

On the Homology of Several Number-Theoretic Set Families

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ABSTRACT: This paper describes the homology of various simplicial complexes associated with set families from combinatorial number theory, including primitive sets, pairwise coprime sets, product-free sets, and coprime-free sets. We present a condition on a set family that results in easy computation of the homology groups and show that the first three examples, among many others, admit such a structure. We then extend our techniques to address the complexes associated with coprime-free sets and a generalization of primitive sets.

Keywords: Coprime-free sets; Homology; Primitive sets; Product-free sets

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1. Introduction

HOMOLOGY THEORY has been a fundamental tool in enumerative combinatorics since G.-C. Rota's watershed 1964 paper [14] established links between the topological data of a poset and its combinatorial properties. Combinatorial problems, reciprocally, give rise to a wealth of topological spaces to study. In particular, many families of sets that arise in the field of combinatorial number theory possess interesting topological structures.

In this paper, we consider a number of simplicial complexes associated with set families defined by conditions of a number-theoretic nature. It turns out that many of these complexes are in some sense the same, topologically speaking, and we introduce a definition—namely, that of a partition-intersecting family—to capture a condition that engenders this structure. The remainder of the introduction will be devoted to this task. In the second section, we detail well-known relationships between the homology groups of a poset and alternating sums associated with its elements. Three that fall into our definition of partition-intersecting are primitive sets, pairwise coprime sets, and product-free sets. In Section 3 we describe these homologies without breaking a sweat and enumerate a few more examples that can be dealt with in the same way. On the other hand, the family of coprime-free sets is not partition-intersecting, and in this case, we can only fully characterize the zeroth and first homology groups. This is done in Section 4. In Section 5, we tackle an analogous generalisation of primitive sets, whose corresponding homology groups are derivable nearly free of charge from the machinery used to deal with coprime-free sets.

The number-theoretic properties of the set families underlying this paper have been extensively studied; we have chosen to defer the exposition of this background to Section 3, when these sets are defined in more detail.

1.1 Definitions and notation

For integers $n \geq 1$ we denote by $[n]$ the discrete interval $\{1, \dots, n\}$. The main object of study will be some family \mathcal{F} of finite subsets of integers, so for brevity, we shall let \mathcal{F}_n denote the finite set $\mathcal{F} \cap 2^{[n]}$. Consider the partially ordered set (poset) obtained by taking \mathcal{F}_n as a ground set and ordering its elements by inclusion. If $\emptyset \in \mathcal{F}_n$, then it is the bottom element of the poset, since $\emptyset \subseteq x$ for all $x \in \mathcal{F}_n$. It will become important later to add the restriction $[n] \notin \mathcal{F}_n$ for all $n \geq 2$, so that \mathcal{F}_n does not have a top element, but in this case, we may adjoin an artificial top element to obtain a lattice; we shall denote this element by $\hat{1}$. An element y is said to *cover* an element $x \neq y$ if $x \leq y$ and $x \leq z \leq y$ implies that $z = x$ or $z = y$. In a lattice, an element covered by $\hat{1}$ is called a *coatom*.

1.2 Cross-cuts

A *chain* in a poset X is a set $\{x_1, x_2, \dots, x_k\} \subseteq X$ such that $x_1 < x_2 < \dots < x_k$. A chain is *maximal* if adding any new element to the chain causes the chain property to be violated. Let L be a lattice with top element $\hat{1}$ and bottom element $\hat{0}$. A *cross-cut* C is a subset of $L \setminus \{\hat{1}, \hat{0}\}$ such that no two elements of C are comparable and every chain in L contains some element of C . It is easy to see that the set of coatoms in a lattice is a cross-cut.

A subset $S \subseteq L$ is said to be *spanning* if the join of all its elements is $\hat{1}$ and the meet of all its elements is $\hat{0}$. The cross-cut complex $\Delta(C)$ of a cross-cut C is the simplicial complex obtained from taking C as the vertex set and adding a face for any subset of C that is *not* spanning. (For us, a *simplicial complex* is a set of subsets of a finite set that is closed under intersection.) It is a classical result that the homology groups of $\Delta(C)$ depend only on the lattice L and not the choice of cross-cut C , so it makes sense to speak of the *homology* of L . For the rest of the paper, by homology, we shall mean reduced homology (i.e., the zeroth homology of a connected space is trivial) over the group \mathbf{Z} . So we write $\tilde{H}_k(\Delta)$ for $\tilde{H}_k(\Delta, \mathbf{Z})$.

1.3 Partition-intersecting families

We now define a criterion on a poset that leads to easily-described homology groups. Let \mathcal{F} be a family of subsets of \mathbf{N} . For $n \geq 2$ in the definition below, let $\mathcal{F}_n = \mathcal{F} \cap 2^{[n]}$ for brevity and let M_n be the set of maximal elements in \mathcal{F}_n . These are the elements $S \in \mathcal{F}_n$ such that for all $x \in [n] \setminus S$, $S \cup \{x\} \notin \mathcal{F}_n$. We shall say that \mathcal{F} is *partition-intersecting with m components* if there exists m such that for all $n \geq 2$, the set $M_n \subseteq \mathcal{F}_n$ of maximal elements can be partitioned into m disjoint nonempty classes $M_n = C_1 \sqcup \dots \sqcup C_m$ with the property that for all i , the intersection $\bigcap_{S \in C_i} S$ is nonempty, and for all $i \neq j$ and any $S \in C_i$ and $T \in C_j$, $S \cap T = \emptyset$.

The following lemma describes the homology groups of partition-intersecting families.

Lemma 1.1. *Let \mathcal{F} be a downward closed set family that is partition-intersecting with m components. Suppose further that $[n] \notin \mathcal{F}_n$ for all $n \geq 2$. Then for any cross-cut C of the lattice $L = (\mathcal{F} \cap 2^{[n]}) \cup \{\hat{1}\}$, we have $\tilde{H}_0(\Delta(C)) = \mathbf{Z}^{m-1}$ and for all $k > 0$, $\tilde{H}_k(\Delta(C)) = 0$.*

Proof. Fix $n \geq 2$. By hypothesis, the set M_n of coatoms of \mathcal{F}_n can be partitioned into $M_n = C_1 \sqcup C_2 \sqcup \dots \sqcup C_m$ where each C_i is nonempty, and for all i , and all subsets C' of C_i , the intersection $\bigcap_{S \in C'} S$ is nonempty. This tells us that for any i , any subset of C_i is not spanning and is thus a face in $\Delta(M_n)$. In other words, the simplicial complex $\Delta(M_n)$ is the union of m disjoint simplices, which has $\tilde{H}_0(\Delta(M_n)) = \mathbf{Z}^{m-1}$ and $\tilde{H}_k(\Delta(M_n)) = 0$ for all $k > 0$. This completes the proof. \square

The proof of this lemma is shorter than the very definition of partition-intersecting family, and from this one might suspect that the definition is too artificial or convoluted to be applicable. However, we shall see that in fact, many set families arising in combinatorial number theory are partition-intersecting. Hence Lemma 1.1 can be used to find the homology groups of the resulting lattices and in turn, the homology groups give alternating-sum identities for the cardinalities of the sets in the family.

2. The Möbius function and alternating sums

This short section relates Lemma 1.1 to the cardinalities of the sets in a partition-intersecting family. Recall that the *Möbius function* μ_X of a poset X is defined on pairs of elements $(x, y) \in X^2$ with $x \leq y$, and is given by the recursive formula $\mu_X(x, x) = 1$ for all $x \in X$ and

$$\mu_X(x, y) = - \sum_{x \leq z < y} \mu_X(x, z).$$

The following theorem, which appears in Rota’s 1964 paper [14], relates the Möbius function to the homological data of a poset.

Theorem (Cross-cut theorem). *Let L be a lattice with bottom element $\hat{0}$ and top element $\hat{1}$. For any cross-cut C of L ,*

$$\mu_L(\hat{0}, \hat{1}) = \sum_{k=0}^n (-1)^k \text{rank } \tilde{H}_k(\Delta(C)),$$

where $\tilde{H}_k(\Delta(C), \mathbf{Z})$ is the k th reduced homology group of $\Delta(C)$.

The Möbius function generalizes the counting method used to derive the inclusion-exclusion formula; indeed, letting B_n denote the lattice of all subsets of $[n]$, ordered by inclusion, one checks that $\mu_{B_n}(\emptyset, S) = (-1)^{|S|}$ for any $S \in B_n$. This fact will be used in the proof of the following proposition, which is a corollary of Lemma 1.1. Combined with the proof of the earlier lemma, the overall method employed is similar to one used by M. K. Goh, J. Hamdan, and J. Saks [9] to show that the Möbius function of the poset of all arithmetic progressions contained in $[n]$ equals the number-theoretic Möbius function evaluated at $n - 1$.

Theorem 2.1. *Let \mathcal{F} be a family of integer sets that is partition-intersecting with m components. Suppose furthermore that \mathcal{F} is downward closed and that for all $n \geq 2$, $\mathcal{F}_n = \mathcal{F} \cap 2^{[n]}$ does not contain $[n]$ itself. Then, letting $F_{n,k}$ denote the number of elements of cardinality k in \mathcal{F}_n for $0 \leq k < n$, we have*

$$\sum_{k=1}^{n-1} (-1)^k F_{n,k} = 1 - m$$

for all $n \geq 2$.

Proof. Fix $n \geq 2$ and let L_n be the lattice obtained by adjoining a top element to \mathcal{F}_n . Since \mathcal{F}_n is downward closed, for all $x \in \mathcal{F}_n$ the induced subposet

$$\{z \in L_n : 0 \leq z \leq x\}$$

is isomorphic to the Boolean lattice $B_{|x|}$ and we have $\mu_{L_n}(0, x) = (-1)^{|x|}$. Then since $[n] \notin \mathcal{F}_n$, we have

$$\sum_{k=1}^{n-1} (-1)^k F_{n,k} = \sum_{x \in L_n \setminus \{\widehat{1}\}} \mu_{L_n}(\emptyset, x) = -\mu_{L_n}(\emptyset, \widehat{1}).$$

But letting C be the cross-cut of coatoms in L_n , by Theorem A we have

$$\mu_{L_n}(\emptyset, \widehat{1}) = \sum_{k=0}^n (-1)^k \text{rank } \widetilde{H}_k(\Delta(C)),$$

and since \mathcal{F} is partition-intersecting, Lemma 1.1 tells us that

$$\sum_{k=0}^n (-1)^k \text{rank } \widetilde{H}_k(\Delta(C)) = \text{rank } \widetilde{H}_0(\Delta(C)) = m - 1,$$

which is what we wanted. □

3. Primitive, pairwise coprime, and product-free sets

We now show that many families of integer sets that arise in combinatorial number theory are partition-intersecting. As motivation, observe that the set $\{2, 3, 5, 7, 11, \dots\}$ of primes possesses many pleasant properties that may be generalized to other sets of integers. For instance, a set S of integers is said to be

- i) *primitive* if for any two distinct $i, j \in S$, i does not divide j ;
- ii) *pairwise coprime* if any two distinct $i, j \in S$ have $\text{gcd}(i, j) = 1$; and
- iii) *product-free* if for any $i, j \in S$ not necessarily distinct, $ij \notin S$.

It is easily seen that the set of prime numbers satisfies the requirements for all three of these definitions, and of course, so does any finite subset of the primes. Denote $\{1, 2, \dots, n\}$ by $[n]$ for short and let P_n be the number of primitive subsets of $[n]$. Likewise, let Q_n be the number of pairwise coprime subsets of $[n]$ and let R_n be the number of product-free subsets of $[n]$. These quantities have been the subject of interest of several papers over the past thirty years. All of them were treated by a 1990 paper of P. J. Cameron and P. Erdős [6]. They showed that

$$c_1^n \leq P_n \leq c_2^n$$

where $c_1 = 1.44967\dots$ and $c_2 = 1.59\dots$,

$$2^{\pi(n)} e^{(1/2+o(1))\sqrt{n}} \leq Q_n \leq 2^{\pi(n)} e^{(2+o(1))\sqrt{n}}$$

where π is the prime-counting function. This result was subsequently improved to

$$Q(n) = 2^{\pi(n)} e^{\sqrt{n}(1+O(\log \log n / \log n))}$$

by N. J. Calkin and A. Granville [5]. The paper of Cameron and Erdős also gives the lower bound $R_n \geq 2^{n-\sqrt{n}}$, and the authors note that any product-free sequence has an upper density of less than 1, but there are product-free sequences with density more than $1 - \epsilon$ for any $\epsilon > 0$.

Regarding primitive sets, Cameron and Erdős conjectured that the limit $\lim_{n \rightarrow \infty} P(n)^{1/n}$ exists; this was proven by R. Angelo in 2017 [3]. A separate proof was presented in a paper by H. Liu, P. P. Pach, and R. Palincza [12]; it yielded constants $c_1 = 1.571068$ and $c_2 = 1.574445$. Shortly afterward, an independent approach of N. McNew [13] further improved the lower bound c_1 to 1.572939.

A great deal of attention has been given to infinite primitive sets. In 1993, P. Erdős and Z. Zhang [8] proved that for any primitive set A ,

$$\sum_{a \in A} \frac{1}{a \log a} \leq 1.84.$$

The sum $\sum_p 1/(p \log p)$ equals $1.6366\dots$ when p ranges over all primes, and Erdős conjectured that the bound he and Zhang gave could be improved to this value. This was proven very recently, in a 2022 preprint of J. D. Lichtman [11].

Let $P_{n,k}$ be the number of primitive subsets of $[n]$ of cardinality exactly k , so that $P_n = \sum_{k=0}^n P_{n,k}$. Define $Q_{n,k}$ and $R_{n,k}$ analogously for pairwise coprime sets and product-free sets. None of these quantities have any known formula, as far as the authors are aware. In this section, we shall prove alternating-sum identities for these three counts. First, we list some simple facts about these quantities in special cases.

Proposition 3.1. *For $n \geq 2$ we have*

- i) $P_{n,1} = Q_{n,1} = n$ and $R_{n,1} = n - 1$;
- ii) $P_{n,2} = \sum_{i=2}^n (i - d(i))$, where $d(i)$ counts the number of divisors of i ;
- iii) $Q_{n,2} = \sum_{i=2}^n \varphi(i)$, where φ is Euler's totient function;
- iv) $R_{n,2} = \binom{n}{2} - n - \lfloor \sqrt{n} \rfloor + 2$; and
- v) for any p prime and $k \geq 3$, $F(p, k) = F(p - 1, k) + F(p - 1, k - 1)$, where F can be any of P , Q , or R .

Proof. Assertion (i) is obvious, as are assertions (ii) and (iii) once one notes that in both cases the sum runs over the larger element in each 2-element set. To prove (iv), we start with all 2-element subsets of $[n]$, then remove the $n - 1$ subsets containing 1 as well as the $\lfloor \sqrt{n} \rfloor - 1$ subsets that contain a square and its square root.

For the final assertion, note that every primitive k -subset of $[p]$ either does not contain p or it does. In the first case, it is counted by $P_{p-1,k}$, and in the second case, it is counted by $P_{p-1,k-1}$, since we can add p to any primitive $(k - 1)$ -subset of $[p - 1]$ without violating primitivity (for this we need $k \geq 3$, otherwise the $(k - 1)$ -subset could be $\{1\}$). The same argument works in the case of relatively coprime and product-free sets, since p is coprime to all smaller integers. \square

Theorem 3.1. *The families of primitive sets, pairwise coprime sets, and product-free sets are all partition-intersecting, with two components in the first case and one component in the other two cases.*

Proof. Fix $n \geq 2$ and let p be the largest prime less than or equal to n . By the famous 1852 theorem of P. Chebyshev [7], we have $2p > n$. This will be important in two of the cases. We now show that each of these families is partition-intersecting, computing the number m of components along the way.

We deal with primitive sets first. In this case, the collection C of maximal primitive sets contains the singleton $\{1\}$, but no other element of C contains 1, since 1 divides every other positive integer. So we set $C_1 = \{\{1\}\}$ and $C_2 = C \setminus \{1\}$. To any subset of $\{2, 3, \dots, n\}$, we may add the element p without violating primitivity, since no element divides p and $2p > n$. Thus every element of C_2 contains p and we see that $m = 2$.

For the family of pairwise coprime sets, every maximal set contains 1, since $\gcd(1, i) = 1$ for all $1 \leq i \leq n$. So we have $C_1 = C$ and $m = 1$.

Lastly, we note that no product-free set contains 1, since $1 \cdot 1 = 1$, and every maximal product-free subset of $[n]$ must also contain p , since p is not the product of any two elements of $\{2, 3, \dots, n\}$ and $2p > n$. Hence again we have $C_1 = C$ and $m = 1$ in this case as well. \square

The interval $[n]$ is neither primitive, nor pairwise coprime (for $n \geq 4$), nor product-free. Furthermore, these three properties are all closed under taking subsets, so the families are downward closed, which gives us the following corollary of Theorem 2.1. (Note that since [2] and [3] are pairwise coprime, the cases $n = 2$ and $n = 3$ must be verified separately in the case of pairwise coprime sets.)

Corollary 3.1. *For $n \geq 2$, we have*

- i) $\sum_{k=0}^n (-1)^k P_{n,k} = -1;$
- ii) $\sum_{k=0}^n (-1)^k Q_{n,k} = 0;$ and
- iii) $\sum_{k=0}^n (-1)^k R_{n,k} = 0.$

Values of $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ for small n and k are presented in the appendix. For the purpose of exposition, we have chosen to focus on three specific set families in this section, but many more multiplicative conditions can be placed on integer sets to yield partition-intersecting families. Two more examples from [6] are treated by the following proposition.

Proposition 3.2. *If \mathcal{F} is the family of all finite subsets S of \mathbf{N} such that*

- i) *for all $i, j, k, l \in S$ all distinct, $ij \neq kl$;*
- ii) *for all $i, j, k \in S$ with $i \notin \{j, k\}$, i does not divide jk ; or*
- iii) *for all $i, j \in S$ with $i < j$, i divides j ,*

then \mathcal{F} is partition-intersecting with one component and, letting $F_{n,k}$ denote the number of sets in $\mathcal{F} \cap 2^{[n]}$ of cardinality k , we have

$$\sum_{k=0}^n (-1)^k F_{n,k} = 0,$$

for n large enough depending on the case.

Proof. Cases (i) and (ii) are proved in a fashion similar to the case of product-free sets. Case (iii) is similar to the case of pairwise coprime sets. The details are left to the reader. □

Note that if m is a constant not depending on n , then for a family to be partition-intersecting with m components we must have m equal to either 1 or 2, since there are at most two maximal elements in \mathcal{F}_2 . But by relaxing $n \geq 2$ to $n \geq N$ for some larger N in the definition of partition-intersecting, the value of m would then range between 1 and $\binom{N}{N/2}$. One could also imagine extending the definition to let m vary as a function of n , which may allow more set families to be treated.

Lastly, note that for all of these examples, the alternating-sum identities themselves are not so difficult to prove from scratch. For six of the seven families mentioned above, let p be a prime in the interval $(n/2, n]$ and note that any $S \subseteq [n] \setminus \{p\}$ is in the family if and only if $S \cup \{p\}$ is. The one exception is in the case of primitive sets, when $S = \{1\}$ is primitive but $\{1, p\}$ is not, which explains why the alternating sum equals -1 in this case.

4. Coprime-free sets

In part (iii) of Proposition 3.2 above, we negated the condition “ i does not divide j ” from the definition of primitive sets and ended up with another partition-intersecting family. In this section, we show what happens if one negates the condition $\gcd(i, j) = 1$, that is, if we require that $\gcd(i, j) > 1$ for all $i \neq j$ in each of our sets. These sets are said to be *coprime-free*, and they are not partition-intersecting in general. The asymptotic number of coprime-free sets was treated by the paper [6] of Cameron and Erdős, and subsequently in the paper [5] by Calkin and Granville. (Both papers were mentioned in the previous section.) The latter paper showed that the number of subsets $S \subseteq [n]$ for which $\gcd(i, j) \neq 1$ for all distinct $i, j \in S$ is

$$2^{\lfloor n/2 \rfloor} + 2^{\lfloor n/2 \rfloor - N} + O\left(2^{\lfloor n/2 \rfloor - N} \exp\left(-C \frac{n}{\log^2 n \log \log n}\right)\right),$$

for an absolute constant $C > 0$, where $N = (e^{-\gamma} + o(1))n / \log \log n$. (Here γ denotes the Euler-Mascheroni constant $0.5772\dots$)

In the proof of the proposition below we will make use of the notion of a nerve complex, which we define here, alongside some more terminology concerning simplicial complexes. Let $C = \{U_i\}_{i \in I}$ be a set family. The *nerve complex* of C is a simplicial complex defined on the vertex set I , where a set $J \subseteq I$ forms a face if and only if $\bigcap_{i \in J} U_i \neq \emptyset$. Borsuk’s nerve theorem [4] states that for a simplicial complex Δ and a set $\{\Delta_i\}_{i \in I}$ of subcomplexes of Δ , if

- i) $\bigcup_{i \in I} \Delta_i = \Delta$; and
- ii) any nonempty intersection of finitely many Δ_i is contractible,

then the nerve complex of $\{\Delta_i\}_{i \in I}$ is homotopy equivalent to Δ .

The *simplicial star* of a vertex x in a simplicial complex is the set of all faces containing x . The *link* of x is the set of all faces F such that $x \notin F$ but $F \cup \{x\}$ is a face.

Also of use to us is the the Mayer–Vietoris sequence, which for reduced simplicial homology says that

$$\begin{aligned} \cdots \rightarrow \tilde{H}_s(A) \oplus \tilde{H}_s(B) \rightarrow \tilde{H}_s(X) \rightarrow \tilde{H}_{s-1}(A \cap B) \rightarrow \tilde{H}_{s-1}(A) \oplus \tilde{H}_{s-1}(B) \rightarrow \\ \cdots \rightarrow \tilde{H}_0(A \cap B) \rightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) \rightarrow \tilde{H}_0(X) \rightarrow 0, \end{aligned}$$

whenever A and B are simplicial subcomplexes whose union is the simplicial complex X (see, e.g., Remark 5.18 of [10]).

The following proposition computes the zeroth homology of the posets corresponding to coprime-free subsets of $[n]$, and shows that the first homology is trivial.

Proposition 4.1. *Let \mathcal{F} be the family of finite coprime-free subsets of \mathbf{N} . Let \mathcal{F}_n denote $\mathcal{F} \cap 2^{[n]}$ and let Δ_n be the simplicial complex corresponding to the cross-cut of coatoms in the lattice $\mathcal{F}_n \cup \bar{1}$. Then we have*

$$\tilde{H}_0(\Delta_1) = 0, \quad \tilde{H}_0(\Delta_2) = \mathbf{Z}, \quad \tilde{H}_0(\Delta_3) = \mathbf{Z}^2,$$

and for $n \geq 4$ we have $\tilde{H}_0(\Delta_n) = \mathbf{Z}^{C(n)+1}$, where $C(n)$ is the number of primes in the range $(n/2, n]$. Furthermore, $\tilde{H}_1(\Delta_n)$ is trivial.

Proof. The assertions for $1 \leq n \leq 3$ easily follow from the observation that the only nonempty coprime-free subsets of $\{1, 2, 3\}$ are the singleton sets $\{1\}$, $\{2\}$, and $\{3\}$.

Now fix $n \geq 4$. Any prime p in the range $(n/2, n]$ is coprime to every integer in $[n] \setminus \{p\}$, so the set $\{p\}$ is a maximal coprime-free subset of $[n]$ and hence is an isolated vertex of Δ_n . The set $\{1\}$ is also an isolated vertex for the same reason, giving us $C(n) + 1$ isolated points in Δ_n . To prove that $\tilde{H}_0(\Delta_n) = \mathbf{Z}^{C(n)+1}$ we must show that the rest of Δ_n forms one nonempty connected component. Note first that for every prime $p \in [2, n/2]$ (there are such primes because $n \geq 4$), the set $x_p = p\mathbf{N} \cap [n]$ is a non-singleton maximal coprime-free subset of $[n]$. Since $2p \in x_p$ for all $p \in (2, n/2]$, $x_p \cap x_2 \neq \emptyset$ for all these coatoms, and thus these vertices are connected by some nontrivial face in Δ_n . Lastly, consider an arbitrary vertex of Δ_n that is not one of the $C(n) + 1$ singletons described above. This coatom must contain some multiple of a prime p in $[2, n/2]$. Hence $x \cap x_p \neq \emptyset$.

We will now prove that $\tilde{H}_1(\Delta_n)$ is trivial for all n . To do so, we will use Borsuk’s nerve theorem to reduce the problem to studying a simpler simplicial complex. For all $i \in [n]$, let U_i be the simplicial subcomplex of Δ_n induced on the vertices that contain i . Note that the set $\{U_i\}_{i \in [n]}$ covers Δ_n . This is true because faces of Δ_n are present wherever a set of maximal coprime-free subsets have nonempty intersection, and this intersection must contain some element of $[n]$. Moreover, for any $J \subseteq [n]$, let $U_J = \bigcap_{i \in J} U_i$. A face of Δ_n belongs to this intersection precisely when each of its vertices contains every element of J , so this is the subcomplex of Δ_n induced on vertices x such that $J \subseteq x$. If J is not coprime-free, then clearly U_J is empty, and if J is coprime-free and nonempty, then since all vertices in U_J are supersets of J , they all intersect and thus the complex U_J is a simplex and *a fortiori* contractible. Applying Borsuk’s nerve theorem, we have shown that Δ_n is homotopy equivalent to the simplicial complex X_n whose vertex set is $[n]$ and in which a subset $J \subseteq [n]$ is a face if and only if J is coprime-free.

We proceed by induction on n . That $\tilde{H}_1(X_n) = 0$ for $1 \leq n \leq 4$ can easily be verified by hand. Now let $n > 4$. If n is prime, then X_n is simply the union of X_{n-1} and a new isolated vertex, so $\tilde{H}_0(X_n) = \tilde{H}_0(X_{n-1}) \oplus \mathbf{Z}$ but all other homologies remain unchanged, and by the induction hypothesis, $\tilde{H}_1(X_n) = 0$.

The interesting case is when $n > 4$ is not prime. We let A be the simplicial star of the vertex n in X_n , and let $B = X_{n-1}$, considered as a subset of X_n . It is clear that the union of A and B is X_n , since A contains all faces containing n and B contains all other faces. The relevant section of the Mayer–Vietoris sequence is

$$\cdots \rightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \rightarrow \tilde{H}_1(X_n) \rightarrow \tilde{H}_0(A \cap B) \rightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) \rightarrow \tilde{H}_0(X_n) \rightarrow 0.$$

Note that A , being a simplicial star, is contractible. Moreover, we proved above that $\tilde{H}_0(X_n) = C(n) + 1$ and $\tilde{H}_0(B) = C(n - 1) + 1$, and by the induction hypothesis, $\tilde{H}_1(B) = 0$.

Next we count the connected components in $A \cap B$, whose vertices are the integers in $[n - 1]$ that share a prime factor with n . Let $i, j \in A \cap B$, let p be a prime that divides both i and n , and let q be a prime that divides both j and n . Writing $n = kp = lq$, we have $k, l \geq 2$, since n is not prime. If, furthermore, $n/2$ is not prime, then $k, l \geq 3$, so both $2p$ and $2q$ are in $A \cap B$ and we have the path $i \rightarrow 2p \rightarrow 2q \rightarrow j$ along the 1-skeleton of $A \cap B$. In the case that $n/2$ is prime, the case that k or l is equal to 2 corresponds to either i or j being $n/2$, and we already know that the prime $n/2$ is an isolated vertex of X_{n-1} , so it is also an isolated vertex of $A \cap B$. We have shown that

$$\text{rank } \tilde{H}_0(A \cap B) = \mathbf{1}_{n/2 \text{ is prime}}.$$

Putting all these facts into the exact sequence above, we have

$$\cdots \rightarrow 0 \rightarrow \tilde{H}_1(X_n) \rightarrow \mathbf{Z}^{\mathbf{1}_{n/2 \text{ is prime}}} \rightarrow 0 \oplus \mathbf{Z}^{C(n-1)+1} \rightarrow \mathbf{Z}^{C(n)+1} \rightarrow 0.$$

But since n is composite, we have

$$C(n-1) = \begin{cases} C(n) + 1, & \text{if } n/2 \text{ is prime;} \\ C(n), & \text{otherwise.} \end{cases}$$

Thus we see that $C(n-1) + 1 = C(n) + 1 + \mathbf{1}_{n/2 \text{ is prime}}$, and can conclude that $\text{rank } \tilde{H}_1(X_n) = 0$. \square

This proof illustrates that the situation with coprime-free sets is rather different from the partition-intersecting families we considered before. From Proposition 4.1 we cannot conclude anything about the alternating sum corresponding to coprime-free sets, and next we will show that the proposition cannot be improved in general.

In the following description of Δ_{143} , which was communicated to us by M. Adamaszek [1], we shall use terminology established in the work of J. A. Barmak and E. G. Minian [2]. A vertex x in a simplicial complex X is said to be *dominated* by another vertex y if the link of x is the simplicial star of y in the subcomplex $X \setminus \{x\}$. In other words, for every face F of X that contains x , $F \cup \{y\}$ is also a face of X . Hence removing the vertex x from the complex does not change the homotopy type and the replacement of X with $X \setminus \{x\}$ is called an *elementary strong collapse* of X .

Proposition 4.2. *Let \mathcal{F} be the family of finite coprime-free subsets of \mathbf{N} . Let \mathcal{F}_{143} denote $\mathcal{F} \cap 2^{[143]}$ and let Δ_{143} be the simplicial complex corresponding to the cross-cut of coatoms in the lattice $\mathcal{F}_{143} \cup \hat{1}$. We have $\tilde{H}_2(\Delta_{143}) \neq 0$.*

Proof. By Borsuk’s nerve theorem as we employed it in the proof of Proposition 4.1, Δ_{143} is homotopy equivalent to the simplicial complex X_{143} whose vertex set is $[143]$ and in which a nonempty subset $S \subseteq [143]$ forms a face if and only if S is coprime-free.

We now describe a sequence of elementary strong collapses. We check every pair $(i, j) \in [143]$ with $i \neq j$ to see if j dominates i . For each such pair, we may remove i from the complex without altering the homology groups. Note that in our context, j dominates i if and only if the set of primes dividing i is a subset of the set of primes dividing j . It is possible that both i dominates j and j dominates i (this happens if they have the same set of distinct prime factors with different multiplicities). In this case, it is valid to remove either i or j , but we shall make the choice to eliminate the larger integer. When the algorithm terminates, we are left with a simplicial complex X' whose vertex set is the set of all maximal squarefree integers in $[143]$, that is, squarefree integers i such that no squarefree multiple of i belongs to $[143]$.

We can safely ignore vertices in X' corresponding to primes in $(143/2, 143]$ as these isolated vertices have no bearing on $\tilde{H}_2(X')$. The main component in X' is the simplicial complex on the vertex set

$$\begin{aligned} &\{30, 42, 58, 62, 66, 70, 74, 77, 78, 82, 85, 86, 87, 91, 93, \\ &94, 95, 102, 105, 106, 110, 111, 114, 115, 118, 119, \\ &122, 123, 129, 130, 133, 134, 138, 141, 142, 143\}, \end{aligned}$$

where, again, a set of vertices forms a face if and only if it is coprime-free. Consider the subcomplex on the vertices $\{42, 66, 77, 78, 91, 143\}$. From the factorizations

$$\begin{aligned} 42 &= 2 \cdot 3 \cdot 7, & 66 &= 2 \cdot 3 \cdot 11, & 77 &= 7 \cdot 11, \\ 78 &= 2 \cdot 3 \cdot 13, & 91 &= 7 \cdot 13, & \text{and} & \quad 143 = 11 \cdot 13, \end{aligned}$$

it is easy to check by hand that these vertices form the surface of an octahedron. For this to be a nontrivial generator of $\tilde{H}_2(X')$, we must rule out the possibility that it is the boundary of a larger cycle. The complex X' is small enough that this is amenable to machine-assisted verification. \square

In Appendix B, we include a Sage program that performs the computation alluded to in the proof of Proposition 8. This program reports that $\tilde{H}_2(X_{143}) = \mathbf{Z}$, and $\tilde{H}_2(X_n) = 0$ for all $n \leq 142$.

5. A generalisation of primitive sets

A primitive set is a set S that does not contain more than one multiple of any integer in the set. We now generalize the condition to forbid having more than s multiples of an integer in the set. We will refer to sets satisfying this condition as *s-multiple sets*, and when $s = 1$ we recover the definition of a primitive set. In the more general case, we still have the useful property that any maximal s -multiple set contains 1 or it contains

every prime in the range $(n/2, n]$. The following theorem describes the homology of lattices associated with s -multiple set families.

As with coprime-free sets, s -multiple families are not partition-intersecting in general, but unlike in the previous section, we are able to fully describe the homology of the simplicial complexes corresponding to s -multiple set families.

Theorem 5.1. *Let $s \geq 2$ be an integer and let \mathcal{F}_s be the family of s -multiple subsets of \mathbf{N} . Fix $n > s$, let $\mathcal{F}_{s,n} = \mathcal{F}_s \cap 2^{[n]}$, and let Δ_n is the simplicial complex formed by the cross-cut of coatoms in $L_{s,n} = \mathcal{F}_{s,n} \cup \hat{1}$. Then we have*

$$\text{rank } \tilde{H}_t(\Delta_n) = \begin{cases} \binom{n-2}{s-1}, & \text{if } t = s - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Borsuk’s nerve theorem once again tells us that it suffices to study the simplicial complex X_n on the vertex set $\{1, \dots, n\}$, with a face for every s -multiple set of vertices. We again use the Mayer–Vietoris sequence, so let A be the simplicial star of 1 in X_n , and let B be the full simplicial subcomplex on $\{2, \dots, n\}$. The intersection $A \cap B$ is the link of 1, in which every subset of $\{2, \dots, n\}$ of cardinality $s - 1$ is a maximal face, since 1 may be added to any such set without violating the s -multiple condition, but any subset of $\{2, \dots, n\}$ of cardinality s already contains s multiples of 1. Thus $A \cap B$ is the $(s - 2)$ -skeleton of an $(n - 2)$ -simplex, and we have

$$\text{rank } \tilde{H}_t(A \cap B) = \begin{cases} \binom{n-2}{s-1}, & \text{if } t = s - 2; \\ 0, & \text{otherwise.} \end{cases}$$

The subcomplex A , being the simplicial star of 1, is evidently contractible. Now let p be a prime in the range $(n/2, n]$. Since 1 is not a vertex of B , and since any multiple of p is greater than n , we see that for any F in B such that $p \notin F$, the set $F \cup \{p\}$ is also a face of B . This means that the entire simplicial complex B is the simplicial star of p , and is thus contractible as well. But since $\tilde{H}_k(A) \oplus \tilde{H}_k(B) \cong 0$ for all k , the Mayer–Vietoris sequence informs us that

$$\tilde{H}_k(X_n) \cong \tilde{H}_{k-1}(A \cap B)$$

for all $k \geq 1$. This completes the proof. □

Finally, by reasoning as in the proof of Theorem 2.1, we have a corollary concerning alternating sums of counts of s -primitive sets.

Corollary 5.1. *Let $s \geq 2$ be an integer, let $n > s$, and let $P_{s,n,k}$ denote the number of s -multiple subsets of $[n]$ with cardinality exactly k . Then*

$$\sum_{k=0}^n (-1)^k P_{s,n,k} = (-1)^s \binom{n-2}{s-1}.$$

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A. Numerical tables

This appendix contains values of $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ for small values of n and k . The row sums of $P_{n,k}$, which we called P_n in section 3, appear as A051026 in the On-line Encyclopedia of Integer Sequences (OEIS). The row sums of $Q_{n,k}$ and $R_{n,k}$ were referred to as Q_n and R_n in section 3 and appear as A084422 and A326489, respectively. The triangular numbers $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ are A355145, A355146, and A355147 in the OEIS, respectively.

Table 1: The number $P_{n,k}$ of primitive k -subsets of $[n]$

n	$P_{n,0}$	$P_{n,1}$	$P_{n,2}$	$P_{n,3}$	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	$P_{n,8}$	$P_{n,9}$
1	1	1	0	0	0	0	0	0	0	0
2	1	2	0	0	0	0	0	0	0	0
3	1	3	1	0	0	0	0	0	0	0
4	1	4	2	0	0	0	0	0	0	0
5	1	5	5	2	0	0	0	0	0	0
6	1	6	7	3	0	0	0	0	0	0
7	1	7	12	10	3	0	0	0	0	0
8	1	8	16	15	5	0	0	0	0	0
9	1	9	22	26	13	2	0	0	0	0
10	1	10	28	38	22	4	0	0	0	0
11	1	11	37	66	60	26	4	0	0	0
12	1	12	43	80	76	35	6	0	0	0
13	1	13	54	123	156	111	41	60	0	0
14	1	14	64	161	227	180	74	12	0	0
15	1	15	75	206	323	299	161	47	6	0
16	1	16	86	253	425	421	242	75	10	0
17	1	17	101	339	678	846	663	317	85	10

Table 2: The number $Q_{n,k}$ of pairwise coprime k -subsets of $[n]$

n	$Q_{n,0}$	$Q_{n,1}$	$Q_{n,2}$	$Q_{n,3}$	$Q_{n,4}$	$Q_{n,5}$	$Q_{n,6}$	$Q_{n,7}$	$Q_{n,8}$
1	1	1	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0
4	1	4	5	2	0	0	0	0	0
5	1	5	9	7	2	0	0	0	0
6	1	6	11	8	2	0	0	0	0
7	1	7	17	19	10	2	0	0	0
8	1	8	21	25	14	3	0	0	0
9	1	9	27	37	24	6	0	0	0
10	1	10	31	42	26	6	0	0	0
11	1	11	41	73	68	32	6	0	0
12	1	12	45	79	72	33	6	0	0
13	1	13	57	124	151	105	39	6	0
14	1	14	63	138	167	114	41	6	0
15	1	15	71	159	192	128	44	6	0
16	1	16	79	183	228	157	56	8	0
17	1	17	95	262	411	385	213	64	8

Table 3: The number $R_{n,k}$ of product-free k -subsets of $[n]$

n	$R_{n,0}$	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$R_{n,5}$	$R_{n,6}$	$R_{n,7}$	$R_{n,8}$	$R_{n,9}$
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0
4	1	3	2	0	0	0	0	0	0	0
5	1	4	5	2	0	0	0	0	0	0
6	1	5	9	6	1	0	0	0	0	0
7	1	6	14	15	7	1	0	0	0	0
8	1	7	20	29	22	8	1	0	0	0
9	1	8	26	43	38	17	3	0	0	0
10	1	9	34	68	76	47	15	2	0	0
11	1	10	43	102	144	123	62	17	2	0
12	1	11	53	143	234	238	149	55	11	1

B. Computer verification of Proposition 8

The following Sage program verifies that $n = 143$ is the first example for which X_n , as defined in Proposition 8, has nontrivial second homology, and in fact, it shows that $\tilde{H}_2(X_{143}) = \mathbf{Z}$. We did not pay particular attention to the efficiency of runtime; nonetheless the entire computation took less than eight minutes on a standard laptop computer with 16 GB of RAM and a 2.4 GHz Intel Core i5 processor.

```
def coprime_free_second_homology(n):
    F = set()
    # build set of maximal squarefree integers in [n]
    for i in reversed(range(1,n+1)):
        if not is_squarefree(i):
            continue
        dominated = False
```

```
    for j in F:
        if j % i == 0:
            dominated = True
            break
    if not dominated:
        F.add(i)
# build graph on composite numbers in F,
# where  $i - j$  if and only if  $\gcd(i, j) > 1$ 
G = Graph()
for i in F:
    if not is_prime(i):
        G.add_vertex(i)
for i in G.vertices():
    for j in G.vertices():
        if i < j and  $\gcd(i, j) > 1$ :
            G.add_edge(i, j)
S = G.clique_complex()
return(S.homology(2))

# verify that the smallest example with nontrivial
# second homology is when  $n = 143$ 
for n in range(1, 144):
    print(n, coprime_free_second_homology(n))
```