Counting Partitions by Genus
I. Genus 0 to 2

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ABSTRACT: The counting of partitions according to their genus is revisited. The case of genus 0 –non-crossing partitions– is well known. Our approach relies on two pillars: first a functional equation between generating functions, originally written in genus 0 and interpreted graphically by Cvitanovic, is generalized to higher genus; secondly, we show that all partitions may be reconstructed from the "(semi)-primitive" ones introduced by Cori and Hetyei. Explicit results for the generating functions of all types of partitions are obtained in genus 1 and 2.

1. Introduction

1.1 Partitions, their genus and their census

Consider the set $\mathcal{P}(n)$ of partitions of the *set* $[\![n]\!] := \{1, \cdots, n\}$. If $\alpha \in \mathcal{P}(n)$ is made of α_1 parts of length 1, α_2 parts of length 2, etc, we say that α is of *type* $[\alpha] = [1^{\alpha_1}, \cdots, n^{\alpha_n}]$, which may be regarded as a partition of the *integer n*: $[\alpha] \vdash n$.

Let $C_{n,[\alpha]}$ denote the number of partitions of type $[\alpha]$

$$C_{n,[\alpha]} = \frac{n!}{\prod_{\ell=1}^{n} \alpha_\ell! (\ell!)^{\alpha_\ell}} \, .$$

The census of partitions may be subject to different conditions. In particular, it is well known, as we recall below in sect. 2.2, that any partition α may be assigned a *genus* $g(\alpha)$ by a formula descending from Euler's relation. Curiously, the census of partitions according to their genus is still an open problem, in spite of several fundamental contributions, [5, 6, 15, 20, 21]. Except for a few particular cases, only the case of genus 0 is thoroughly known: the *non-crossing partitions* (or planar) have been enumerated by Kreweras [15], before reappearing in various contexts, matrix integrals [1, 7], free probability [18, 19], and more recently, out-of-equilibrium quantum systems [12, 17] or radiation entropy [22].

Let $C_{n,[\alpha]}^{(g)}$ denote the number of partitions in $\mathcal{P}(n)$ of type $[\alpha]$ and genus g. Obviously $\sum_g C_{n,[\alpha]}^{(g)} = C_{n,[\alpha]}$.

We find it convenient to use generating functions (GF) to encode these numbers. Introduce a set of indeterminates κ_n, $n \in \mathbb{N}_+$, their GF

$$W(x) = \sum_{n \geq 1} \kappa_n x^n$$

and then

$$
\begin{aligned}
Z(x) &= 1 + \sum_{n \geq 1} \sum_{[\alpha] \vdash n} C_{n,[\alpha]} \kappa_{[\alpha]} x^n = \sum_g Z^{(g)}(x) \\
Z^{(g)}(x) &= \delta_{g0} + \sum_{n \geq 1} \sum_{[\alpha] \vdash n} C_{n,[\alpha]}^{(g)} \kappa_{[\alpha]} x^n
\end{aligned}
$$

where

$$\kappa_{[\alpha]} := \prod_{\ell=1}^{n} \kappa_{\ell}^{\alpha_{\ell}}. \tag{1}$$

There is a well-known relation between $Z^{(0)}$ and W, which has been found in different avatars [1, 7, 18], see (12) below. To extend such a relation to higher genus, we rely on a proven method. The diagrams encoding the partitions are first reduced to basic diagrams, in finite number at a given genus. In a second step, all diagrams –all partitions– are reconstructed by "dressing" the basic ones. This method is well-known in combinatorics and quantum field theory ("skeleton diagrams"). In the context of the enumeration of unicellular maps and partitions of a given genus, it has been explored by Chapuy [3] and Cori–Hetyei [6], who call the basic diagrams *schemes* and *primitive*, respectively.

In this paper, explicit formulae relating $Z^{(g)}$ to W and its derivatives will be found for $g = 1, 2$, and the corresponding expressions of $Z^{(g)}(x)$ are given by Theorem 1, (18), and Theorem 2, (29). Extension to higher genera is in principle feasible, if the list of their primitive diagrams is known.

1.2   Genus dependent cumulant expansion

The question of the relation between Z and W also arises in probability theory and statistics. There, it is common practice to associate *cumulants* to moments of random variables. If X is a random variable with moments $m_n = \mathbb{E}(X^n)$ of arbitrary order, we decompose these moments on cumulants κ_m and their products associated with partitions $\alpha \in \mathcal{P}(n)$

$$m_n = \sum_{\alpha \in \mathcal{P}(n)} \kappa_\alpha, \tag{2}$$

where

$$\kappa_\alpha = \prod_{\substack{C_i \\ \text{parts of } \alpha}} \kappa_{|C_i|}$$

with $|C_i|$ the cardinal of part C_i. Thus each term in (2) may be regarded as associated with a splitting of $[\![n]\!]$ into parts described by the partition α: in statistical mechanics, the terms κ_α are dubbed the *connected parts* of the moment m_n. Since the κ_α depend only on the type of the partition α, we may rewrite (2) as

$$m_n = \sum_{[\alpha] \vdash n} C_{n,[\alpha]} \kappa_{[\alpha]} \tag{3}$$

with $\kappa_{[\alpha]}$ as in (1). For example, $m_4 = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4$.
Thus the indeterminates κ_ℓ have acquired the meaning of cumulants, and $Z(x)$ and $W(x)$ are the GF of moments and cumulants, respectively.

Then, making use of the genus $g(\alpha)$ mentioned above, it is natural to modify the expansion (3) by weighting the various terms according to their genus. Introducing a parameter ϵ, we write

$$m_n(\epsilon) = \sum_{\alpha \in \mathcal{P}(n)} \epsilon^{g(\alpha)} \kappa_\alpha \tag{4}$$

or

$$m_n(\epsilon) = \sum_{[\alpha] \vdash n} \sum_{g=0}^{g_{max}([\alpha])} C_{n,[\alpha]}^{(g)} \epsilon^g \kappa_{[\alpha]},.$$

For example, $m_4(\epsilon) = \kappa_4 + 4\kappa_3\kappa_1 + (2+\epsilon)\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4$, see below.

Obviously for $\epsilon = 1$, we recover the usual expansion (3), whereas for $\epsilon = 0$, we have an expansion on *non-crossing* (or free, or planar) cumulants. Thus (4) provides an interpolation between the usual cumulant expansion and that on non-crossing ones.

1.3   Eliminating or reinserting singletons

In a partition, parts of size 1 are called *singletons*. It is natural and easy to remove them in the counting, or to relate the countings of partitions with or without singletons. Let us denote with a hat the GF of partitions without singletons: $\hat{Z}^{(g)}(x)$, and derive the relation between $\hat{Z}^{(g)}(x)$ and $Z^{(g)}(x)$. This is particularly easy in the language of statistics, where discarding singletons amounts to going to a centered variable: $X = \hat{X} + \mathbb{E}(X) = \hat{X} + m_1 = \hat{X} + \kappa_1$

$$m_n = \mathbb{E}(X^n) = \mathbb{E}((\hat{X} + \kappa_1)^n) = \sum_{r=0}^{n} \binom{n}{r} \hat{m}_{n-r} \kappa_1^r$$

and, since singletons do not affect the genus, see below Section 2.6,

$$C_{n,[\alpha',1^r]}^{(g)} = \binom{n}{r} C_{n-r,[\alpha']}^{(g)}$$

where the partition α' is *singleton free* (s.f.). For example,

$$
\begin{aligned}
m_1 &= \kappa_1 \\
m_2 &= \kappa_2 + \kappa_1^2 \\
m_3 &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 \\
m_4 &= \kappa_4 + (2+\epsilon)\kappa_2^2 + 4\kappa_3\kappa_1 + 6\kappa_2\kappa_1^2 + \kappa_1^4 \\
m_5 &= \kappa_5 + 5\,\kappa_4\,\kappa_1 + 5(1+\epsilon)\kappa_3\kappa_2 + 10\,\kappa_3\,\kappa_1{}^2 + 5(2+\epsilon)\,\kappa_2{}^2\kappa_1 + 10\,\kappa_2\,\kappa_1{}^3 + \kappa_1{}^5\,,
\end{aligned}
$$

etc.

Then

$$
\begin{aligned}
Z^{(g)}(x) &= \sum_{n\ge 0} x^n \sum_{\substack{[\alpha] \\ \alpha \in \mathcal{P}(n)}} C_{n,[\alpha]}^{(g)} \kappa_{[\alpha]} \\
&= \sum_{n\ge 0} x^n \sum_{r=0}^{n} \sum_{\substack{[\alpha'] \\ \alpha' \in \mathcal{P}(n-r),\, \text{s.f.}}} C_{n,[1^r,\alpha']}^{(g)} \kappa_{[\alpha']} \kappa_1^r \\
&= \sum_{n'\ge 0} x^{n'} \sum_{\substack{[\alpha'] \\ \alpha' \in \mathcal{P}(n'),\, \text{s.f.}}} C_{n',[\alpha']}^{(g)} \kappa_{[\alpha']} \sum_{r\ge 0} \binom{n'+r}{r} \kappa_1^r x^r \\
&= \sum_{n'\ge 0} \sum_{\substack{[\alpha'] \\ \alpha' \in \mathcal{P}(n'),\, \text{s.f.}}} C_{n',[\alpha']}^{(g)} \kappa_{[\alpha']} \frac{x^{n'}}{(1-\kappa_1 x)^{n'+1}} \\
&= \frac{1}{1-\kappa_1 x}\, \hat{Z}^{(g)}\Big(\frac{x}{1-\kappa_1 x}\Big)\,,
\end{aligned}
\tag{5}
$$

and conversely

$$\hat{Z}^{(g)}(u) = \frac{1}{1+\kappa_1 u}\, Z^{(g)}\Big(\frac{u}{1+\kappa_1 u}\Big)\,. \tag{6}$$

2. Partitions and their genus

In this section, we recall some standard notions on partitions, show how to associate a graphical representation to a partition and introduce its genus in a natural way.

2.1 Parts of a partition

As explained in sect. 1, we are interested in partitions of the *set* $[\![n]\!]$. Note that when listing the parts of a partition $\alpha = (\{i_1\}, \cdots \{i_{\alpha_1}\}, \{j_1, j_2\}. \cdots)$,
(i) the ordering of elements in each part is immaterial, and we thus choose to write them in increasing order;
(ii) the relative position of parts is immaterial.
   For example, consider the partition $(\{1,3,4,6,7\}, \{2,5,9\}, \{8\}, \{10\})$ of $[\![10]\!]$. It is of type $[1^2, 3, 5]$ with two singletons $\{8\}$ and $\{10\}$. Clearly, the order of elements within each part is irrelevant, *e.g.* parts $\{1,3,4,6,7\}$ and $\{3,4,1,7,6\}$ describe the same subset of $[\![10]\!]$. One may thus order the elements of each part. Likewise the relative order of the parts is immaterial:
$(\{1,3,4,6,7\}, \{2,5,9\}, \{8\}, \{10\})$ and $(\{2,5,9\}, \{8\}, \{1,3,4,6,7\}, \{10\})$ describe the same partition.

2.2 Combinatorial and graphical representations of a partition and its genus

A general partition α of $\mathcal{P}(n)$ may be described in terms of a pair of *permutations* σ and τ, both in $\mathcal{S}_n$: σ is the cyclic permutation $(1, 2, \cdots, n)$; τ belongs to the class $[\alpha]$ of $\mathcal{S}_n$, and its *cycles* are described by the parts of α, thus subject to the condition (i) above: *each cycle is an increasing list of integers*.
The genus g of the partition is then defined by [13]

$$n + 2 - 2g = \#\mathrm{cy}(\tau) + \#\mathrm{cy}(\sigma) + \#\mathrm{cy}(\sigma \circ \tau^{-1})$$

or in the present case,

$$-2g = \sum \alpha_\ell - 1 - n + \#\text{cy}(\sigma \circ \tau^{-1}). \tag{7}$$

since here $\#\text{cy}(\sigma) = 1$ and $\#\text{cy}(\tau) = \sum \alpha_k$. Since $\#\text{cy}(\sigma \circ \tau^{-1}) \geq 1$, we find an upper bound on g

$$g \leq g_{\max} := \left\lfloor \frac{1}{2}\left(n - \sum \alpha_k\right) \right\rfloor,$$

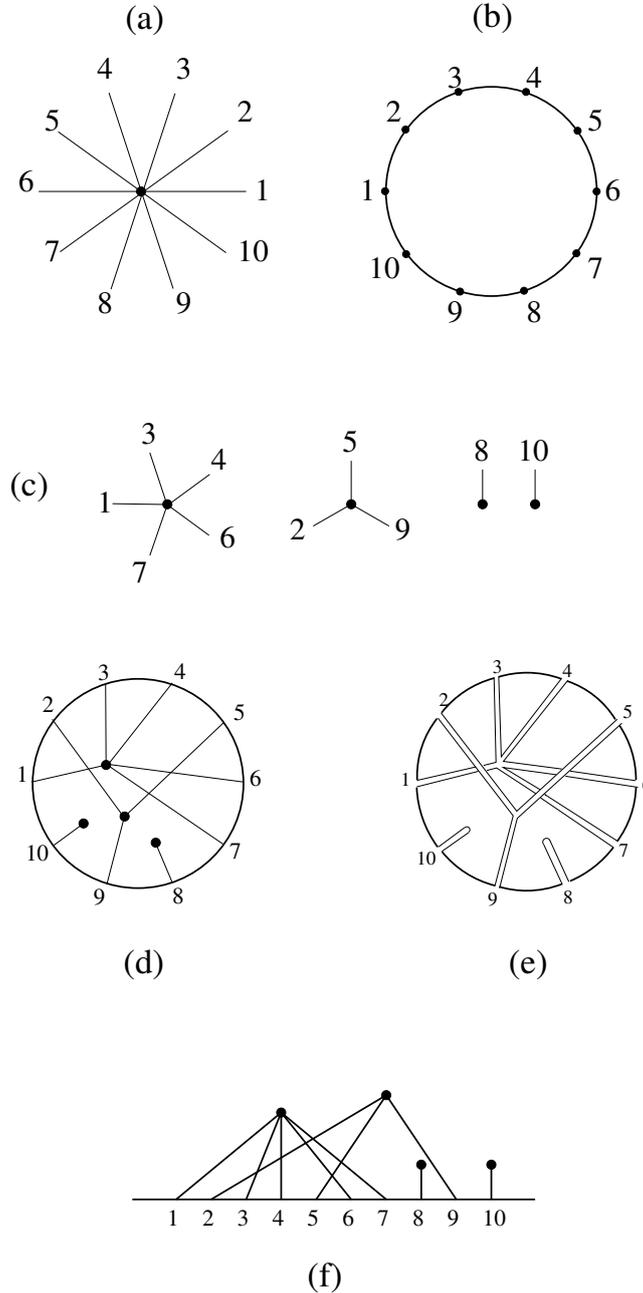see also [23]. We recall below why this definition of the genus is natural.

Figure 1: The partition $(\{1, 3, 4, 6, 7\}, \{2, 5, 9\}, \{8\}, \{10\})$ of $[\![10]\!]$. (a) and (b): two equivalent representations of the special 10-vertex; (c) the four other vertices; (d) a contribution to $C_{10,[1^2\,3\,5]}^{(g)}$; (e) the double line version of (d), with three faces and thus genus $g = 2$; (f) the linear version of (d)

Example 2.1. *For the above partition of* $[\![10]\!]$, $\sigma = (1, 2, \cdots, 10)$, $\tau = (1, 3, 4, 6, 7)(2, 5, 9)(8)(10)$, $\sigma \circ \tau^{-1} = (1, 8, 9, 6, 5, 3, 2, 10)(4)(7)$. *Thus* $2g = 11 - 4 - 3 = 4$, $g = 2$, *while* $g_{\max} = 3$.

To a given partition, we may also attach a map: it has α_ℓ ℓ-valent vertices, in short ℓ-vertices [*], for $\ell = 1, 2, \cdots$, whose edges are numbered clockwise by the elements of the partition, and a special n-valent

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[*]Remember that α_ℓ are the multiplicities introduced in (1)

vertex, with its n edges numbered *anti*-clockwise from 1 to n, see Figure 1a,c. Edges are connected pairwise by matching their indices. Two maps are regarded as topologically equivalent if they encode the same partition. In fact it is topologically equivalent and more handy to attach n points *clockwise* on a circle, and to connect them pairwise by arcs of the circle, see Figure 1b. Now the permutation σ describes the connectivity of the n points on the circle, while τ describes how these points are connected through the other vertices. It is readily seen that the permutation $\sigma \circ \tau^{-1}$ describes the circuits bounding clockwise the faces of the map. This is even more clearly seen if one adopts a double line notation for each edge [11], thus transforming the map into a "fat graph", see Figure 1e . Thus the number of cycles of $\sigma \circ \tau^{-1}$ is the number f of faces of the map. Since each face is homeomorphic to a disk, gluing a disk to each face transforms the map into a closed Riemann surface, to which we may apply Euler's formula

$$2 - 2g = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 1 + \sum_\ell \alpha_\ell - n + f \tag{8}$$

with $f = \#\text{cy}(\sigma \circ \tau^{-1})$, and we have reproduced (7).

Remark 2.1. *This coding of a map, or here of a partition, by a pair of permutations, with a resulting expression of its genus, is an old idea originating in the work of Jacques, Walsh, and Lehman [13, 20, 21] and rediscovered and used with variants by many authors since then [8].*

Remark 2.2. *The diagrammatic representation that we adopt here differs from that of other authors [6, 23]: in fact, it is a dual picture, with our vertices corresponding to the faces of these authors. Our preference for the former is due to its analogy with Feynman diagrams.*

2.3 Glossary

It may be useful to list some elements of the terminology used below. It is convenient to represent a partition of $\mathcal{P}(n)$ by a diagram. It may be a *circular* diagram, with n points equidistributed clockwise, as on Figure 1-d, and it has a genus as explained above. We distinguish the *points* on the circle from the *vertices* which lie inside the disk.

Occasionally we use a *linear* diagram, with n points labelled from 1 to n on a line (or an arc), and vertices above the line.

Note that if we give each point of the circle a weight x and each k-vertex the weight κ_k, the sum of diagrams of genus g builds the GF $Z^{(g)}(x)$.

In a (circular) diagram, we call *2-line* a pair of edges attached to a 2-vertex. In the following, the middle 2-vertex will be omitted on 2-lines, to avoid overloading the figures. A 2-line is then just a straight line between two points of the circle.

In a diagram, we call *adjacent* a pair of edges joining a vertex to adjacent points on the circle. For example, on Figure 2, the edges ending at 1 and 3 are not adjacent, those ending at 3 and 4 are.

In the following discussion, it will be important to focus on a point on the circle, say point 1, and see what it is connected to. We shall refer to it as the *marked point*.

If α is a partition of $\mathcal{P}(n)$ of a given type, all its conjugates by powers of the cyclic permutation σ have the same type. Counting partitions of a given type thus amounts to counting *orbits* of diagrams under the action of σ, while recording the length (cardinality) of each orbit. Diagrammatically, the point 1 being marked, we list orbits under rotations of the inner pattern of vertices and edges by the cyclic group $\mathbb{Z}_n$, and record the length of each orbit. An orbit has a weight equal to its length $n/\mathfrak{s}$, where $\mathfrak{s}$ is the order of the stabilizer of the diagram – a subgroup of the rotation group. For example, the left-most diagram of Figure 8 has $\mathfrak{s} = 2$, the right-most $\mathfrak{s} = 8$, the others have $\mathfrak{s} = 1$.

2.4 The coefficients $C^{(g)}_{n,[\alpha]}$

We now return to our problem of determining the coefficients $C^{(g)}_{n,[\alpha]}$ in (4). From the previous discussion, if we denote $\mathcal{O}_n([\alpha]) \subset \mathcal{S}_n$ the subset of permutations of class $[\alpha]$, whose cycles involve only increasing sequences of integers, we have

$$C^{(g)}_{n,[\alpha]} = \# \left\{ \tau \middle| \tau \in \mathcal{O}_n([\alpha]), \ g = \frac{1}{2}\Big(n + 1 - \sum \alpha_\ell - \#\text{cy}(\sigma \circ \tau)\Big) \right\} . \tag{9}$$

Alternatively, one may use the diagrammatic language to write

$$C^{(g)}_{n,[\alpha]} = \sum_{\text{orbits}} \text{length of orbit} = n \sum_{\text{orbits}} \frac{1}{\mathfrak{s}}, \tag{10}$$

with a sum over orbits of diagrams of type $[\alpha]$ and genus g.

2.5 Remark on matrix integrals

As 't Hooft's double line notation [11] suggests, the coefficient

$$C_{n,[\alpha]}(\epsilon) = \sum_g C_{n,[\alpha]}^{(g)} \epsilon^g$$

could be defined and computed in matrix integrals

(i) as the coefficient of $\prod_\ell \kappa_\ell^{\alpha_\ell}$ in the computation of $\langle \frac{1}{N} : \text{tr}\, M^n : \rangle_{rc}$ in a matrix theory with action $S = -\frac{1}{2} N \text{tr}\, M^2 + N \sum_\ell \kappa_\ell \, \text{tr}\, M^\ell/\ell$; the notation $:\ :$ and the subscript "rc" will be explained shortly;

(ii) as the value of $\langle : \frac{1}{N} \text{tr}\, M^n :: \prod_\ell \frac{(N \text{tr}\, M^\ell/\ell)^{\alpha_\ell}}{\alpha_\ell!} : \rangle_{rc}$ in a Gaussian matrix theory.

In both cases, $\epsilon = \frac{1}{N^2}$, if N is the size of the (Hermitian) matrices; $C_{n,[\alpha]}(N^{-2})$ is given by a sum of Feynman diagrams (in fact, of "fat graphs", or of maps) with $1 + \sum_\ell \alpha_\ell$ vertices, n edges ("propagators") joining the n-vertex $\text{tr}\, M^n$ to the other ℓ-vertices, and f faces associated with each closed index circuit. The double dots $:\ :$ is a standard notation in quantum field theory, where it denotes the *normal* or *Wick product*, that forbids edges from a vertex to itself: here it forces all edges to reach the n-vertex. The crucial point is that we impose a restricted crossing ("rc") condition: the edges connecting each ℓ-vertex to the n-vertex cannot cross one another, thus respecting their original cyclicity and ordering. Only crossings of edges emanating from distinct vertices are allowed.

It is that constraint, a direct consequence of rule 2.1 (i) above, that makes the computation of the coefficients $C_{n,[\alpha]}^{(g)}$ by matrix integrals or group theoretical techniques, and the writing of recursion formulae between them, quite non trivial. For partitions into doublets, however, one deals only with 2-vertices for which the constraint is irrelevant, and $C_{n=2p,[2^p]}^{(g)}$ is computable by these techniques [2, 10, 20].
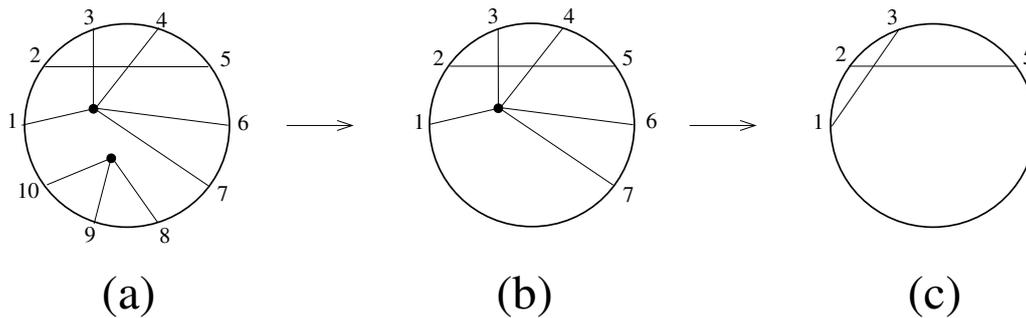

(a)   (b)   (c)

Figure 2: (a) Diagram for the partition of $[\![10]\!]$ into $(\{1,3,4,6,7\},\{2,5\},\{8,9,10\})$, $f = 6$ hence genus $g = 1 - (3+1-10+6)/2 = 1$; (b) after removal of the three adjacent edges coming from the "centipede" $\{8,9,10\}$, here a 3-vertex, now $n' = 7$, $f' = 4$, $g' = 1$; (c) after reduction of two sets of adjacent edges to points 3 and 4, and 6, 7 and 1: now $n'' = 4$, $f'' = 1$, $g'' = 1 + (2 + 1 - 4 + 1)/2 = 1$
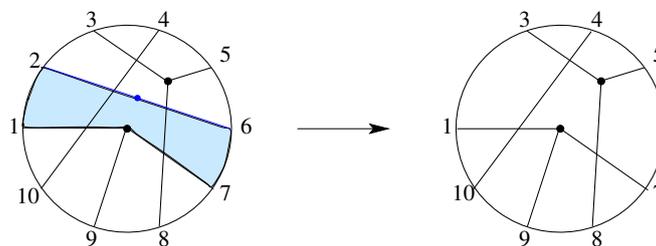

Figure 3: Removing the blue parallel pair of edges and the light blue face does not affect the genus: Variations $\Delta n = -2$, $\Delta f = -1$, $\Delta \sum \alpha_k = -1$, hence $\Delta g = 0$

2.6 Reducing the diagrams

In this subsection, we show that certain modifications of a diagram associated with a partition do not modify its genus. The present discussion follows closely that of Cori and Hetyei [6].

(i) Removing singletons. Removing p singletons changes the number of parts $\sum \alpha_k$ by $-p$, n by $-p$ and the number of faces f is unchanged, hence according to (8) the genus remains unchanged.

(ii) Removing centipedes. **Definition**. A *centipede* is a planar linear subdiagram made of a p-vertex, all the edges of which are attached in a consecutive way to the outer circle, Figure 2. In other words, it corresponds to a part of the partition with consecutive integers (modulo n), $\{j, j+1, \cdots, j+p\}$. Removing it changes the number of parts $\sum \alpha_k$ by -1, n by $-p$ and the number of faces f by $-(p-1)$, see the figure, hence the genus remains unchanged.

(iii) Removing adjacent edges. If two edges emanating from a vertex go to two consecutive points of the circle, (adjacent pair), see Figure 2b-c, removing one of them does not change $\sum \alpha_k$ but changes n and f by -1, hence does not change the genus. One may iterate this operation on the same vertex until one meets a crossing with an edge emanating from another vertex. (If no crossing occurs, this means that the vertex and its edges formed a centipede in the sense of (ii) and may be erased without changing the genus.) To allow an unambiguous reconstruction of all diagrams later in the dressing process, we adopt the following **Convention 1**: in removing such adjacent edges, one keeps the edge attached to the marked point 1, or the first edge encountered clockwise starting from 1, and one removes the others. See Figure 2 for illustration.

(iv) Removing parallel lines. **Definition**. Suppose two pairs of edges join respectively points i and $\tau(i) = j+1$ on the circle, and j and $\tau(j) = i+1$. We call two such pairs *parallel*.
Note that this is equivalent to saying that they form a 2-cycle of the permutation $\sigma \circ \tau^{-1}$. For example, on Figure 3, $\sigma \circ \tau^{-1}$ has the 2-cycle $(2, 7)$. And conversely, any such 2-cycle is associated with two parallel pairs of edges.

    (a) If one of these two vertices is a 2-vertex, one may remove the corresponding pair of edges and the 2-vertex without changing the genus, since $\sum \alpha_\ell$ and f have decreased by 1 and n by 2, see Figure 3 for illustration. If both pairs of edges are attached to 2-vertices, we choose by **Convention 2** to keep the pair attached to the point of the circle of the smallest label. In particular, if one of the pairs is attached to the marked point 1, it is kept and the other removed.

    (b) If both pairs of edges are attached to vertices of valency larger that 2, we keep them both. See Figure 13 below for an example.
After carrying these removals of parallel lines, we are left with *primitive* or *semi-primitive* diagrams (or partitions), following Cori–Hetyei's terminology: in primitive diagrams, no parallel pair is left; therefore, by the remark above, all cycles of $\sigma \circ \tau^{-1}$ have length larger than 2. Semi-primitive diagrams still have parallel pairs of type (b).

Now Cori and Hetyei have proved some fundamental results:

Proposition 2.1. *To an arbitrary diagram corresponds a unique primitive (or semi-primitive) diagram obtained by a sequence of reductions as above, and independent of the order of these reductions.*

Our new observation is that, conversely, any diagram may be recovered by "dressing" a primitive (or semi-primitive) diagram, as we shall see below.
Moreover [6],

Proposition 2.2. *For a given genus, there are only a finite number of primitive diagrams.*

This follows from two inequalities: $f = \#\mathrm{cy}(\sigma \circ \tau^{-1}) \leq \frac{n}{3}$, since in a primitive diagram all cycles of $\sigma \circ \tau^{-1}$ are of length larger or equal to three (see above); and $\sum \alpha_i \leq n/2$ after eliminating the singletons. Hence plugging these inequalities in (7), we get for a primitive diagram

$$n \leq 6(2g-1). \tag{11}$$

As for the semi-primitive diagrams, it was shown in [6] that they are all obtained by a finite number of operations from the primitive ones, hence are themselves in finite number.
For example

Proposition 2.3. *The primitive diagrams of genus 1 are the two diagrams of Figure 4, which have two, resp. three 2-lines. No semi-primitive occurs in genus 1.*

The proof of that statement is given in [6], sect. 8, where the two primitive partitions or diagrams are referred to as β_1 and β_2.

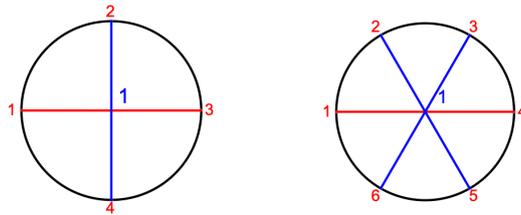Figure 4: The two "primitive" diagrams of genus 1. The blue figure in the middle is the weight of the diagram in (9), namely the length of its orbit

3. From genus 0 to genus 1 . . .

3.1 Non-crossing partitions and the genus 0 generating function

Recall first that in genus 0, the formula given by Kreweras [15] on the census of non-crossing partitions may be conveniently encoded in the following functional relation between the genus 0 GF of moments $Z^{(0)}(x)$ and that of cumulants $W(x)$ defined above [†]

$$Z^{(0)}(x) = 1 + W(xZ^{(0)}(x)). \tag{12}$$

Indeed by application of Lagrange inversion formula, one recovers Kreweras' result

$$C^{(0)}_{n,[\alpha]} = \frac{n!}{(n + 1 - \sum \alpha_k)! \ \prod_k \alpha_k!},$$

as proved in [1].

There is a simple diagrammatical interpretation of the relation (12) due to Cvitanovic [7], see Figure 5, which reads: *in an arbitrary planar (i.e., non-crossing) diagram, the marked point 1 on the exterior circle is necessarily connected to a n-vertex, $n \geq 1$, between the n edges of which lie arbitrary insertions of other (linear) diagrams of $Z^{(0)}$.* Our aim is to extend this kind of relation to higher genus.
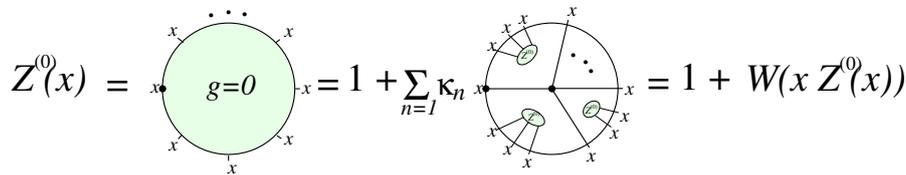

Figure 5: A graphical representation of identity $Z^{(0)}(j) = W(j \, Z^{(0)}(j))$

3.2 Dressing the genus 1 primitive diagrams

We have seen that genus 1 diagrams may be reduced to the two primitive ones of Figure 4. We now write a relation à la Cvitanovic between the generating functions W, $Z^{(0)}$ and $Z^{(1)}$, depicted in Figure 6

$$Z^{(1)}(x) = \sum_{n \geq 2} \kappa_n n x^n (Z^{(0)})^{n-1} Z^{(1)} + \text{sum of dressed diagrams of Figure 4}, \tag{13}$$

which reads: in a generic diagram of genus 1, the marked point 1 is attached **(a)** either to an edge of an n-vertex, between the non-crossing edges of which are inserted one (linear) subdiagram of genus 1 and $(n-1)$ subdiagrams of genus 0 [‡], **(b)** or to an edge of a dressed primitive diagram of genus 1.

Let us concentrate on the case **(b)** and make explicit what is meant by dressing.

The dressing consists in reinserting the elements removed in steps (iv)–(i) of Section 2.6, in that reverse order. First, additional 2-lines are introduced, "parallel" to the two, resp. three 2-lines of the primitive diagrams of Figure 4. Each of these 2-lines carries by definition a 2-vertex. Then to reinsert "adjacent" edges removed in step (iii), each these 2-vertices may be transformed into a k-vertex, whose $k-2$ additional edges may fall,

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[†] Recall this relation is equivalent to the functional identity $X \circ Y = \text{id}$, where $Y(x) := x^{-1} Z^{(0)}(x^{-1})$ and $X(y) := y^{-1}(1+W(y))$, and $R(y) = X(y) - \frac{1}{y} = \frac{1}{y} W(y)$ is the celebrated Voiculescu R function [18, 19].

[‡] Remember that by convention, $Z^{(0)}(x)$ starts with 1, hence these subdiagrams of genus 0 may be trivial
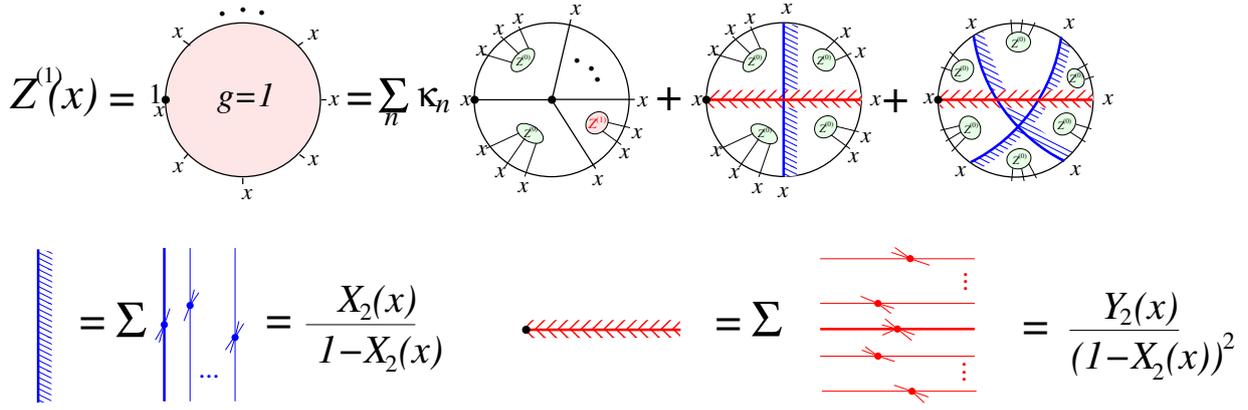
Figure 6: Top: A graphical representation of identity (14). Bottom: a schematic representation of the dressing of the (red) 2-line attached to the marked point 1; or to another (blue) 2-line. In the latter case, according to Convention 1, additional edges may be "emitted" from the central vertex to go to clockwise adjacent points on the circle, and their contribution to the generating function is $X_2(x)$. For the red line, these additional edges may connect to either side of the marked point, and they contribute $Y_2(x)$ to the GF

by Convention 1, on either of the two arcs of the circle adjacent to the endpoints of the 2-line and "clockwise downstream", and without crossing one another: there are $k-1$ partitions of $k-2$ into two parts, one of them possibly empty, hence we attach a weight $X_2(x) := \sum_{k \geq 1}(k-1)\kappa_k x^k$ to each of these parallel lines. Since there is an arbitrary number $r \geq 0$ of parallel lines, they contribute $X_2(x)^r$, and their geometric series sums up to $1/(1 - X_2(x))$. The same applies to the original blue 2-lines of the primitive diagram of Figure 6, which thus gives each a factor $X_2(x)$. The red 2-line, which is the one attached to the marked point 1, has a different weight, as the $k-2$ edges emanating from its k-vertex may fall on either side of the marked point or on the rightmost part of the diagram (see Convention 2 above): this is associated with a partition of the $k-2$ edges into three parts (two of them possibly empty), in number $k(k-1)/2$, which gives the red 2-line a weight $Y_2(x) = \sum_k \frac{k(k-1)}{2}\kappa_k x^k$, while its dressing by parallel lines leads to a factor $1/(1 - X_2(x))^2$, because again, parallel lines above or below the red 2-line are possible. The last step is to reinsert "centipedes" and (possibly) singletons, namely in changing everywhere x into $\tilde{x} = xZ^{(0)}(x)$.

In that way, we have reinstated all features that had been erased in the reduction to primitive diagrams, and constructed the contribution to the GF $Z^{(1)}(x)$ of *all diagrams in which the marked point 1 is attached to an edge that belongs to a dressed primitive diagram.* Indeed in the resulting diagrams, the marked point 1 may be attached to any of the edges, as it should: this is clear whenever that edge is an edge of the primitive diagram; this is also true if the edge is one of the parallel lines added, or one of the added adjacent edges: that was the role of the factors in the definition of X_2 or Y_2 to count these cases. It is thus clear that all possible diagrams of type **(b)** contributing to $Z^{(1)}$ have been obtained by the dressing procedure, and that they are generated once and only once, hence with the right weight. Finally, the cases **(a)** where 1 is not attached to a dressed primitive, but to some genus 0 subdiagram, are accounted for by the first term in (13).

3.3   The genus 1 generating function

Define $\tilde{x} = xZ^{(0)}(x)$ .Gathering all the contributions of Section 3.2 we have

$$Z^{(1)}(x) = \sum_{n \geq 2} \kappa_n n x^n (Z^{(0)}(x))^{n-1} Z^{(1)}(x) + \frac{Y_2(\tilde{x})X_2(\tilde{x})}{(1 - X_2(\tilde{x}))^3} + \frac{Y_2(\tilde{x})X_2^2(\tilde{x})}{(1 - X_2(\tilde{x}))^4} , \tag{14}$$

i.e.,

$$(1 - V(x))Z^{(1)}(x) = \frac{Y_2(\tilde{x})X_2(\tilde{x})}{(1 - X_2(\tilde{x}))^4}$$

where

$$X_2(x) = \sum_{k \geq 2}(k-1)\kappa_k x^k = xW'(x) - W(x), \tag{15}$$

$$Y_2(x) = \sum_{k \geq 2}\frac{k(k-1)}{2}\kappa_k x^k = \frac{1}{2}x^2 W''(x) \tag{16}$$

$$V(x) = \sum_k k\kappa_k x^k Z^{(0)\,k-1} = xW'(\tilde{x}) \,. \tag{17}$$

This is summarized in the following theorem.

Theorem 3.1. *If $\tilde{x} = xZ^{(0)}(x)$, the generating function of genus 1 partitions is given by*

$$Z^{(1)}(x) = \frac{X_2(\tilde{x})Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))^4\,(1 - V(x))} \,. \tag{18}$$

Alternatively, if we introduce

$$\tilde{X}_2(x) := \frac{X_2(\tilde{x})}{(1 - X_2(\tilde{x}))}, \qquad \tilde{Y}_2(x) := \frac{Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))^2} \tag{19}$$

we have the simple expression

$$Z^{(1)}(x) = \frac{\tilde{Y}_2(x)\tilde{X}_2(x)(1 + \tilde{X}_2(x))}{(1 - V(x))} \,.$$

3.4 Examples and applications

3.4.1 $n = 2p$, $[2^p] \vdash n$

If all κ_i vanish but $\kappa_2 = 1$, *i.e.*, if we consider partitions of $n = 2p$ into p doublets, which is the celebrated case considered in [10, 20], we have $W(x) = x^2$, hence

$$Z^{(0)}(x; \kappa_2 = 1, \kappa_{i\neq2} = 0) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2}$$

as the solution of (12). Then following Theorem 1, we find

$$Z^{(1)}(x; \kappa_2 = 1, \kappa_{i\neq2} = 0) = \frac{x^4}{(1 - 4x^2)^{5/2}} \,, \tag{20}$$

in accordance with known results.

3.4.2 $n = 3p$, $[3^p] \vdash n$

In that case, we take $\kappa_3 = 1$, $W(x) = x^3$, hence $Z^{(0)}$ satisfies the third degree equation,

$$(xZ)^3 - Z + 1 = 0 \tag{21}$$

and it is the GF of Fuss–Catalan numbers. We may write it as

$$Z^{(0)}(x; \kappa_3 = 1, \kappa_{i\neq3} = 0) = \frac{2}{\sqrt{3x^3}} \sin\left(\frac{1}{3}\mathrm{Arcsin}\left(\frac{3}{2}\sqrt{3x^3}\right)\right) \,.$$

Then following Theorem 1, one finds, after some algebra,

$$Z^{(1)}(x; \kappa_3 = 1, \kappa_{i\neq3} = 0) = \frac{1152\, x^3 \sin^6\left(\frac{1}{3}\mathrm{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)}{\left(2\cos\left(\frac{1}{3}\mathrm{Arccos}\left(1 - \frac{27x^3}{2}\right)\right) - 1\right)\left(9\sqrt{x^3} - 4\sqrt{3}\sin\left(\frac{1}{3}\mathrm{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)\right)^4} \tag{22}$$

with a Taylor expansion

$$6x^6 + 102x^9 + 1212x^{12} + 12330x^{15} + 114888x^{18} + 1011486x^{21} + 8558712x^{24} + 70324884x^{27} + 564931230x^{30} + \cdots$$

in agreement with direct calculation, see [4]. Note that the closest singularity of $Z^{(1)}$ is at the vanishing point of the discriminant of (21), namely $x^3 = 4/27$:

$$Z^{(1)}(x; \kappa_3 = 1, \kappa_{i\neq3} = 0) \sim \frac{\mathrm{const.}}{(\frac{4}{27} - x^3)^{5/2}} \,,$$

when $x^3 \to 4/27$, with the same exponent $5/2$ as in (20).

3.4.3 Total number of partitions of genus 0 and 1

Let all κ be equal to 1, resp. all κ's but $\kappa_1 = 0$. Then the previous expressions yield the GF of the numbers of partitions of genus 0 or 1, with, resp. without singletons:

$$Z^{(0)}(x; \kappa_i = 1) = \frac{1 - \sqrt{1 - 4x}}{2x} \tag{23}$$

$$\hat{Z}^{(0)}(x) := Z^{(0)}(x; \kappa_1 = 0, \kappa_{i \geq 2} = 1) = \frac{1 - \sqrt{1 - \frac{4x}{1+x}}}{2x} = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1 + x)} \qquad \text{no singleton}$$

$$Z^{(1)}(x; \kappa_i = 1) = \frac{x^4}{(1 - 4x)^{5/2}} \tag{24}$$

$$\hat{Z}^{(1)}(x) := Z^{(1)}(x; \kappa_1 = 0, \kappa_{i \geq 2} = 1) = \frac{x^4}{(1 - 2x - 3x^2)^{5/2}} \qquad \text{no singleton} \tag{25}$$

on which we may verify the relations (5-6) above.

Proof. If all $\kappa_i = 1$, $W(x) = x/(1 - x)$ as a formal series, and $Z^{(0)}(x)$, solution of $Z^{(0)}(x) = W(xZ^{(0)}(x))$ as a formal series, is given by (23), (the GF of the Catalan numbers). Likewise, if $\kappa_1 = 0$, the others equal to 1, $W(x) = x^2/(1 - x)$, etc. For genus 1, we then make use of Theorem 1 to derive (24-25). $\qquad \square$

3.4.4 Number of partitions with a fixed number of parts, in genus 0 and 1

Let all κ be equal to y, then $W(x) = xy/(1 - x)$, and $Z^{(g)}(x, y) = \sum_{n,k} p^{(g)}(n, k) x^n y^k$ is the GF of the numbers $p^{(g)}(n, k)$ of genus g partitions of n with k parts. $Z^{(0)}$ is the solution of (12)

$$Z^{(0)}(x, y) = \frac{1 + x - xy - \sqrt{(1 + x - xy)^2 - 4x}}{2x}.$$

which is the GF of Narayana numbers, and then we compute by (18)

$$Z^{(1)}(x, y) = \frac{x^4 y^2}{((1 + x - xy)^2 - 4x)^{5/2}}$$

which is the expression given by Yip [23], and Cori and Hetyei [5].

If we exclude singletons, $W(x; \kappa_1 = 0) = x^2 y/(1 - x)$, and the GF read now

$$\hat{Z}^{(0)}(x, y) := Z^{(0)}(x, y; \kappa_1 = 0) = \frac{1 + x - \sqrt{(1 - x)^2 - 4x^2 y}}{2x(1 + xy)}$$

$$\hat{Z}^{(1)}(x, y) := Z^{(1)}(x, y; \kappa_1 = 0) = \frac{x^4 y^2}{((1 - x)^2 - 4x^2 y)^{5/2}}.$$

4.   ...to genus 2

4.1   Primitive and semi-primitive diagrams of genus 2

The list of primitive and semi-primitive diagrams of genus 2 is known, thanks to the work of Cori and Hetyei [6]. This has been confirmed independently, in the present work, by generating on the computer all partitions of genus 2 of a given type, and then eliminating all those that involve adjacent or parallel edges. By (11) these primitive diagrams have at most 18 points (*i.e.*, $n \leq 18$), and either up to 9 2-vertices, or one or two 3-vertices, or one 4-vertex. In Table 1, are listed their number for increasing total number of points n. [§]

[§]In Table 1 of [6] there is the unfortunate omission of the 175 primitive diagrams with one 3-vertex (a 3-cycle in their terminology), while those diagrams are properly taken into account in the ensuing formulae. These missing diagrams are listed in Figure 10.

n	2-vertices	one 3-vertex	two 3-vertices	two 3-vertices semi-prim.	one 4-vertex
6	0	0	1	0	0
7	0	14	0	0	0
8	21	0	20	0	6
9	0	141	0	0	0
10	168	0	65	15	15
11	0	407	0	0	0
12	483	0	52	36	9
13	0	455	0	0	0
14	651	0	0	21	0
15	0	175	0	0	0
16	420	0	0	0	0
17	0	0	0	0	0
18	105	0	0	0	0

Table 1. Number of (semi-)primitive diagrams of genus 2

Based on this list of primitive diagrams, we may now write an equation similar to (14)

$$Z^{(2)}(x) = \sum_n n\kappa_n x^n (Z^{(0)}(x))^{n-1} Z^{(2)}(x) \qquad (26)$$
$$+ \quad \text{dressing of (semi}-)\text{primitive diagrams of genus 2}$$

as illustrated in Figure 7.

Remark 4.1. *It might seem natural to also have in the r.h.s. of (26) a term with two insertions of genus 1 subdiagrams. In fact, such diagrams will be included in the set of primitives and their dressings. An example is given by the first diagram of Figure 8.*
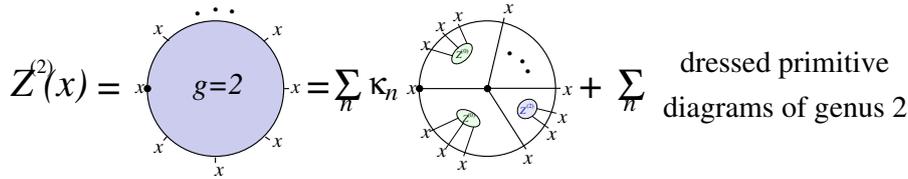

Figure 7: A graphical representation of relation (26)

4.2 Dressing of primitive diagrams of genus 2

The dressing of primitive diagrams with only 2-lines (Column 2 of Table 1) involves the same functions $\tilde{X}_2$ and $\tilde{Y}_2$ defined above in sect. 3.3: $\tilde{Y}_2$ is assigned to the line attached to point 1, while the other lines carry the weight $\tilde{X}_2$. Hence their contribution to the r.h.s. of (26) reads

$$z_2 = \tilde{Y}_2(x)\Big(21\tilde{X}_2^3(x) + 168\tilde{X}_2^4(x) + 483\tilde{X}_2^5(x) + 651\tilde{X}_2^6(x) + 420\tilde{X}_2^7(x) + 105\tilde{X}_2^8(x)\Big)$$

with the notations of (19).

For the dressing of primitive diagrams with 3- or 4-vertices, we must introduce new functions that generalize X_2 and Y_2 defined in (15-16)

$$X_\ell(x) = \sum_{k \geq \ell} \binom{k-1}{\ell-1} \kappa_k x^k \qquad (27)$$

$$Y_\ell(x) = \sum_{k \geq \ell} \binom{k}{\ell} \kappa_k x^k$$

$$\ell > 2 \qquad \tilde{X}_\ell(x) := \frac{X_\ell(\tilde{x})}{(1 - X_2(\tilde{x}))^\ell} \qquad ; \qquad \tilde{Y}_\ell(x) := \frac{Y_\ell(\tilde{x})}{(1 - X_2(\tilde{x}))^\ell} .$$

with, as before, $\tilde{x} = xZ^{(0)}(x)$. (Beware that the power of $(1 - X_2(\tilde{x}))$ in the denominator of $\tilde{X}_\ell$ does not apply to $\ell = 2$, compare with (19).) These functions too may also be expressed in terms of derivatives of W: for example, $Y_3(x) = \frac{1}{6}x^3 W'''(x)$, etc.

Consider first a primitive diagram with one 3-vertex, like those depicted in Figure 9. Remember that all distinct rotated diagrams must be considered and hence, the marked point 1 may be attached to the 3-vertex or to any one of the 2-lines.

(i) In the case where the marked point 1 is attached to one of the 2-lines, its 2-vertex may be changed into a k vertex, $k > 2$ and as in Section 3.2, this yields a weight $Y_2(x)/(1 - X_2(x))^2$, while the lines emanating from the 3-vertex or parallel to it contribute $X_3(x)/(1 - X_2(x))^3$. And again, a final change of x into $\tilde{x}$ completes the dressing.

(ii) In the former case, 1 attached to the 3-vertex, this 3-vertex may be promoted to a k-vertex, $k > 3$, with $k - 3$ lines ending on four different arcs of the circle: there are $\binom{k}{3}$ ways of distributing them, whence a weight $Y_3(x)$. Then adding parallel lines may be done in 3 ways, whence a weight $1/(1 - X_2(x))^3$. The 2-lines, on the other hand, carry a weight $X_2(x)/(1 - X_2(x))$, just like in Section 3.2. Finally, again as in Section 3.2, the variable x has to be substituted for the dressed one $\tilde{x} = xZ^{(0)}$ to take into account all possible insertions of genus 0 subdiagrams.

(iii) There is, however, a case not yet accounted for by the previous dressing. When the marked point 1 is attached to a 2-line parallel to a pair of edges of the 3-vertex, that line has been erased in the reduction process and must be restored. A weight $2Y_2(x)/(1 - X_2(x))$ is attached to that new line, with a factor 2 comes from the two ends of the 2-line, and a single factor $1/(1 - X_2(x))$ as compared with what we saw in Section 3.2, because the counting of parallel lines between the new line and the 3-vertex has already been taken into account in the term $\tilde{X}_3(x)$.

Now each of the previous contributions must be weighted by its number of occurrences when the diagram is rotated. For example, each of the two diagrams of Figure 9 contributes $+ 4\tilde{Y}_2\tilde{X}_2\tilde{X}_3$ (since marked point 1 may be at any of the four end-points of the 2-lines) $+ 3\tilde{Y}_3\tilde{X}_2^2$ (3 ways of attaching point 1 to the 3-vertex) $+3\tilde{X}_3\tilde{X}_2^2(2Y_2(\tilde{x})/(1 - X_2(\tilde{x})))$ (when 1 lies on a line parallel to two edges of the 3-vertex). More generally, for a primitive diagram of an orbit of symmetry order $\mathfrak{s}$, with one 3-vertex and p 2-lines, $n = 3 + 2p$, the weight is

$$\frac{1}{\mathfrak{s}}\left(2p\tilde{Y}_2\tilde{X}_2^{p-1}\tilde{X}_3 + 3\tilde{Y}_3\tilde{X}_2^p + 3\tilde{X}_3\tilde{X}_2^p(2Y_2(\tilde{x})/(1 - X_2(\tilde{x})))\right),$$

where we write $\tilde{X}_\ell$ and $\tilde{Y}_\ell$ in short for $\tilde{X}_\ell(x)$ and $\tilde{Y}_\ell(x)$. Thus the orbits of partitions of $[\![n]\!]$ with a primitive diagram with a single 3-vertex contribute

$$\sum_{\text{orbits}} \frac{1}{\mathfrak{s}}\left((n - 3)\tilde{Y}_2\tilde{X}_2^{\frac{n-5}{2}}\tilde{X}_3 + 3\tilde{X}_2^{\frac{n-3}{2}}\left(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\right)\right).$$

But as we saw in (10), for a given n, $\sum_{\text{orbits}} \frac{1}{\mathfrak{s}} = \frac{N}{n}$, where N is the number listed in Table 1, column 3, row n. In total, the diagrams with a single 3-vertex contribute to the r.h.s. of (26) the amount z_3 listed below in (31).

The dressing of primitive diagrams with two 3-vertices or one 4-vertex (columns 4 and 6 of Table 1) is done along similar lines. Thus for an orbit of primitive diagram with two 3-vertices and p 2-lines, with now $n = 2p+6$, we get

$$\frac{1}{\mathfrak{s}}\left(2p\tilde{Y}_2\tilde{X}_3^2\tilde{X}_2^{p-1} + 6\tilde{X}_2^p\tilde{X}_3\left(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\right)\right) \tag{28}$$

and the total contribution z_{33} of such diagrams is given in (32).

For a primitive diagram with one 4-vertex and p 2-lines, (and $n = 2p + 4$), likewise, we get

$$\frac{1}{\mathfrak{s}}\left(2p\tilde{Y}_2\tilde{X}_4\tilde{X}_2^{p-1} + 4\tilde{X}_2^p\left(\tilde{Y}_4 + \tilde{X}_4\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\right)\right)$$

and the total contribution z_4 is given in (34).

Finally, the dressing of semi-primitive diagrams (see a sample in Figure 13) requires special care to avoid double counting. Consider such a semi-primitive diagram, thus with two 3-vertices and p 2-lines, $n = 2p + 6$. First, when the point 1 is attached to one of the 2-lines or one of the two 3-vertices, we have a contribution like the first two terms in (28), but multiplied by $(1 - X_2(\tilde{x}))$ not to count twice the set of lines between the two parallel lines. Moreover, when the point 1 is attached to an added line parallel to one of the branches of the two 3-vertices, there are 5 locations for that line, whence a contribution $\frac{5}{\mathfrak{s}}\tilde{X}_3^2\tilde{X}_2^p \times 2Y_2(\tilde{x})$, with no further factor $1/(1 - X_2(\tilde{x}))$. In total, a semi-primitive diagram contributes

$$\frac{1}{\mathfrak{s}}\left((1 - X_2(\tilde{x}))\left(2p\tilde{Y}_2\tilde{X}_3^2\tilde{X}_2^{p-1} + 6\tilde{Y}_3\tilde{X}_3\tilde{X}_2^p\right) + 5\tilde{X}_2^p\tilde{X}_3^2(2Y_2(\tilde{x}))\right)$$

and the total from semi-primitive diagrams appears as z_{33s} in (33).

Remark 4.2. *As noticed by Cori and Hetyei [6], the semi-primitive diagrams may be obtained from the primitive ones by "splitting" a vertex of valency larger than 3. For example the three diagrams of Figure 13 may be obtained from those of Figure 14 by splitting their 4-vertex as in Figure 15. One might thus consider only primitive diagrams and include the splitting operation in the dressing procedure. The benefit is that primitive diagrams are easy to characterize: they are such that, in genus 2, the permutation τ has no 1-cycle and $\sigma \circ \tau^{-1}$ no 2-cycle.*

4.3 General case of genus 2

Collecting all the contributions of the previous subsection, we can now make (26) more explicit in the form of

Theorem 4.1. *The generating function of genus 2 partitions is given by*

$$Z^{(2)}(x)(1 - V(x)) = z_2 + z_3 + z_{33} + z_{33s} + z_4 \tag{29}$$

where $V(x)$ has been given in (17) and $z_2, \cdots, z_4$ are the contributions of dressing the (semi-)primitive diagrams listed in Table 1.

$$
\begin{align}
z_2 &= \tilde{Y}_2(21\tilde{X}_2^3 + 168\tilde{X}_2^4 + 483\tilde{X}_2^5 + 651\tilde{X}_2^6 + 420\tilde{X}_2^7 + 105\tilde{X}_2^8)\,; \tag{30} \\
z_3 &= \tilde{X}_3\tilde{Y}_2(8\tilde{X}_2 + 94\tilde{X}_2^2 + 296\tilde{X}_2^3 + 350\tilde{X}_2^4 + 140\tilde{X}_2^5) \tag{31} \\
&\quad + \tilde{X}_2(6\tilde{X}_2 + 47\tilde{X}_2^2 + 111\tilde{X}_2^3 + 105\tilde{X}_2^4 + 35\tilde{X}_2^5)\Big(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big)\,; \\
z_{33} &= \tilde{X}_3^2\tilde{Y}_2(5 + 26\tilde{X}_2 + 26\tilde{X}_2^2) \tag{32} \\
&\quad + \tilde{X}_3(1 + 15\tilde{X}_2 + 39\tilde{X}_2^2 + 26\tilde{X}_2^3)\Big(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big)\,; \\
z_{33s} &= \tilde{Y}_2\tilde{X}_3^2\tilde{X}_2(6 + 18\tilde{X}_2 + 12\tilde{X}_2^2)(1 - X_2(\tilde{x})) \tag{33} \\
&\quad + \tilde{Y}_3\tilde{X}_3\tilde{X}_2^2(9 + 18\tilde{X}_2 + 9\tilde{X}_2^2)(1 - (X_2(\tilde{x})) + \tilde{X}_3^2\tilde{X}_2^2(15 + 30\tilde{X}_2 + 15\tilde{X}_2^2)Y_2(\tilde{x})\,; \\
z_4 &= \tilde{Y}_2\tilde{X}_4(3\tilde{X}_2 + 9\tilde{X}_2^2 + 6\tilde{X}_2^3) + (3\tilde{X}_2^2 + 6\tilde{X}_2^3 + 3\tilde{X}_2^4)\Big(\tilde{Y}_4 + \tilde{X}_4\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big)\,, \tag{34}
\end{align}
$$

and we recall that $\tilde{X}_\ell$ and $\tilde{Y}_\ell$ stand for $\tilde{X}_\ell(x)$ and $\tilde{Y}_\ell(x)$ defined in (27).

The resulting expressions for the numbers $C^{(2)}_{n,[\alpha]}$ have been tested up to $n = 15$ and all $[\alpha]$ against direct enumeration using formulae (9) or (10), and for some higher values of n for a few particular cases.
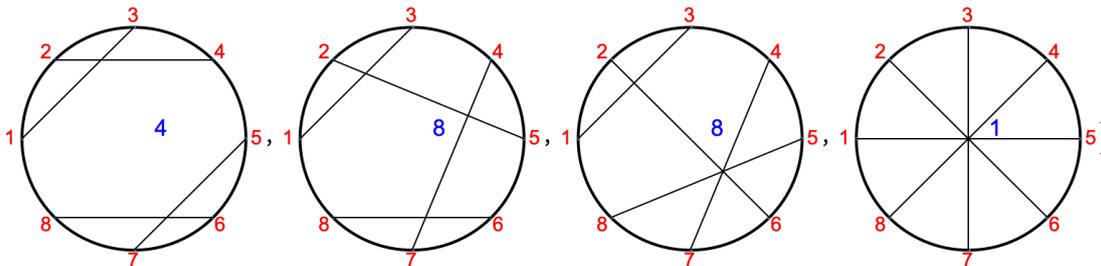

Figure 8:   The primitive diagrams of order 8, type $[2^4]$ and genus 2, with their weight in blue

4.4 Particular cases

4.4.1 Genus 2 partitions of $n = 2p$ into p doublets

In the simplest case where only $\kappa_2 \neq 0$ (and set equal to 1 with no loss of generality), the primitive diagrams are of order $n \leq 18$ – a sample of which is shown in Figure 8[¶]. They involve only 2-lines and their dressing is given by the expression (30) above. Thus

$$
\begin{align*}
Z^{(2)}(x; \kappa_2 = 1, \kappa_{i\neq 2} = 0) &= \\
\frac{\tilde{Y}_2(x)}{(1 - 2x^2Z^{(0)}(x))}\Big(21\tilde{X}_2^3(x) &+ 168\tilde{X}_2^4(x) + 483\tilde{X}_2^5(x) + 651\tilde{X}_2^6(x) + 420\tilde{X}_2^7(x) + 105\tilde{X}_2^8(x)\Big)
\end{align*}
$$

[¶]All genus 2 primitive and semi-primitive diagrams may be found on `https://www.lpthe.jussieu.fr/~zuber/Z_UnpubPart.html`
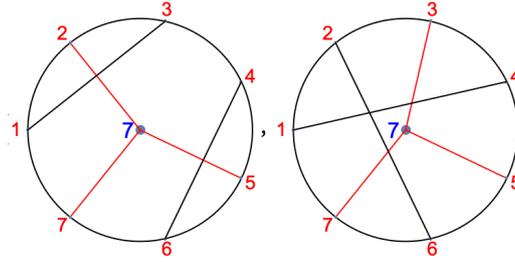
Figure 9: The primitive diagrams of order 7, type $[2^2\,3]$ and genus 2, with the sum of weights equal to 14
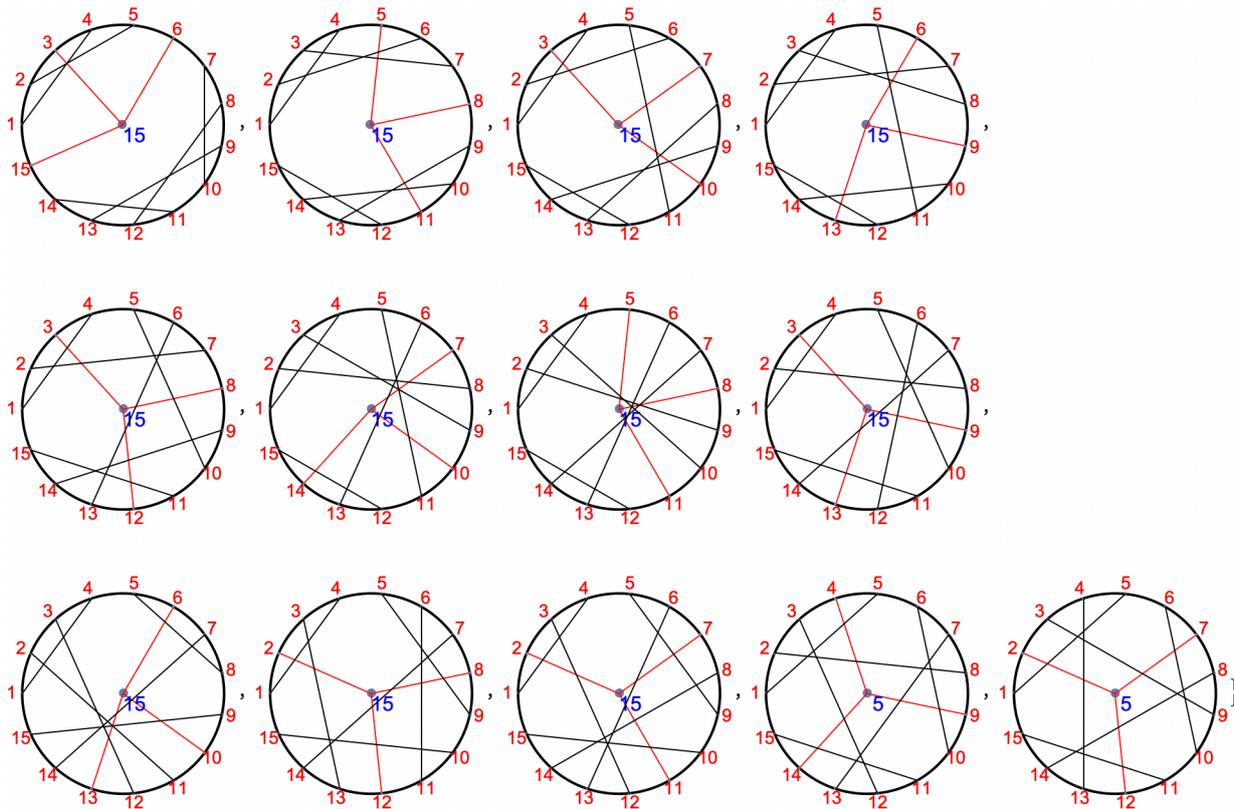

Figure 10: The primitive diagrams of order 15, type $[2^6\,3]$ and genus 2, with the sum of weights equal to 175
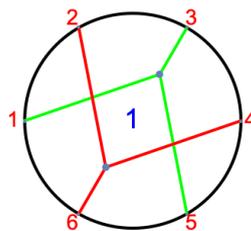

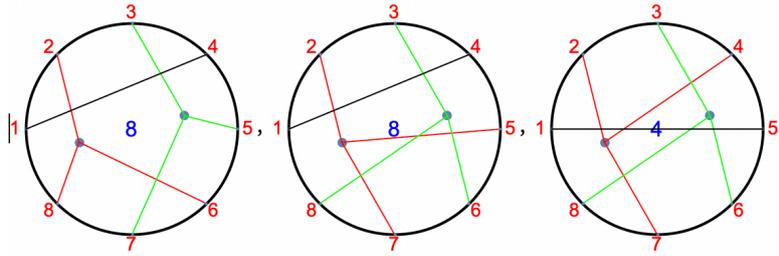Figure 11: The primitive diagram of order 6, type $[3^2]$ and genus 2, of weight 1

Figure 12: The primitive diagrams of order 8, type $[2\,3^2]$ and genus 2, of total weight 20
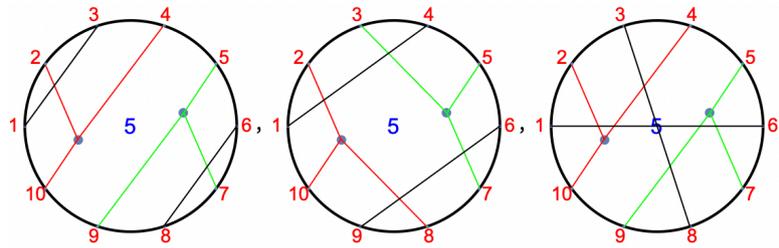

Figure 13: The 3 semi-primitive diagrams of order 10, type $[2^2\,3^2]$, and genus 2, with the sum of weights equal to 15
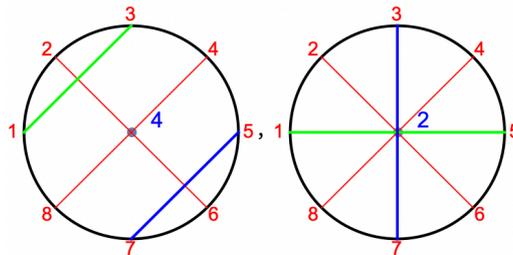

Figure 14: The 2 primitive diagrams of order 8, type $[2^2\,4]$, and genus 2, with the sum of weights equal to 6
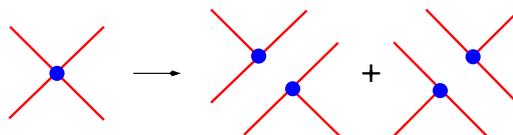

Figure 15: The splitting procedure, by which here a 4-vertex is split into two 3-vertices

with the notations of (19). After some substantial algebra (carried out by Mathematica), one finds

$$Z^{(2)}(x; \kappa_2 = 1, \kappa_{i \neq 2} = 0) = \frac{21x^8(1 + x^2)}{(1 - 4x^2)^{11/2}} \tag{35}$$

in agreement with the results of [10].

4.4.2 Genus 2 partitions of $n = 3p$ into p triplets

We now assume as in Section 3.4.2 that only $\kappa_3 \neq 0$ (and equals 1 with no loss of generality). Let $s := \sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3}x^{3/2}\right)\right)$. Then, following (29), $Z^{(2)}$ takes the fairly cumbersome form

$$Z^{(2)}(x; \kappa_3 = 1; \kappa_{i \neq 3} = 0) =$$
$$\frac{192s^6x^6\left(8s^3\left(128\left(11264s^9 + 8676\sqrt{3}s^6x^{3/2} + 3105s^3x^3\right) + 9315\sqrt{3}x^{9/2}\right) + 729x^6\right)}{\left(2\cos\left(\frac{1}{3}\text{Arccos}\left(1 - \frac{27x^3}{2}\right)\right) - 1\right)\left(9\sqrt{x^3} - 4\sqrt{3}\sin\left(\frac{1}{3}\text{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)\right)^{10}}$$

(compare with the denominator of $Z^{(1)}$ in (22). The first terms of the series expansion read

$$x^6 + 144x^9 + 6046x^{12} + 149674x^{15} + 2771028x^{18} + 42679084x^{21} + \cdots$$

One finds again a singular behaviour of the form

$$Z^{(2)}(x; \kappa_3 = 1; \kappa_{i \neq 3} = 0) \sim \frac{\text{const.}}{\left(\frac{4}{27} - x^3\right)^{11/2}}.$$

4.4.3 Total number of genus 2 partitions

Taking all κ's equal to 1 (and possibly $\kappa_1 = 0$), as in Section 3.4.3, hence $W(x) = x/(1-x)$ or $\widehat{W}(x) = x^2/(1-x)$, we compute by (7) the GF of the total number of genus 2 partitions (with or without singletons), and we recover the result of Cori and Hetyei [6]

$$Z^{(2)}(x; \kappa_i = 1) = \frac{x^6(1 + 6x - 19x^2 + 21x^3)}{(1 - 4x)^{11/2}},$$

and also

$$Z^{(2)}(x; \kappa_1 = 0; \kappa_{i>1} = 1) = \frac{x^6(1 + 10x + 5x^2 + 5x^3 + 9x^4)}{(1 - 2x - 3x^2)^{11/2}}$$

in accordance with (5).

4.4.4 Genus 2 partitions into r parts

The two-variable GF of the number of genus 2 partitions into a given number of parts is obtained as in Section 3.4.4 by setting all $\kappa_i = y$. Theorem 2 leads to

$$Z^{(2)}(x, y) = \frac{x^6y^2\, p(x, y)}{((1 + x - xy)^2 - 4x)^{11/2}} \tag{36}$$
$$p(x, y) = 1 - x(4 - 10y) + x^2(6 - 10y - 15y^2) - x^3(4 + 10y - 39y^2 + 4y^3)$$
$$+ x^4(1 + 10y - 15y^2 - 4y^3 + 8y^4)$$

as first derived by Cori–Hetyei [6]. Similar formulae are obtained if singletons are excluded

$$\widehat{Z}^{(2)}(x, y) = \frac{x^6y^2\, \hat{p}(x, y)}{((1 - x)^2 - 4x^2y)^{11/2}}$$
$$\hat{p}(x, y) = 1 + x(-4 + 14y) + x^2(6 - 22y + 21y^2) + x^3(-4 + 2y + 7y^2) + x^4(1 + 6y - 19y^2 + 21y^3).$$

The counting of genus 2 partitions into r parts is then obtained by identifying the coefficient of y^r in (36). For example, for $r = 2$ (partitions into two parts with or without singleton)

$$Z^{(2)}(x; r = 2) = \widehat{Z}^{(2)}(x; r = 2) = \frac{x^6}{(1 - x)^7} = \sum_{n \geq 6} \binom{n}{6} x^n = \sum_{n \geq 6} x^n \sum_{p=2}^{n-2} \frac{n}{6} \binom{p-1}{2} \binom{n-p-1}{2}$$

in agreement with a general result for $r = 2$ and arbitrary genus [4]. For $r = 3$ (partitions into three parts without singleton)

$$Z^{(2)}(x; r = 3) = \frac{14x^7(1 + 2x)}{(1 - x)^9} = 14 \sum_{n \geq 7} \binom{n}{7} \frac{3n - 13}{8} x^n.$$

5. Conclusion and perspectives

In principle the method could be extended to higher genus, but at the price of an increasing number of (semi)primitive diagrams, whose set remains to be listed, with an Ansatz of the form

$$Z^{(g)}(x) = \frac{\sum \text{dressing of (semi)primitive diagrams of genus } g}{1 - \sum_n n\kappa_n x^n (Z^{(0)}(x))^{n-1}} \,.$$

For instance, in genus 3, primitive diagrams may occur up to $n = 30$ and they start at order $n = 12$. An Ansatz for partitions into doublets (*i.e.*, of type $[2^p]$), for $g = 3$ is thus

$$Z^{(3)}(x; \kappa_2 = 1, \kappa_{i \neq 2} = 0) = \frac{\tilde{Y}_2(x)\tilde{X}_2^5(x)}{(1 - 2x^2 Z^{(0)}(x))} \sum_{j=0}^{9} a_j \tilde{X}_2^j(x)$$

in which the numerical coefficients a_j count the primitives of type $[2^{j+6}]$ and may be determined against the known result of $[10, 20]$

$$Z^{(3)}(x; \kappa_2 = 1, \kappa_{i \neq 2} = 0) = \frac{11x^{12}(135 + 558x^2 + 158x^4)}{(1 - 4x^2)^{17/2}} \,. \tag{37}$$

Hence

$$
\begin{aligned}
Z^{(3)}(x; \kappa_2 = 1, \kappa_{i \neq 2} = 0) &= \frac{11\tilde{Y}_2(x)\tilde{X}_2^5(x)}{(1 - 2x^2 Z^{(0)}(x))} \Big(135 + 2313\tilde{X}_2(x) + 15728\tilde{X}_2^2(x) + 57770\tilde{X}_2^3(x) \\
&\quad + 128985\tilde{X}_2^4(x) + 183955\tilde{X}_2^5(x) + 169078\tilde{X}_2^6(x) + 97188\tilde{X}_2^7(x) + 31850\tilde{X}_2^8(x) + 4550\tilde{X}_2^9(x) \Big) \,.
\end{aligned}
$$

Likewise, in genus 4,

$$
\begin{aligned}
Z^{(4)}(x; \kappa_2 = 1, \kappa_{i \neq 2} = 0) &= \frac{143x^{16}(1575 + 13689x^2 + 18378x^4 + 2339x^6)}{(1 - 4x^2)^{23/2}} \tag{38} \\
&= \frac{143\tilde{Y}_2\tilde{X}_2^7}{(1 - 2x^2 Z^{(0)}(x))} \Big(1575 + 43614\tilde{X}_2 + 497277\tilde{X}_2^2 + 3194702\tilde{X}_2^3 + 13162499\tilde{X}_2^4 \\
&\quad + 37212840\tilde{X}_2^5 + 74956749\tilde{X}_2^6 + 109645557\tilde{X}_2^7 + 117063972\tilde{X}_2^8 + 90449979\tilde{X}_2^9 \\
&\quad + 49312410\tilde{X}_2^{10} + 18008865\tilde{X}_2^{11} + 3956750\tilde{X}_2^{12} + 395675\tilde{X}_2^{13} \Big) \,.
\end{aligned}
$$

We end this paper with a few remarks on some intriguing issues. There is some evidence of a universal singular behaviour of all generating functions,

$$Z^{(g)}(x) \sim (x_0 - x)^{\frac{1}{2} - 3g}$$

as can be seen on the partitions into doublets (20), (35), (37), (38), and for $g = 1, 2$ on other cases. This would imply a large n behaviour of coefficients $C_{n,[\alpha]}^{(g)}$ (for appropriately rescaled patterns α) of the form

$$C_{n,[\alpha]}^{(g)} \sim \text{const } x_0^{-n - 3g + \frac{1}{2}} \, n^{3g - \frac{1}{2}} \qquad \text{as } n, [\alpha] \text{ grow large} \,.$$

This type of singularity of the GF and the associated asymptotic behaviour have been observed in the parallel problem of enumeration of unicellular maps by Chapuy [3], who interpreted the number $6g - 1$ as the number of edges in his dominant "schemes" (the analogues of our primitives). That the same behaviour appears in the present context of partitions indicates that the restriction of maps due to the restricted crossing constraint discussed in Section 2.5 is "irrelevant" (in the sense of critical phenomena), *i.e.*, does not affect the singular behaviour. The "critical exponent" $\frac{1}{2} - 3g$ is also familiar to physicists in the context of boundary loop models and Wilson loops [14]. Such a connection is natural in the case of partitions into doublets, since it is known that in that case, the counting amounts to computing the expectation value of $\operatorname{tr} M^n$ in a Gaussian matrix integral, hence for large n, of a large loop. That the same singular or asymptotic behaviour takes place in (all ?) other cases seems to indicate that an effective Gaussian theory takes place in that limit[‖].

A natural question is whether the Topological Recurrence of Chekhov, Eynard, and Orantin [9] is relevant for the counting of partitions and is related to or independent of the approach of this paper [**].

[‖]I'm grateful to Ivan Kostov for discussions on that point.

[**]In a recent paper `http://arxiv.org/abs/2306.16237`, Hock has been able to recast the results of Theorems 3.1 and 4.1 in a compact form, using the function X of footnote † and its derivatives, as motivated by Topological Recurrence.

As mentioned in the introduction, the formulae derived in this paper yield an interpolation between expansions on ordinary and on free cumulants. What is the relevance of this interpolation? How does it compare with other existing interpolations?

All these questions are left for future investigation.

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