mathcal{O}_S \oplus \mathfrak{d}\mathcal{O}_S \oplus \mathcal{O}_S^{|J|-r(J)-1}.\end{aligned}$$

Therefore $\rho_{\mathcal{A}_S} = \operatorname{lcm}\left\{ \mathfrak{d}_{J,r(J)}\mathcal{O}_S \;\middle|\; \varnothing \neq J \subseteq [n] \right\} = \rho_{\mathcal{A}}\mathcal{O}_S$.

(ii) By Lemma 6.1(ii), $\chi^{\mathrm{quasi}}_{\mathcal{A}_S}(\mathfrak{A}) = |M(\mathcal{A}_S(\mathcal{O}_S/\mathfrak{A}))| = |M(\mathcal{A}(\mathcal{O}/(\mathfrak{A}\cap\mathcal{O})))| = \chi^{\mathrm{quasi}}_{\mathcal{A}}(\mathfrak{A}\cap\mathcal{O})$.

(iii) Let $\mathfrak{A} \in I(\mathcal{O}_S)$ be any ideal satisfying $\mathfrak{A} + \rho_{\mathcal{A}_S} = \delta$. Then $(\mathfrak{A}\cap\mathcal{O}) + \rho_{\mathcal{A}} = \delta \cap \mathcal{O}$. Therefore

$$f^{\delta}_{\mathcal{A}_S}(N(\mathfrak{A})) = \chi^{\mathrm{quasi}}_{\mathcal{A}_S}(\mathfrak{A}) = \chi^{\mathrm{quasi}}_{\mathcal{A}}(\mathfrak{A}\cap\mathcal{O}) = f^{\delta\cap\mathcal{O}}_{\mathcal{A}}(N(\mathfrak{A}\cap\mathcal{O})) = f^{\delta\cap\mathcal{O}}_{\mathcal{A}}(N(\mathfrak{A})).$$

Hence we obtain $f^{\delta}_{\mathcal{A}_S}(t) = f^{\delta\cap\mathcal{O}}_{\mathcal{A}}(t)$. □

6.2 The torsion arrangement $\mathcal{A}_S(K/\mathcal{O}_S)$

The torsion arrangement $\mathcal{A}_S(K/\mathcal{O}_S)$ consists of

$$T_{j,S} := \left\{ \pi_S(\boldsymbol{x}) \in (K/\mathcal{O}_S)^\ell \;\middle|\; \boldsymbol{x}c_j \equiv 0 \pmod{\mathcal{O}_S} \right\} \qquad (j \in [n]),$$

where $\pi_S\colon K^\ell \to (K/\mathcal{O}_S)^\ell$ is the natural projection. For each $\varnothing \neq J \subseteq [n]$, let $T_{J,S} := \bigcap_{j\in J} T_{j,S}$ and let $T_{\varnothing,S} := (K/\mathcal{O}_S)^\ell$.

The posets of layers of $\mathcal{A}(K/\mathcal{O})$ and $\mathcal{A}_S(K/\mathcal{O}_S)$ are as follows:

$$\begin{aligned} L(\mathcal{A}(K/\mathcal{O})) &= \{ \pi(\boldsymbol{x}) + \pi(H_J) \mid J \subseteq [n], \pi(\boldsymbol{x}) \in T_J \}, \\ L(\mathcal{A}_S(K/\mathcal{O}_S)) &= \{ \pi_S(\boldsymbol{x}) + \pi_S(H_J) \mid J \subseteq [n], \pi_S(\boldsymbol{x}) \in T_{J,S} \}. \end{aligned}$$

Let $\eta_S\colon (K/\mathcal{O})^\ell \to (K/\mathcal{O}_S)^\ell$ be the canonical $\mathcal{O}$-homomorphism. Since $\eta_S \circ \pi = \pi_S$, the $\mathcal{O}$-homomorphism η_S induces the following order-preserving map:

$$\begin{array}{rccl} \eta_S\colon & L(\mathcal{A}(K/\mathcal{O})) & \longrightarrow & L(\mathcal{A}(K/\mathcal{O}_S)) \\ & Z & \longmapsto & \eta_S(Z) = \{ \eta_S(z) \mid z \in Z \}. \end{array}$$

Note that $\dim Z = \dim \eta_S(Z)$ since if $Z = \pi(\boldsymbol{x}) + \pi(H_J)$ then $\eta_S(Z) = \pi_S(\boldsymbol{x}) + \pi_S(H_J)$.

The map η_S induces isomorphisms between torsion subposets as follows:

Theorem 6.2. *For any ideal $\delta \mid \rho_{\mathcal{A}_S}$, the restriction*

$$\eta_S\colon L(\mathcal{A}(K/\mathcal{O}))[\delta\cap\mathcal{O}] \longrightarrow L(\mathcal{A}_S(K/\mathcal{O}_S))[\delta]$$

is an isomorphism as finite posets. In particular,

$$L(\mathcal{A}_S(K/\mathcal{O}_S)) \simeq L(\mathcal{A}(K/\mathcal{O}))[\rho_{\mathcal{A}}\mathcal{O}_S \cap \mathcal{O}].$$

Proof. First, we show that if $Z \in L(\mathcal{A}(K/\mathcal{O}))[\delta\cap\mathcal{O}]$, then $\eta_S(Z) \in L(\mathcal{A}_S(K/\mathcal{O}_S))[\delta]$. Suppose that $Z = \pi(\boldsymbol{x}) + \pi(H_Z)$. Then $\eta_S(Z) = \pi_S(\boldsymbol{x}) + \pi_S(H_Z)$.

Assume $a/s \in \delta$ ($a \in \mathcal{O}, s \in S$). Since $a \in \delta\cap\mathcal{O}$ and $Z \in L(\mathcal{A}(K/\mathcal{O}))[\delta\cap\mathcal{O}]$, we have $\pi(a\boldsymbol{x}) \in \pi(H_Z)$ and hence $\pi_S(a\boldsymbol{x}) \in \pi_S(H_Z)$. Then there exists $\boldsymbol{v} \in H_Z$ such that $a\boldsymbol{x} - \boldsymbol{v} \in \mathcal{O}_S^\ell$. Therefore $(a/s)\boldsymbol{x} - (1/s)\boldsymbol{v} \in \mathcal{O}_S^\ell$. Since $(1/s)\boldsymbol{v} \in H_Z$, we have $\pi_S((a/s)\boldsymbol{x}) \in \pi_S(H_Z)$. Thus $\eta_S(Z) \in L(\mathcal{A}_S(K/\mathcal{O}_S))[\delta]$.

Secondly, we prove the injectivity. Suppose $Z_1, Z_2 \in L(\mathcal{A}(K/\mathcal{O}))[\delta\cap\mathcal{O}]$ and $Z_i = \pi(\boldsymbol{x}_i)+\pi(H_{Z_i})$ for $i \in \{1,2\}$ and assume that $\eta_S(Z_1) = \eta_S(Z_2)$. Then $\pi_S(\boldsymbol{x}_1) + \pi_S(H_{Z_1}) = \pi_S(\boldsymbol{x}_2) + \pi_S(H_{Z_2})$ and hence $H := H_{Z_1} = H_{Z_2}$ by Lemma 4.3. Therefore there exists $\boldsymbol{v} \in H$ such that $\boldsymbol{x}_1 - \boldsymbol{x}_2 - \boldsymbol{v} \in \mathcal{O}_S^\ell$. Hence there exists $s \in S$ such that $s\boldsymbol{x}_1 - s\boldsymbol{x}_2 - s\boldsymbol{v} \in \mathcal{O}^\ell$. Then we have $\pi(s\boldsymbol{x}_1) \equiv \pi(s\boldsymbol{x}_2) \pmod{\pi(H)}$.

By Lemma 6.1(i), we have $(\delta\cap\mathcal{O})+\langle s\rangle = \langle 1\rangle$ and hence there exist $d \in \delta\cap\mathcal{O}$ and $a \in \mathcal{O}$ such that $d+as = 1$. Since $\pi(d\boldsymbol{x}_1), \pi(d\boldsymbol{x}_2) \in \pi(H)$ by the assumption $Z_1, Z_2 \in L(\mathcal{A}(K/\mathcal{O}))[\delta\cap\mathcal{O}]$, we have

$$\begin{aligned}\pi(\boldsymbol{x}_1) &= \pi(d\boldsymbol{x}_1) + \pi(as\boldsymbol{x}_1) \equiv a\pi(s\boldsymbol{x}_1) \\ &\equiv a\pi(s\boldsymbol{x}_2) = \pi(d\boldsymbol{x}_2) + \pi(as\boldsymbol{x}_2) = \pi(\boldsymbol{x}_2) \pmod{\pi(H)}.\end{aligned}$$

Thus $Z_1 = \pi(\boldsymbol{x}_1) + \pi(H) = \pi(\boldsymbol{x}_2) + \pi(H) = Z_2$ and hence the map η_S is injective.

Finally, we prove the surjectivity. A layer in $L(\mathcal{A}_S(K/\mathcal{O}_S))[\delta]$ belongs to $(T_{J,S}/\pi_S(H_J))[\delta]$ for some $J \subseteq [n]$. From the discussion so far, when we restrict the map η_S to $(T_J/\pi(H_J))[\delta \cap \mathcal{O}]$, we have an injection $\eta_J : (T_J/\pi(H_J))[\delta \cap \mathcal{O}] \to (T_{J,S}/\pi_S(H_J))[\delta]$. By Theorem 4.2 and Lemma 6.1(ii),

$$|(T_J/\pi(H_J))[\delta \cap \mathcal{O}]| = \prod_{i=1}^{r(J)} N((\delta \cap \mathcal{O}) + \mathfrak{d}_{J,i}) = \prod_{i=1}^{r(J)} N(\delta + \mathfrak{d}_{J,i}\mathcal{O}_S) = |(T_{J,S}/\pi_S(H_J))[\delta]|.$$

Therefore η_J is a bijection and hence η_S is surjective. □

Corollary 6.1. *Let $\delta \mid \rho_{\mathcal{A}_S}$. Then*

$$\chi^{\delta}_{\mathcal{A}_S(K/\mathcal{O}_S)}(t) = \chi^{\delta\cap\mathcal{O}}_{\mathcal{A}(K/\mathcal{O})}(t).$$

Proof. It follows immediately from Theorem 6.2. □

Remark 6.1. *Corollary 6.1 also can be obtained by Theorem 4.3 and Theorem 6.1(iii).*

7. Computing methods for the characteristic quasi-polynomials

7.1 The LCM-periods

Recall that the LCM-period $\rho_{\mathcal{A}}$ is defined by

$$\rho_{\mathcal{A}} = \operatorname{lcm}\left\{ \mathfrak{d}_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \right\},$$

where the ideals $\mathfrak{d}_{J,i}$ are described in Lemma 3.1. In this subsection, we will prove that the size of J can be restricted as follows:

Proposition 7.1 (See also [6, p.323 Formula (11)])**.**

$$\rho_{\mathcal{A}} = \operatorname{lcm}\left\{ \mathfrak{d}_{J,r(J)} \;\middle|\; J \subseteq [n],\, 1 \leq |J| \leq \min\{\ell, n\} \right\}.$$

In order to prove Proposition 7.1, we show the following lemma.

Lemma 7.1 (See also [6, Lemma 2.3])**.** *Let J_1 and J_2 be two subsets of $[n]$ such that $J_1 \supseteq J_2$ and $r(J_1) = r(J_2)$. Then $\mathfrak{d}_{J_1,r(J_1)} \mid \mathfrak{d}_{J_2,r(J_2)}$.*

Proof. Since $J_1 \supseteq J_2$, we have $T_{J_1} \subseteq T_{J_2}$ and $H_{J_1} \subseteq H_{J_2}$. Moreover, since $r(J_1) = r(J_2)$, we have $H := H_{J_1} = H_{J_2}$. Then we obtain an inclusion $T_{J_1}/\pi(H) \subseteq T_{J_2}/\pi(H)$. Since $\mathfrak{d}_{J_i,r(J_i)} = \operatorname{Ann}(T_{J_i}/\pi(H))$ for each $i \in \{1,2\}$ by Lemma 4.1, we have $\mathfrak{d}_{J_2,r(J_2)} \subseteq \mathfrak{d}_{J_1,r(J_1)}$. Thus $\mathfrak{d}_{J_1,r(J_1)} \mid \mathfrak{d}_{J_2,r(J_2)}$. □

Proof of Proposition 7.1. Assume that $\ell < n$. Let J be a subset of $[n]$ with $\ell < |J| \leq n$. Then we can choose a subset $J' \subseteq J$ such that $r(J') = |J'| = r(J) \leq \ell$. Lemma 7.1 implies that $\mathfrak{d}_{J,r(J)} \mid \mathfrak{d}_{J',r(J')}$. This leads to the desired result. □

7.2 Coefficients

Recall that the κ-constituent of $\chi^{\text{quasi}}_{\mathcal{A}}$ is given by

$$f^{\kappa}_{\mathcal{A}}(t) = \sum_{J \subseteq [n]} (-1)^{|J|}\, m(J,\kappa) t^{\ell - r(J)}$$

by Theorem 3.1, where $m(J,\kappa) = \prod_{i=1}^{r(J)} N\left(\kappa + \mathfrak{d}_{J,i}\right)$.

Proposition 7.2 (See also [8, Corollary 2.3])**.** *Suppose that $\kappa, \kappa' \in I(\mathcal{O})$ both divide $\rho_{\mathcal{A}}$ and assume that there exists $s \in \mathbb{Z}_{>0}$ such that $\kappa + \mathfrak{d}_{J,r(J)} = \kappa' + \mathfrak{d}_{J,r(J)}$ for all $J \subseteq [n]$ with $|J| \leq s$. Then*

$$\deg\left(f^{\kappa}_{\mathcal{A}}(t) - f^{\kappa'}_{\mathcal{A}}(t)\right) < \ell - s.$$

In particular, we have $\deg\left(f^{\kappa}_{\mathcal{A}}(t) - f^{\langle 1\rangle}_{\mathcal{A}}(t)\right) < \ell - s$ *if $\mathfrak{d}_{J,r(J)} + \kappa = \langle 1\rangle$ for all $J \subseteq [n]$ with $|J| \leq s$.*

Proof. It is sufficient to show that $m(J,\kappa)=m(J,\kappa')$ for each $J\subseteq[n]$ with $r(J)\le s$. Let J' be a subset of J such that $r(J')=|J'|=r(J)\le s$. By the assumption, we have $\kappa+\mathfrak{d}_{J',r(J')}=\kappa'+\mathfrak{d}_{J',r(J')}$. By Lemma 7.1, $\mathfrak{d}_{J,r(J)}\mid\mathfrak{d}_{J',r(J')}$. Hence we obtain

$$\begin{aligned} m(J,\kappa) &= \prod_{i=1}^{r(J)} N(\kappa+\mathfrak{d}_{J,i}) = \prod_{i=1}^{r(J)} N(\kappa+\mathfrak{d}_{J,r(J)}+\mathfrak{d}_{J,i}) = \prod_{i=1}^{r(J)} N(\kappa'+\mathfrak{d}_{J,r(J)}+\mathfrak{d}_{J,i}) \\ &= \prod_{i=1}^{r(J)} N(\kappa'+\mathfrak{d}_{J,i}) = m(J,\kappa'). \end{aligned}$$

□

Proposition 7.3 (See also [8, Corollary 2.4]). *Suppose that $\kappa,\kappa'\in I(\mathcal{O})$ both divide $\rho_{\mathcal{A}}$ and that $\kappa+\kappa'=\langle 1\rangle$. Assume that there exists $s\in\mathbb{Z}_{>0}$ such that $\kappa+\mathfrak{d}_{J,r(J)}=\langle 1\rangle$ or $\kappa'+\mathfrak{d}_{J,r(J)}=\langle 1\rangle$ for all $J\subseteq[n]$ with $|J|\le s$. Then*

$$\deg\left(f_{\mathcal{A}}^{\langle 1\rangle}(t)+f_{\mathcal{A}}^{\kappa\kappa'}(t)-f_{\mathcal{A}}^{\kappa}(t)-f_{\mathcal{A}}^{\kappa'}(t)\right)<\ell-s.$$

Proof. For each $J\subseteq[n]$ with $r(J)\le s$, it is sufficient to show that

$$1+m(J,\kappa\kappa')-m(J,\kappa)-m(J,\kappa')=0.$$

Let J' be a subset of J such that $r(J')=|J'|=r(J)\le s$. By the assumption, we have $\kappa+\mathfrak{d}_{J',r(J')}=\langle 1\rangle$ or $\kappa'+\mathfrak{d}_{J',r(J')}=\langle 1\rangle$. By Lemma 7.1, $\mathfrak{d}_{J,r(J)}\mid\mathfrak{d}_{J',r(J')}$. Hence we have $\kappa+\mathfrak{d}_{J,r(J)}=\langle 1\rangle$ or $\kappa'+\mathfrak{d}_{J,r(J)}=\langle 1\rangle$. Since $\mathfrak{d}_{J,i}\mid\mathfrak{d}_{J,r(J)}$ for each i, we have

$$m(J,\kappa)=\prod_{i=1}^{r(J)} N(\kappa+\mathfrak{d}_{J,i})=1 \qquad\text{or}\qquad m(J,\kappa')=\prod_{i=1}^{r(J)} N(\kappa'+\mathfrak{d}_{J,i})=1.$$

Thus we have

$$\begin{aligned} 0 &= (1-m(J,\kappa))\,(1-m(J,\kappa')) = 1-m(J,\kappa)-m(J,\kappa')+m(J,\kappa)m(J,\kappa') \\ &= 1-m(J,\kappa)-m(J,\kappa')+\prod_{i=1}^{r(J)} N\left((\kappa+\mathfrak{d}_{J,i})\,(\kappa+\mathfrak{d}_{J,i})\right). \end{aligned}$$

Since $\kappa+\kappa'=\langle 1\rangle$,

$$(\kappa+\mathfrak{d}_{J,i})\,(\kappa'+\mathfrak{d}_{J,i})=\kappa\kappa'+\mathfrak{d}_{J,i}\,(\kappa+\kappa'+\mathfrak{d}_{J,i})=\kappa\kappa'+\mathfrak{d}_{J,i}\,(\langle 1\rangle+\mathfrak{d}_{J,i})=\kappa\kappa'+\mathfrak{d}_{J,i}.$$

Therefore

$$\prod_{i=1}^{r(J)} N\left((\kappa+\mathfrak{d}_{J,i})\,(\kappa+\mathfrak{d}_{J,i})\right)=\prod_{i=1}^{r(J)}(\kappa\kappa'+\mathfrak{d}_{J,i})=m(J,\kappa\kappa'),$$

which leads to the desired result. □

Proposition 7.4 (See also [8, Corollary 2.5]). *Suppose that $\kappa,\kappa'\in I(\mathcal{O})$ both divide $\rho_{\mathcal{A}}$ and that $\kappa+\kappa'=\langle 1\rangle$. If $\mathfrak{d}_{J,r(J)}$ are powers of prime ideals for all J, we have $f_{\mathcal{A}}^{\kappa\kappa'}(t)=f_{\mathcal{A}}^{\kappa}(t)+f_{\mathcal{A}}^{\kappa'}(t)-f_{\mathcal{A}}^{\langle 1\rangle}(t)$.*

Proof. Since $\kappa+\kappa'=\langle 1\rangle$, it is always true that $\kappa+\mathfrak{p}^m=\langle 1\rangle$ or $\kappa'+\mathfrak{p}^m=\langle 1\rangle$ for any prime ideal $\mathfrak{p}$ and $m\in\mathbb{Z}_{\ge 0}$. Therefore the claim follows immediately from Proposition 7.3. □

8. The characteristic quasi-polynomials of non-crystallographic root systems of types H_2, H_3, and H_4

Let Φ be an irreducible crystallographic root system and Φ^+ a positive system of Φ. Every positive root is expressed as a linear combination of the simple roots with integral coefficients. Gathering the coefficient column vectors, we obtain the set $\mathcal{A}_\Phi$ consisting of integral column vectors. Kamiya, Takemura, and Terao [5,7] computed the characteristic quasi-polynomial of $\mathcal{A}_\Phi$ and its LCM-period explicitly by using the classification of root systems. Note that Suter [13] gave essentially the same calculation in terms of the number of lattice points in the fundamental alcoves (the Ehrhart quasi-polynomials).

Now, by the results in this paper, we can consider the non-crystallographic cases. The irreducible non-crystallographic root systems are classified into one infinite family $\mathrm{I}_2(m)$ together with two exceptional types H_3 and H_4. We consider the root systems of types $\mathrm{H}_2 := I_2(5), \mathrm{H}_3$, and H_4.

We assume that every root in Φ_{H_n} is normalized. Then every positive root is represented as a linear combination of the simple roots over $\mathcal{O} := \mathbb{Z}[\tau]$, where τ denotes the golden ratio $\tau := \frac{1+\sqrt{5}}{2}$. The ring $\mathcal{O}$ is a residually finite Dedekind domain since it is the ring of integers of the quadratic field $K := \mathbb{Q}(\sqrt{5})$.

We calculate the characteristic quasi-polynomials and the LCM-periods (the minimum periods) of the root systems of types $\mathrm{H}_2, \mathrm{H}_3$, and H_4, using with SageMath [14] and propositions obtained in Section 7. The minimum periods are shown in Table 1 with the exponents and the Coxeter numbers.

	Exponents	Coxeter number	Minimum period
H_2	$1, 4$	5	$\langle 1\rangle$
H_3	$1, 5, 9$	10	$\langle 2\rangle$
H_4	$1, 11, 19, 29$	30	$\langle 6\sqrt{5}\rangle$

Table 1: Minimum periods for $\mathrm{H}_2, \mathrm{H}_3$, and H_4

8.1 The characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathrm{H}_2}$

Suppose that

$$\Phi_{\mathrm{H}_2} = \left\{ \pm\left(\cos\frac{2\pi k}{5}, \sin\frac{2\pi k}{5}\right) \;\middle|\; 0 \le k \le 4 \right\}.$$

Note that the convex hull of Φ_{H_2} is the regular decagon. One may choose simple roots as follows.

$$\alpha_1 = (1, 0), \qquad \alpha_2 = \left(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}\right).$$

The coefficient matrix of the positive roots with respect to $\{\alpha_1, \alpha_2\}$ is

$$\begin{pmatrix} 1 & 0 & \tau & 1 & \tau \\ 0 & 1 & 1 & \tau & \tau \end{pmatrix}.$$

The LCM-period is $\rho_{\mathrm{H}_2} = \langle 1\rangle$ and the constituent of $\chi^{\mathrm{quasi}}_{\mathrm{H}_2}$ is as follows:

$$f^{\langle 1\rangle}_{\mathrm{H}_2}(t) = t^2 - 5t + 4 = (t-1)(t-4).$$

8.2 The characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathrm{H}_3}$

Suppose that

$$\Phi_{\mathrm{H}_3} = \left\{ \begin{array}{cl} (\pm 1, 0, 0) & \text{and all permutations} \\ \frac{1}{2}(\pm\tau, \pm 1, \pm\tau^{-1}) & \text{and all even permutations} \end{array} \right\}.$$

Note that the convex hull of Φ_{H_3} is known as the icosidodecahedron. One may choose simple roots as follows:

$$\alpha_1 = \frac{1}{2}(\tau, -1, \tau^{-1}), \qquad \alpha_2 = \frac{1}{2}(-\tau, 1, \tau^{-1}), \qquad \alpha_3 = \frac{1}{2}(1, \tau^{-1}, -\tau).$$

The coefficient matrix of the positive roots with respect to $\{\alpha_1, \alpha_2, \alpha_3\}$ is

$$\begin{pmatrix} 1 & 0 & \tau & 1 & \tau & 0 & 0 & \tau & \tau & \tau^2 & 1 & \tau & \tau & \tau^2 & \tau^2 \\ 0 & 1 & 1 & \tau & \tau & 0 & 1 & 1 & \tau^2 & \tau^2 & \tau & \tau & \tau^2 & \tau^2 & 2\tau \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \tau & \tau & \tau & \tau & \tau \end{pmatrix}.$$

The LCM-period is $\rho_{\mathrm{H}_3} = \langle 2\rangle$ and the constituents of $\chi^{\mathrm{quasi}}_{\mathrm{H}_3}$ is as follows:

$$f^{\langle 1\rangle}_{\mathrm{H}_3}(t) = t^3 - 15t^2 + 59t - 45 = (t-1)(t-5)(t-9),$$
$$f^{\langle 2\rangle}_{\mathrm{H}_3}(t) = t^3 - 15t^2 + 59t - 60 = (t-4)(t^2 - 11t + 15).$$

8.3 The characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathrm{H}_4}$

Suppose that

$$\Phi_{\mathrm{H}_4} = \left\{ \begin{array}{cl} (\pm 1, 0, 0, 0) & \text{and all permutations} \\ \frac{1}{2}(\pm 1, \pm, 1, \pm 1, \pm 1) & \text{and all permutations} \\ \frac{1}{2}(\pm\tau, \pm 1, \pm\tau^{-1}, 0) & \text{and all even permutations} \end{array} \right\}.$$

Note that the convex hull of Φ_{H_4} is known as the 600-cell. One may choose simple roots as follows:

$$\alpha_1 = \frac{1}{2}(\tau, -1, \tau^{-1}, 0), \qquad \alpha_2 = \frac{1}{2}(-\tau, 1, \tau^{-1}, 0),$$
$$\alpha_3 = \frac{1}{2}(1, \tau^{-1}, -\tau, 0), \qquad \alpha_4 = \frac{1}{2}(-1, -\tau, 0, \tau^{-1}).$$

The coefficient matrix of the positive roots with respect to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is

$$\left(\begin{array}{cccccccccc}
1 & 0 & 1 & \tau & \tau & 0 & 0 & \tau & \tau & \tau+1 \\
0 & 1 & \tau & 1 & \tau & 0 & 1 & 1 & \tau+1 & \tau+1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$
$$\begin{array}{cccccccccc}
1 & \tau & \tau & \tau+1 & \tau+1 & 0 & 0 & 0 & \tau & \tau \\
\tau & \tau & \tau+1 & \tau+1 & 2\tau & 0 & 0 & 1 & 1 & \tau+1 \\
\tau & \tau & \tau & \tau & \tau & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}$$
$$\begin{array}{cccccccccc}
\tau+1 & \tau & \tau+1 & \tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+1 & 1 & \tau & \tau \\
\tau+1 & \tau+1 & \tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+2 & 2\tau+2 & \tau & \tau & \tau+1 \\
1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+2 & \tau & \tau & \tau \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \tau & \tau & \tau
\end{array}$$
$$\begin{array}{cccccccccc}
\tau+1 & \tau+1 & \tau & \tau+1 & \tau+1 & 2\tau+1 & 2\tau+1 & \tau+1 & \tau+1 & 2\tau+1 \\
\tau+1 & 2\tau & \tau+1 & \tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+2 & 2\tau & 2\tau+1 & 2\tau+1 \\
\tau & \tau & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & 2\tau & 2\tau & 2\tau \\
\tau & \tau & \tau & \tau & \tau & \tau & \tau & \tau & \tau & \tau
\end{array}$$
$$\begin{array}{cccccccccc}
2\tau+1 & 2\tau+2 & 2\tau+1 & 2\tau+1 & 2\tau+2 & 3\tau+1 & 2\tau+2 & 2\tau+1 & 2\tau+1 & 2\tau+1 \\
3\tau+1 & 3\tau+1 & 2\tau+2 & 3\tau+1 & 3\tau+1 & 3\tau+2 & 3\tau+2 & 2\tau+2 & 2\tau+2 & 3\tau+1 \\
2\tau & 2\tau & 2\tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+1 & \tau+2 & 2\tau+1 & 2\tau+1 \\
\tau & \tau & \tau & \tau & \tau & \tau & \tau & \tau+1 & \tau+1 & \tau+1
\end{array}$$
$$\left.\begin{array}{cccccccccc}
2\tau+2 & 2\tau+2 & 3\tau+1 & 2\tau+2 & 3\tau+1 & 3\tau+1 & 3\tau+2 & 3\tau+2 & 3\tau+2 & 3\tau+2 \\
3\tau+1 & 3\tau+2 & 3\tau+2 & 3\tau+2 & 3\tau+2 & 3\tau+3 & 3\tau+3 & 4\tau+2 & 4\tau+2 & 4\tau+2 \\
2\tau+1 & 2\tau+1 & 2\tau+1 & 2\tau+2 & 2\tau+2 & 2\tau+2 & 2\tau+2 & 2\tau+2 & 3\tau+1 & 3\tau+1 \\
\tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & \tau+1 & 2\tau
\end{array}\right).$$

The LCM-period is $\rho_{\mathrm{H}_4} = \langle 6\sqrt{5} \rangle$ and the constituents of $\chi^{\mathrm{quasi}}_{\mathrm{H}_4}$ is as follows:

$$\begin{aligned}
f^{\langle 1 \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 7140t + 6061 = (t-1)(t-11)(t-19)(t-29), \\
f^{\langle 3 \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 7140t + 9261 = (t-9)(t-21)(t^2-30t+49), \\
f^{\langle \sqrt{5} \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 7140t + 14125 = (t-5)(t-25)(t^2-30t+113), \\
f^{\langle 3\sqrt{5} \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 7140t + 17325, \\
f^{\langle 2 \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 8040t + 17536 = (t-4)(t-16)(t^2-40t+274), \\
f^{\langle 6 \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 8040t + 20736, \\
f^{\langle 2\sqrt{5} \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 8040t + 25600 = (t-20)(t^3-40t^2+338t-1280), \\
f^{\langle 6\sqrt{5} \rangle}_{\mathrm{H}_4}(t) &= t^4 - 60t^3 + 1138t^2 - 8040t + 28800.
\end{aligned}$$

8.4 Observations

Kamiya, Takemura, and Terao [7, Theorem 3.1] gave an explicit formula of the generating function $\Gamma_\Phi := \sum_{q=1}^{\infty} \chi^{\mathrm{quasi}}_{\Phi}(q) t^q$ for an irreducible crystallographic root system Φ in terms of the coefficient of the highest root and the Coxeter number. We obtain the following corollary.

Corollary 8.1 (Kamiya-Takemura-Terao [7, Corollary 3.4]). *Let h be the Coxeter number of Φ. Then $\chi_{\Phi}^{\mathrm{quasi}}(q) > 0$ if and only if $q \geq h$.*

In order to consider the analogue, we list the Coxeter numbers and possible values of the absolute norms of nonzero ideals of $\mathcal{O}$. The Coxeter numbers are

$$h_{\mathrm{H}_2} = 5, \qquad h_{\mathrm{H}_3} = 10, \qquad h_{\mathrm{H}_4} = 30.$$

The possible values of absolute norms are positive integers of the form $a^2 + ab - b^2$:

$$1, 4, 5, 9, 11, 16, 19, 20, 25, 29, 31, \dots.$$

From the lists above, we obtain the following theorem.

Theorem 8.1. *Let $\ell \in \{2, 3, 4\}$ and $\mathfrak{a}$ a nonzero ideal of $\mathcal{O}$. Then $\chi_{\mathrm{H}_\ell}^{\mathrm{quasi}}(\mathfrak{a}) > 0$ if and only if $N(\mathfrak{a}) \geq h_{\mathrm{H}_\ell}$.*

It is well known that the exponents of a root system satisfy duality with respect to the Coxeter number. Namely, if d is an exponent, then $h - d$ is also an exponent, where h denotes the Coxeter number. The exponents for $\mathrm{H}_2, \mathrm{H}_3$, and H_4 are

$$(1, 4), \qquad (1, 5, 9), \qquad (1, 11, 19, 29),$$

respectively. They appear in the factorization of the characteristic polynomials.

The characteristic quasi-polynomial of an irreducible crystallographic root system also has duality with respect to the Coxeter number. The duality can be shown from the explicit expressions given by Kamiya, Takemura, and Terao [5], or Suter [13]. Yoshinaga [17] gave a classification-free proof.

Theorem 8.2 (Yoshinaga [17, Corollary 3.8]). *Let Φ be an irreducible crystallographic root system of rank ℓ and h its Coxeter number. Then $\chi_{\Phi}^{\mathrm{quasi}}(q) = (-1)^{\ell} \chi_{\Phi}^{\mathrm{quasi}}(h - q)$.*

Note that the duality holds as quasi-polynomials but not the level of the constituents. Yoshinaga [16] studied the condition for the constituents to hold the duality in detail.

We consider an analogue of Theorem 8.2 for our cases. One can see that if $\kappa \mid \rho_{\mathrm{H}_\ell}$ is not a multiple of $\langle 2 \rangle$, then the κ-constituent satisfies the duality. Namely $f_{\mathrm{H}_\ell}^{\kappa}(t) = (-1)^{\ell} f_{\mathrm{H}_\ell}^{\kappa}(h_{\mathrm{H}_\ell} - t)$ when $\langle 2 \rangle \nmid \kappa$. However, unfortunately, it seems that the κ-constituent does not satisfy the duality when κ is a multiple of $\langle 2 \rangle$.

Question 8.1. *Are there any reasons for Theorem 8.1 and the partial duality above?*

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