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New Proofs of Interlacing of Zeros of Eulerian Polynomials. III

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ABSTRACT: Many generating functions of combinatorial systems have palindromic coefficients. A notable example is the nth Eulerian polynomial $A_n(x)$. It is known that a palindromic polynomial $f(x)$ of degree $2n$ can be expressed as $x^nQ(x+\frac{1}{x})$ for some polynomial $Q(x)$ of degree n. By exploring the real-rootedness of $Q(x)$, we are able to infer the corresponding property of $f(x)$. By representing $A_n(x)$ in the said form, we give new proof of the real-rootedness and interlacing property of $A_n(x)$. This same approach applied to the nth alternating Eulerian polynomial $\widehat{A}_n(x)$ allows us to infer the interlacing/alternating property of the real and imaginary parts of its non-real zeros. The analogous type B results are also presented.

Keywords: Alternating; Alternating descent; Eulerian polynomial; Interlacing; Real-rootedness
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1. Introduction

Let n be a positive integer. Denote by $A_n(x)$ the nth Eulerian polynomial. Two remarkable properties of $A_n(x)$ are the simple $(-\infty,0)$-rootedness and $A_n(x)$ interlacing $A_{n+1}(x)$. The latter interlacing property is a strengthening of the real-rootedness, which in turn implies the unimodality and log-concavity of coefficients.

Variants of Eulerian polynomials are available in the literature. Chebikin [1] considered $\widehat{A}_n(x)$, the alternating analogue of $A_n(x)$. Researchers studied properties of $\widehat{A}_n(x)$ that parallel those of $A_n(x)$. Ma and Yeh [4] showed that all the zeros of $\tilde{A}_n(x)$ are non-real of moduli 1 with their real and imaginary parts exhibiting certain interlacing/alternating properties.

Although $\widehat{A}_n(x)$'s are not real-rooted, they are closely related to some real-rooted polynomials $R_n^a(x)$ by

$$\widehat{A}_n(x) = (1+x)^{\chi(n \text{ even})} x^{d_n^a} R_n^a(x+\tfrac{1}{x}), \tag{1}$$

where $2d_n^a := n-1-\chi(n \text{ even})$. The key to this connection with $R_n^a(x)$ is the palindromicity of the coefficients of $\widehat{A}_n(x)$. The real-rootedness and interlacing/alternating property of $R_n^a(x)$, however, allow us to deduce the interlacing/alternating property of the real, and imaginary, parts of zeros of $\widehat{A}_n(x)$. See Theorem 4.2.

Analogous representations of the Eulerian polynomials $A_n(x)$ allow us to approach their real-rootedness and interlacing property from the palindromic perspective. The alternating Eulerian polynomial $\widehat{A}_n(x)$, although not being real-rooted, fits the present discussion because it falls within the regime of the equation $x+\frac{1}{x}=\alpha$ having non-real zeros; its Eulerian counterparts fall within the regime of having real zeros.

The same can be said about the type B Eulerian polynomial $B_n(x)$ as well as its alternating analogue $\widehat{B}_n(x)$. The organization of this paper is as follows. In Sections 2–3, we look at representations of $A_n(x)$ and $B_n(x)$ similar to (1) and study properties of the concerned polynomials. In Section 4–5, we do the same to the alternating Eulerian polynomial $\widehat{A}_n(x)$ as well as its type B analogue $\widehat{B}_n(x)$.

2. The type A Eulerian case

We study in this section the $A_n(x)$-analogue of (1).

Let $n \in \mathbb{N}$. When n is even, the palindromicity of $A_n(x)$ implies that $x+1$ is a factor of $A_n(x)$. Thus, we let $A_n(x) = (1+x)^{\chi(n \text{ even})}\widetilde{A}_n(x)$ for some polynomial $\widetilde{A}_n(x)$. The first few members of $\widetilde{A}_n(x)$ are:

$$\widetilde{A}_1(x) = \widetilde{A}_2(x) = 1, \quad \widetilde{A}_3(x) = 1+4x+x^2, \quad \widetilde{A}_4(x) = 1+10x+x^2,$$
$$\widetilde{A}_5(x) = 1+26x+66x^2+26x^3+x^4, \quad \widetilde{A}_6(x) = 1+56x+246x^2+56x^3+x^4.$$

Observe that $2d_n^A := \deg \widetilde{A}_n(x) = n-1-\chi(n \text{ even})$.

Lemma 2.1. *For $n \geqslant 1$, $\widetilde{A}_n(x)$ satisfies the following recurrence relations:*

$$\widetilde{A}_{2n+1}(x) = \big(1+(2+2n)x+(2n-1)x^2\big)\widetilde{A}_{2n}(x) + x(1-x^2)\widetilde{A}'_{2n}(x), \tag{2}$$
$$(1+x)\widetilde{A}_{2n}(x) = \big((2n-1)x+1\big)\widetilde{A}_{2n-1}(x) + x(1-x)\widetilde{A}'_{2n-1}(x). \tag{3}$$

Proof. Substituting $A_{2n+1}(x) = \widetilde{A}_{2n+1}(x)$ and $A_{2n}(x) = (1+x)\widetilde{A}_{2n}(x)$ into the recurrence

$$A_{n+1}(x) = (nx+1)A_n(x) + x(1-x)A'_n(x), \quad n = 1,2,\ldots,$$

and simplifying, (2) follows. The proof of (3), being similar, is omitted. □

Example 2.1. *Consider $\widetilde{A}_7(x)$. We have*

$$\widetilde{A}_7(x) = x^3\big((x+\tfrac{1}{x})^3 + 120(x+\tfrac{1}{x})^2 + 1118(x+\tfrac{1}{x}) + 2176\big)$$

so that $\widetilde{A}_7(x) = x^3 Q_7^A(x+\frac{1}{x})$, where $Q_7^A(x) = x^3+120x^2+1118x+2176$.

Supported by the preceding example, we postulate that

$$\widetilde{A}_n(x) = x^{d_n^A} Q_n^A(x+\frac{1}{x}) \tag{4}$$

for some polynomial $Q_n^A(x)$. The first few members of $Q_n^A(x)$ are:

$$Q_1^A(x) = Q_2^A(x) = 1, \quad Q_3^A(x) = x+4, \quad Q_4^A(x) = x+10, \quad Q_5^A(x) = x^2+26x+64,$$
$$Q_6^A(x) = x^2+56x+244, \quad Q_7^A(x) = x^3+120x^2+1118x+2176.$$

The above list suggests that $Q_n^A(x) \in \mathbb{N}[x]$ is of degree d_n^A.

Proposition 2.1. *For $n \geqslant 1$, the polynomial $Q_n^A(x)$ satisfies the following recurrence relations:*

$$Q_{2n+1}^A(x) = (2+2n+nx)Q_{2n}^A(x) + (4-x^2)(Q_{2n}^A)'(x), \tag{5}$$
$$Q_{2n}^A(x) = nQ_{2n-1}^A(x) + (2-x)(Q_{2n-1}^A)'(x), \tag{6}$$

with initial condition $Q_1^A(x) = 1$.

Proof. Substituting $\widetilde{A}_{2n+1}(x) = x^n Q_{2n+1}^A(x+\frac{1}{x})$ and $\widetilde{A}_{2n}(x) = x^{n-1}Q_{2n}^A(x+\frac{1}{x})$ into (2), canceling x^n from both sides, followed by replacing $x+\frac{1}{x}$ by x, (5) follows. The recurrence (6) follows similarly and whose proof is omitted. □

Proposition 2.2. *For $n \geqslant 1$, we have $Q_n^A(x), \widetilde{A}_n(x) \in \mathbb{N}[x]$.*

Proof. Write $Q_n^A(x) = \sum_{k=0}^{d_n^A} c_{n,k}^A x^k$. By extracting the coefficients of x^k in (5)–(6), we have the following recurrences:

$$c_{2n+1,k}^A = (2+2n)c_{2n,k}^A + (n-k+1)c_{2n,k-1}^A + 4(k+1)c_{2n,k+1}^A,$$
$$c_{2n,k}^A = (n-k)c_{2n-1,k}^A + 2(k+1)c_{2n-1,k+1}^A,$$

whose coefficients are all positive. It then follows by induction that $c_{n,k}^A \in \mathbb{N}$, i.e., $Q_n^A(x) \in \mathbb{N}[x]$. By virtue of (4), $\widetilde{A}_n(x) \in \mathbb{N}[x]$ follows. □

Theorem 2.1. *For $n \geqslant 3$, $Q_n^A(x)$ is simply $(-\infty,-2)$-rooted and $Q_{n+1}^A(x)$ strictly alternates left of $Q_n^A(x)$ or $Q_n^A(x)$ strictly interlaces $Q_{n+1}^A(x)$ depending on whether n is odd or even.*

Proof. Induction on n. It is clear that $Q_4^A(x)$ alternates left of $Q_3^A(x)$. Since $Q_5^A(x)=0 \iff x=-13\pm\sqrt{105}$ and $-13-\sqrt{105}<-10<-13+\sqrt{105}<-2$, it follows that $Q_4^A(x)$ interlaces $Q_5^A(x)$. Thus, cases $n=3,4$ hold. Assume that the result holds for $n\geqslant 3$.

If n is odd, then $d_n^A=d_{n+1}^A=(n-1)/2$. Let $x_{n,1}^A<x_{n,2}^A<\cdots<x_{n,(n-1)/2}^A<-2$ be the zeros of $Q_n^A(x)$. Define also $x_{n,0}^A:=-\infty$ so that $\operatorname{sgn} Q_{n+1}(x_{n,0}^A)=(-1)^{(n-1)/2}$. As

$$\operatorname{sgn} Q_{n+1}^A(x_{n,j}^A)=\operatorname{sgn}\bigl((2-x_{n,j}^A)(Q_n^A)'(x_{n,j}^A)\bigr)=(-1)^{(n-1)/2-j},\quad j=1,2,\ldots,\tfrac{n-1}{2},$$

there exist $x_{n+1,j}^A\in(x_{n,j-1}^A,x_{n,j}^A)$ such that $Q_{n+1}^A(x_{n+1,j}^A)=0$. This proves $Q_{n+1}^A(x)$ alternating left of $Q_n^A(x)$.

If n is even, then $d_n^A=n/2-1$ and $d_{n+1}^A=n/2$. Let $x_{n,1}^A<x_{n,2}^A<\cdots<x_{n,n/2-1}^A<-2$ be the zeros of $Q_n^A(x)$. Define also $x_{n,0}^A:=-\infty$ and $x_{n,n/2}^A:=-2$ so that $\operatorname{sgn} Q_{n+1}^A(x_{n,0})=(-1)^{n/2}$. As

$$\operatorname{sgn} Q_{n+1}^A(x_{n,j}^A)=\operatorname{sgn}\bigl((4-(x_{n,j}^A)^2)(Q_n^A)'(x_{n,j}^A)\bigr)=(-1)^{n/2-j+2},\quad j=1,2,\ldots,\tfrac{n}{2}-1,$$

and $\operatorname{sgn} Q_{n+1}^A(x_{n,n/2}^A)=\operatorname{sgn} 2Q_n^A(x_{n,n/2}^A)=1$. Thus, there exist $x_{n+1,j}^A\in(x_{n,j-1}^A,x_{n,j}^A)$ such that $Q_{n+1}^A(x_{n+1,j}^A)=0$, $j=1,2,\ldots,\frac{n}{2}$. This proves $Q_n^A(x)$ interlacing $Q_{n+1}^A(x)$.

This finishes the induction and the proof of the theorem. □

Since $Q_n^A(x)\in\mathbb{N}[x]$, in principal, its coefficients are amenable to a combinatorial interpretation so that it is natural to ask the next question.

Question 2.1. *For $n\geqslant 1$, what do the coefficients of $Q_n^A(x)$ count?*

Lemma 2.2. *Let $\alpha\in(-\infty,-2)$. Then the equation $x+\frac{1}{x}=\alpha$ has distinct negative real roots.*

Proof. We have $x+\frac{1}{x}=\alpha \iff x^2-\alpha x+1=0$, whose roots are $\frac{\alpha\pm\sqrt{\alpha^2-4}}{2}$. Since $0<\sqrt{\alpha^2-4}<|\alpha|=-\alpha$, $\frac{\alpha\pm\sqrt{\alpha^2-4}}{2}\in(-\infty,0)$ follows. □

Lemma 2.3. *The mapping $\Phi\colon(-\infty,-2)\to(-\infty,-1)$ defined by $\Phi(x)=\frac{x-\sqrt{x^2-4}}{2}$ for all $x\in(-\infty,-2)$ is an order preserving bijection.*

Proof. Let $x\in(-\infty,-2)$. Since $\Phi'(x)=\frac{-x+\sqrt{x^2-4}}{2\sqrt{x^2-4}}>0$, $\lim\limits_{x\to-2^-}\Phi(x)=-1$ and $\lim\limits_{x\to-\infty}\Phi(x)=-\infty$, $\Phi(x)$ being an order preserving bijection follows. □

Since the zeros of $\widetilde{A}_n(x)$ satisfy $x+\frac{1}{x}=\alpha^A$, where $\alpha^A\in(-\infty,-2)$ satisfies $Q_n^A(\alpha^A)=0$, we conclude from Theorem 2.1 and Lemmas 2.2–2.3 that all zeros of $\widetilde{A}_n(x)$ are simple and negative. More specifically, let $\alpha_{n,1}^A<\alpha_{n,2}^A<\cdots<\alpha_{n,d_n^A}^A$ be the zeros of $Q_n^A(\alpha)$. Then $x_{n,j}^A:=\Phi(\alpha_{n,j}^A)$, $j=1,2,\ldots,d_n^A$, are the zeros of $\widetilde{A}_n(x)$ in $(-\infty,-1)$ in ascending order.

That $Q_{n+1}^A(x)$ strictly alternating left of $Q_n^A(x)$ or $Q_n^A(x)$ strictly interlacing $Q_{n+1}^A(x)$ depending on whether n is odd or even then translates to become $\widetilde{A}_{n+1}(x)$ strictly alternates left of $\widetilde{A}_n(x)$ or $\widetilde{A}_n(x)$ strictly interlaces $\widetilde{A}_{n+1}(x)$ in $(-\infty,-1)$ depending on whether n is odd or even.

Since $\Phi(\alpha^A)^{-1}=\frac{2}{\alpha^A-\sqrt{(\alpha^A)^2-4}}=\frac{\alpha^A+\sqrt{(\alpha^A)^2-4}}{2}$ is the other root of the equation $x+\frac{1}{x}=\alpha^A$, it follows that $(x_{n,1}^A)^{-1},\ldots,(x_{n,d_n^A}^A)^{-1}$ are the zeros of $\widetilde{A}_n(x)$ in $(-1,0)$ in descending order. Together with the zero $x=-1$ when n is even, $A_n(x)$ interlacing $A_{n+1}(x)$ follows.

3. The type B Eulerian case

Let $n\in\mathbb{N}$. Denote by $B_n(x)$ the nth type B Eulerian polynomial. When n is odd, the palindromicity of $B_n(x)$ implies that $1+x$ is a factor of $B_n(x)$. So, we write $B_n(x)=(1+x)^{\chi(n\text{ odd})}\widetilde{B}_n(x)$ for some polynomial $\widetilde{B}_n(x)$. The first few members of $\widetilde{B}_n(x)$ are:

$$\begin{aligned}&\widetilde{B}_1(x)=1,\quad \widetilde{B}_2(x)=1+6x+x^2,\quad \widetilde{B}_3(x)=1+22x+x^2,\\&\widetilde{B}_4(x)=1+76x+230x^2+76x^3+x^4,\quad \widetilde{B}_5(x)=1+236x+1446x^2+236x^3+x^4.\end{aligned}$$

It is clear that $2d_n^B:=\deg\widetilde{B}_n(x)=n-\chi(n\text{ odd})$. From the recurrence for $B_n(x)$, namely,

$$B_{n+1}(x)=((2n+1)x+1)B_n(x)+2x(1-x)B_n'(x),\quad n=1,2,\ldots,$$

we have

$$\widetilde{B}_{2n}(x) = \big(1 + (2+4n)x + (-3+4n)x^2\big)\widetilde{B}_{2n-1}(x) + 2x(1-x^2)\widetilde{B}'_{2n-1}(x),$$
$$(1+x)\widetilde{B}_{2n+1}(x) = ((4n+1)x+1)\widetilde{B}_{2n}(x) + 2x(1-x)\widetilde{B}'_{2n}(x).$$

We next postulate that

$$\widetilde{B}_n(x) = x^{d_n^B} Q_n^B(x + \frac{1}{x})$$

for some polynomial $Q_n^B(x)$. The first few members of $Q_n^B(x)$ are as follows:

$$Q_1^B(x) = 1, \quad Q_2^B(x) = x+6, \quad Q_3^B(x) = x+22, \quad Q_4^B(x) = x^2+76x+228,$$
$$Q_5^B(x) = x^2+236x+1444, \quad Q_6^B(x) = x^3+722x^2+10540x+22104,$$

which suggest that $Q_n^B(x) \in \mathbb{N}[x]$ is of degree d_n^B.

Proposition 3.1. *For $n \geqslant 1$, the polynomial $Q_n^B(x)$ satisfies the following recurrence relations:*

$$Q_{2n+1}^B(x) = (2n+1)Q_{2n}^B(x) + 2(2-x)(Q_{2n}^B)'(x), \tag{7}$$
$$Q_{2n}^B(x) = \big(2+4n+(-1+2n)x\big)Q_{2n-1}^B(x) + 2(4-x^2)(Q_{2n-1}^B)'(x). \tag{8}$$

From the recurrences (7)–(8), the coefficients of $Q_n^B(x) = \sum_{k=0}^{d_n^B} c_{n,k}^B x^k$ satisfy

$$c_{2n+1,k}^B = (2n-2k+1)c_{2n,k}^B + 4(k+1)c_{2n,k+1}^B,$$
$$c_{2n,k}^B = (4n+2)c_{2n-1,k}^B + (2n-2k+1)c_{2n-1,k-1}^B + 8(k+1)c_{2n-1,k+1}^B.$$

It then follows by induction that $Q_n^B(x) \in \mathbb{N}[x]$. Hence, $\widetilde{B}_n(x) \in \mathbb{N}[x]$.

Since $Q_n^B(x) \in \mathbb{N}[x]$, it is natural to ask the next question.

Question 3.1. *For $n \geqslant 1$, what do the coefficients of $Q_n^B(x)$ count?*

The proof of the next theorem, being similar to that of Theorem 2.1, is omitted.

Theorem 3.1. *For $n \geqslant 2$, $Q_n^B(x)$ is simply $(-\infty,-2)$-rooted and $Q_{n+1}^B(x)$ strictly alternates left of $Q_n^B(x)$ or $Q_n^B(x)$ strictly interlaces $Q_{n+1}^B(x)$ depending on whether n is even or odd.*

Since the zeros of $\widetilde{B}_n(x)$ satisfy $x + \frac{1}{x} = \alpha^B$, where $\alpha^B \in (-\infty,-2)$ satisfies $Q_n^B(\alpha^B) = 0$, we conclude from Theorem 3.1 and Lemmas 2.2–2.3 that all zeros of $\widetilde{B}_n(x)$ are simple and negative. Let $\alpha_{n,1}^B < \alpha_{n,2}^B < \cdots < \alpha_{n,d_n^B}^B$ be the zeros of $Q_n^B(\alpha^B) = 0$. Then $x_{n,j}^B := \Phi(\alpha_{n,j}^B)$, $j = 1,2,\ldots,d_n^B$, are the zeros of $\widetilde{B}_n(x)$ in $(-\infty,-1)$ in ascending order.

The interlacing/alternating conditions on $Q_n^B(x)$ then translates to become $\widetilde{B}_{n+1}(x)$ strictly alternates left of $\widetilde{B}_n(x)$ or $\widetilde{B}_n(x)$ strictly interlaces $\widetilde{B}_{n+1}(x)$ in $(-\infty,-1)$ depending on whether n is even or odd.

Since $\Phi(\alpha^B)^{-1} = \frac{2}{\alpha^B - \sqrt{(\alpha^B)^2-4}} = \frac{\alpha^B + \sqrt{(\alpha^B)^2-4}}{2}$ is the other root of the equation $x + \frac{1}{x} = \alpha^B$, where Φ is the order preserving bijection in Lemma 2.3, it follows that $(x_{n,1}^B)^{-1}, \ldots, (x_{n,d_n^B}^B)^{-1}$ are the zeros of $\widetilde{B}_n(x)$ in $(-1,0)$ in descending order. Together with the zero $x=-1$ when n is odd, $B_n(x)$ interlacing $B_{n+1}(x)$ follows.

4. The (type A) alternating Eulerian polynomials

In this section, we study the representation (1) of $\widehat{A}_n(x)$.

Let $n \in \mathbb{N}$, $[n-1] := \{1,2,\ldots,n-1\}$ and $\mathfrak{S}_n$ denotes the symmetric group of degree n. Chebikin [1] defined the alternating descent set of $\sigma \in \mathfrak{S}_n$ by

$$\operatorname{Altdes}(\sigma) := (\{2i \colon \sigma(2i) < \sigma(2i+1)\} \cup \{2i-1 \colon \sigma(2i-1) > \sigma(2i)\}) \cap [n-1],$$

the nth alternating Eulerian polynomial by

$$\widehat{A}_n(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\operatorname{altdes}(\sigma)},$$

where $\operatorname{altdes}(\sigma) := \#\operatorname{Altdes}(\sigma)$ is the number of alternating descents of σ, and showed the following recurrence holds:

$$2\widehat{A}_{n+1}(x) = (1+n+2x+(n-1)x^2)\widehat{A}_n(x) + (1-x)(1+x^2)\widehat{A}'_n(x), \quad n = 1, 2, \ldots. \tag{9}$$

The first few members of $\widehat{A}_n(x)$ are:

$$\begin{aligned}
&\widehat{A}_1(x) = 1, \quad \widehat{A}_2(x) = 1+x, \quad \widehat{A}_3(x) = 2+2x+2x^2, \quad \widehat{A}_4(x) = 5+7x+7x^2+5x^3,\\
&\widehat{A}_5(x) = 16+26x+36x^2+26x^3+16x^4, \quad \widehat{A}_6(x) = 61+117x+182x^2+182x^3+117x^4+61x^5.
\end{aligned}$$

The palindromicity of $\widehat{A}_n(x)$ implies that $x+1$ is a factor when n is even. So, we let $\widehat{A}_n(x) = (1+x)^{\chi(n \text{ even})}\tilde{A}_n(x)$ for some polynomial $\tilde{A}_n(x)$. It is known that $\widehat{A}_n(x)$ has unimodal coefficients [2, Theorem 1.1].

Proposition 4.1. *For $n \geqslant 1$, $\tilde{A}_n(x)$ satisfies the following recurrence relations:*

$$2\tilde{A}_{2n+1}(x) = 2\big((1+n)(1+x+x^2)+(n-1)x^3\big)\tilde{A}_{2n}(x) + (1-x^4)\tilde{A}'_{2n}(x), \tag{10}$$

$$2(1+x)\tilde{A}_{2n}(x) = 2(n+x+(n-1)x^2)\tilde{A}_{2n-1}(x) + (1-x)(1+x^2)\tilde{A}'_{2n-1}(x). \tag{11}$$

Proof. The recurrence (10) readily follows by substituting $\widehat{A}_{2n+1}(x) = \tilde{A}_{2n+1}(x)$ and $\widehat{A}_{2n}(x) = (1+x)\tilde{A}_{2n}(x)$ into (9). The proof of (11), being similar, is omitted. $\square$

The first few members of $\tilde{A}_n(x)$ are:

$$\begin{aligned}
&\tilde{A}_1(x) = \tilde{A}_2(x) = 1, \quad \tilde{A}_3(x) = 2+2x+2x^2, \quad \tilde{A}_4(x) = 5+2x+5x^2,\\
&\tilde{A}_5(x) = 16+26x+36x^2+26x^3+16x^4, \quad \tilde{A}_6(x) = 61+56x+126x^2+56x^3+61x^4.
\end{aligned}$$

Observe that $2d_n^a := \deg \tilde{A}_n(x) = n-1-\chi(n \text{ even})$. We postulate that

$$\tilde{A}_n(x) = x^{d_n^a} R_n^a(x+\frac{1}{x})$$

for some polynomial $R_n^a(x)$. The first few members of $R_n^a(x)$ are:

$$\begin{aligned}
&R_1^a(x) = R_2^a(x) = 1, \quad R_3^a(x) = 2x+2, \quad R_4^a(x) = 5x+2,\\
&R_5^a(x) = 16x^2+26x+4, \quad R_6^a(x) = 61x^2+56x+4.
\end{aligned}$$

The above list suggests that $R_n^a(x) \in \mathbb{N}[x]$ is of degree d_n^a.

Proposition 4.2. *For $n \geqslant 1$, the polynomial $R_n^a(x)$ satisfies the following recurrence relations:*

$$2R_{2n+1}^a(x) = \big((n-1)x^2+(2n+2)x+4\big)R_{2n}^a(x) + x(4-x^2)(R_{2n}^a)'(x), \tag{12}$$

$$2R_{2n}^a(x) = \big(2+(n-1)x\big)R_{2n-1}^a(x) + x(2-x)(R_{2n-1}^a)'(x). \tag{13}$$

Proof. Substituting $\tilde{A}_{2n+1}(x) = x^n R_{2n+1}^a(x+\frac{1}{x})$ and $\tilde{A}_{2n}(x) = x^{n-1}R_{2n}^a(x+\frac{1}{x})$ into (10), canceling x^n from both sides, followed by replacing $x+\frac{1}{x}$ by x, (12) follows. The recurrence (13) follows similarly and whose proof is omitted. $\square$

Write $R_n^a(x) = \sum_{k=0}^{d_n^a} c_{n,k}^a x^k$. By equating coefficients of x^k on both sides of (12)–(13), we obtain the following recurrences:

$$\begin{aligned}
2c_{2n+1,k}^a &= (4k+4)c_{2n,k}^a + (2n+2)c_{2n,k-1}^a + (n-k+1)c_{2n,k-2}^a,\\
2c_{2n,k}^a &= (2k+2)c_{2n-1,k}^a + (n-k)c_{2n-1,k-1}^a.
\end{aligned}$$

Unlike those cases in Sections 2–3, the integrality of $c_{n,k}^a$'s does not follow immediately due to the presence of the factor 2 on the left sides of these recurrences. The most natural way to establish $R_n^a(x) \in \mathbb{N}[x]$ is the following:

Problem 4.1. *Show that $R_n^a(x)$ is the generating function of a certain combinatorial system.*

Theorem 4.1. *For $n \geqslant 3$, $R_n^a(x)$ is simply $(-2,0)$-rooted and $R_n^a(x)$ strictly alternates left of, or strictly interlaces, $R_{n+1}^a(x)$, depending on whether n is odd or even.*

Proof. Induction on n. Since $R_3^a(x) = 0 \iff x = -1$ and $R_4^a(x) = 0 \iff x = -\frac{2}{5}$, $R_3^a(x)$ alternating left of $R_4^a(x)$ in $(-2, 0)$ follows. Since $R_5^a(x) = 0 \iff x = \frac{-13\pm\sqrt{105}}{16}$ and $-2 < \frac{-13-\sqrt{105}}{16} < -\frac{2}{5} < \frac{-13+\sqrt{105}}{16} < 0$ holds, $R_4^a(x)$ interlacing $R_5^a(x)$ in $(-2, 0)$ follows. Assume that the result holds for $n \geqslant 3$.

When n is odd, $d_n^a = d_{n+1}^a = (n-1)/2$. Let $-2 < x_{n,1}^a < x_{n,2}^a < \cdots < x_{n,(n-1)/2}^a < 0$ be the zeros of $R_n^a(x)$. Define also $x_{n,(n+1)/2}^a := 0$ so that $\operatorname{sgn} R_{n+1}^a(x_{n,(n+1)/2}^a) = 1$. As

$$\operatorname{sgn} 2R_{n+1}^a(x_{n,j}^a) = \operatorname{sgn}\bigl(x_{n,j}^a(2 - x_{n,j}^a)(R_n^a)'(x_{n,j}^a)\bigr) = (-1)^{(n-1)/2-j+1}, \quad j = 1, 2, \ldots, \tfrac{n-1}{2},$$

there exist $x_{n+1,j}^a \in (x_{n,j}^a, x_{n,j+1}^a)$ such that $R_{n+1}^a(x_{n+1,j}^a) = 0$ for $j = 1, 2, \ldots, \frac{n-1}{2}$, i.e., $R_n^a(x)$ alternates left of $R_{n+1}^a(x)$ in $(-2, 0)$.

When n is even, $d_n^a = (n-2)/2$ and $d_{n+1}^a = n/2$. Let $-2 < x_{n,1}^a < x_{n,2}^a < \cdots < x_{n,(n-2)/2}^a < 0$ be the zeros of $R_n^a(x)$. Define also $x_{n,0}^a := -2$ and $x_{n,n/2}^a := 0$ so that $\operatorname{sgn} R_{n+1}^a(x_{n,0}^a) = (-1)^{n/2}$ and $\operatorname{sgn} R_{n+1}^a(x_{n,n/2}^a) = 1$. As

$$\operatorname{sgn} 2R_{n+1}^a(x_{n,j}) = \operatorname{sgn}\bigl(x_{n,j}^a(4 - (x_{n,j}^a)^2)(R_n^a)'(x_{n,j}^a)\bigr) = (-1)^{n/2-j+1}, \quad j = 1, 2, \ldots, \tfrac{n-2}{2},$$

there exist $x_{n+1,j}^a \in (x_{n,j-1}^a, x_{n,j}^a)$ such that $R_{n+1}^a(x_{n+1,j}^a) = 0$ for $j = 1, 2, \ldots, \frac{n}{2}$, i.e, $R_n^a(x)$ interlaces $R_{n+1}^a(x)$ in $(-2, 0)$.

This finishes the induction and the proof of the theorem. □

Lemma 4.1. *Let $\alpha \in (-2, 0)$. Then all roots of $x + \frac{1}{x} = \alpha$ are non-real and of moduli 1.*

Proof. We have $x + \frac{1}{x} = \alpha \iff x^2 - \alpha x + 1 = 0$, whose roots $\frac{\alpha\pm i\sqrt{4-\alpha^2}}{2}$ have moduli $\frac{\sqrt{\alpha^2+(4-\alpha^2)}}{2} = 1$. □

Since the zeros of $\tilde{A}_n(x)$ satisfy $x + \frac{1}{x} = \alpha^a$, where $\alpha^a \in (-2, 0)$ satisfies $R_n^a(\alpha^a) = 0$, we conclude from Lemma 4.1 that all zeros of $\tilde{A}_n(x)$ are non-real and of moduli 1.

Lemma 4.2. *The mappings $\Phi_1, \Phi_2 \colon (-2, 0) \to (-1, 0)$ defined by $\Phi_1(x) = \frac{x}{2}$ and $\Phi_2(x) = -\frac{\sqrt{4-x^2}}{2}$, for all $x \in (-2, 0)$, are bijections with Φ_1 order preserving and Φ_2 order reversing.*

Proof. The mapping Φ_1 being an order-preserving bijection is obvious. Since $\Phi_2'(x) = \frac{x}{2\sqrt{4-x^2}} < 0$ for all $x \in (-2, 0)$, $\lim\limits_{x\to-2^+} \Phi_2(x) = 0^-$, and $\lim\limits_{x\to0^-} \Phi_2(x) = -1^+$, Φ_2 being an order reversing bijection follows. □

Although $\tilde{A}_n(x)$'s are not real-rooted, $R_n^a(x)$'s are. Let $-2 < \alpha_{n,1}^a < \alpha_{n,2}^a < \cdots < \alpha_{n,d_n^a}^a < 0$ be the zeros of $R_n^a(x)$. For $j \in [d_n^a]$, the zeros of $\tilde{A}_n(x)$ corresponding to $\alpha_{n,j}^a$ are $\frac{\alpha_{n,j}^a}{2} + \pm i\frac{\sqrt{4-(\alpha_{n,j}^a)^2}}{2} = \Phi_1(\alpha_{n,j}^a) \pm i\Phi_2(\alpha_{n,j}^a)$. Define $p_n^a(x) = \prod_{j=1}^{d_n^a/2}(x - \Phi_1(\alpha_{n,j}^a))$ and $q_n^a(x) = \prod_{j=1}^{d_n^a/2}(x - \Phi_2(\alpha_{n,j}^a))$. The following is a restatement of [4, Theorem 4].

Theorem 4.2. *For $n \geqslant 3$, $\tilde{A}_n(x)$ has non-real zeros $\Phi_1(\alpha_{n,j}^a) \pm i\Phi_2(\alpha_{n,j}^a)$ of moduli 1, $j = 1, 2, \ldots, d_n^a$, where $-2 < \alpha_{n,1}^a < \alpha_{n,2}^a < \cdots < \alpha_{n,d_n^a}^a < 0$ are the simple zeros of $R_n^a(x)$. Moreover, $p_n^a(x)$ alternates left of or interlaces $p_{n+1}^a(x)$, and $q_{n+1}^a(x)$ alternates left of $q_n^a(x)$ or $q_n^a(x)$ interlaces $q_{n+1}^a(x)$, depending on whether n is odd or even.*

5. The type B alternating Eulerian polynomials

In this section, we study the type B analogue of (1).

Let $n \in \mathbb{N}$, $[0, n-1] := \{0, 1, \ldots, n-1\}$ and B_n denotes the nth hyperoctahedral group. Ma *et al.* [3] defined the type B alternating descent set of $\sigma \in B_n$ by

$$\operatorname{Altdes}_B(\sigma) := (\{2i \colon \sigma(2i) < \sigma(2i+1)\} \cup \{2i-1 \colon \sigma(2i-1) > \sigma(2i)\}) \cap [0, n-1],$$

where $\sigma(0) := 0$, the nth type B alternating Eulerian polynomial by

$$\widehat{B}_n(x) = \sum_{\sigma\in B_n} x^{\operatorname{altdes}_B(\sigma)},$$

where $\operatorname{altdes}_B(\sigma) := \#\operatorname{Altdes}_B(\sigma)$ is the number of type B alternating descents of σ, and showed the following recurrence holds:

$$\widehat{B}_{n+1}(x) = (1 + n + x + nx^2)\widehat{B}_n(x) + (1 - x)(1 + x^2)\widehat{B}_n'(x), \quad n = 1, 2, \ldots,$$

with the initial conditions $\widehat{B}_1(x) = 1 + x$. The first few members of $\widehat{B}_n(x)$'s are as follows:

$$\widehat{B}_1(x) = 1 + x, \quad \widehat{B}_2(x) = 3 + 2x + 3x^2, \quad \widehat{B}_3(x) = 11 + 13x + 13x^2 + 11x^3,$$
$$\widehat{B}_4(x) = 57 + 76x + 118x^2 + 76x^3 + 57x^4.$$

By virtue of the palindromicity, $x+1$ is a factor of $B_n(x)$ when n is odd. So, we let $\widehat{B}_n(x) = (1+x)^{\chi(n \text{ odd})}\check{B}_n(x)$ for some polynomial $\check{B}_n(x)$. The first few members of $\check{B}_n(x)$ are:

$$\check{B}_1(x) = 1, \quad \check{B}_2(x) = 3 + 2x + 3x^2, \quad \check{B}_3(x) = 11 + 2x + 11x^2,$$
$$\check{B}_4(x) = 57 + 76x + 118x^2 + 76x^3 + 57x^4, \quad \check{B}_5(x) = 361 + 236x + 726x^2 + 236x^3 + 361x^4.$$

Proofs in this section are similar to those in Section 5. We simply state the results, and leave their proofs to the interested readers.

Proposition 5.1. *For $n \geqslant 1$, $\check{B}_n(x)$ satisfies the following recurrence relations:*

$$\begin{aligned}(1+x)\check{B}_{2n+1}(x) &= (1 + 2n + x + 2nx^2)\check{B}_{2n}(x) + (1-x)(1+x^2)\check{B}'_{2n}(x),\\ \check{B}_{2n}(x) &= (1 + 2n + 2nx + (1+2n)x^2 + (-2+2n)x^3)\check{B}_{2n-1}(x) + (1-x^4)\check{B}'_{2n-1}(x).\end{aligned}$$

Observe that $2d_n^b := \deg \check{B}_n(x) = n - \chi(n \text{ odd})$. We postulate that

$$\check{B}_n(x) = x^{d_n^b} R_n^b(x + \frac{1}{x})$$

for some polynomial $R_n^b(x)$. The first few members of $R_n^b(x)$ are:

$$R_1^b(x) = 1, \quad R_2^b(x) = 3x + 2, \quad R_3^b(x) = 11x + 2, \quad R_4^b(x) = 57x^2 + 76x + 4,$$
$$R_5^b(x) = 361x^2 + 236x + 4, \quad R_6^b(x) = 2763x^3 + 5270x^2 + 1444x + 8.$$

The above list suggests that $R_n^b(x) \in \mathbb{N}[x]$ is of degree d_n^b.

Proposition 5.2. *For $n \geqslant 1$, the polynomial $R_n^b(x)$ satisfies the following recurrence relations:*

$$\begin{aligned}R_{2n+1}^b(x) &= (nx+1)R_{2n}^b(x) + x(2-x)(R_{2n}^b)'(x),\\ R_{2n}^b(x) &= \big(2 + (2n+1)x + (n-1)x^2\big)R_{2n-1}^b(x) + x(4-x^2)(R_{2n-1}^b)'(x).\end{aligned}$$

Write $R_n^b(x) = \sum_{k=0}^{d_n} c_{n,k}^b x^k$. From the preceding recurrences, $c_{n,k}^b$'s satisfy

$$\begin{aligned}c_{2n+1,k}^b &= (2k+1)c_{2n,k}^b + (n-k+1)c_{2n,k-1}^b,\\ c_{2n,k}^b &= (4k+2)c_{2n-1,k}^b + (2n+1)c_{2n-1,k-1}^b + (n-k+1)c_{2n-1,k-2}^b.\end{aligned}$$

It then follows by induction that $R_n^b(x) \in \mathbb{N}[x]$. Hence, $\check{B}_n(x) \in \mathbb{N}[x]$.

Question 5.1. *For $n \geqslant 1$, what do the coefficients of $R_n^b(x)$ count?*

Theorem 5.1. *For $n \geqslant 2$, $R_n^b(x)$ is simply $(-2,0)$-rooted and $R_n^b(x)$ strictly alternates left of, or strictly interlaces, $R_{n+1}^b(x)$, depending on whether n is even or odd.*

Since the zeros of $\check{B}_n(x)$ satisfy $x + \frac{1}{x} = \alpha^b$, where $\alpha^b \in (-2,0)$ satisfies $R_n^b(\alpha^b) = 0$, we conclude from Lemma 4.1 that all zeros of $\check{B}_n(x)$ are non-real and of moduli 1.

Let $-2 < \alpha_{n,1}^b < \alpha_{n,2}^b < \cdots < \alpha_{n,d_n^b}^b < 0$ be the zeros of $R_n^b(x)$. For $j \in [d_n^b]$, the zeros of $\check{B}_n(x)$ corresponding to $\alpha_{n,j}^b$ are $\frac{\alpha_{n,j}^b}{2} + \pm i\frac{\sqrt{4-(\alpha_{n,j}^b)^2}}{2} = \Phi_1(\alpha_{n,j}^b) \pm i\Phi_2(\alpha_{n,j}^b)$, where Φ_1, Φ_2 are those bijections in Lemma 4.2. Define $p_n^b(x) = \prod_{j=1}^{d_n^b}(x - \Phi_1(\alpha_{n,j}^b))$ and $q_n^b(x) = \prod_{j=1}^{d_n^b}(x - \Phi_2(\alpha_{n,j}^b))$. The type B analogue of Theorem 4.2 is

Theorem 5.2. *For $n \geqslant 2$, $\check{B}_n(x)$ has non-real zeros $\Phi_1(\alpha_{n,j}^b) \pm i\Phi_2(\alpha_{n,j}^b)$ of moduli 1, $j = 1, 2, \ldots, d_n^b$, where $-2 < \alpha_{n,1}^b < \alpha_{n,2}^b < \cdots < \alpha_{n,d_n^b}^b < 0$ are the simple zeros of $R_n^b(x)$. Moreover, $p_n^b(x)$ alternates left of or interlaces $p_{n+1}^b(x)$, and $q_{n+1}^b(x)$ alternates left of $q_n^b(x)$ or $q_n^b(x)$ interlaces $q_{n+1}^b(x)$, depending on whether n is even or odd.*

Ma and Yeh [4] approached the rootedness of $\widehat{A}_n(x)$ by connection with the derivative polynomials $P_n(x)$:

$$2^n(1+x^2)\widehat{A}_n(x) = (1-x)^{n+1}P_n\left(\frac{1+x}{1-x}\right), \quad n = 1, 2, \ldots,$$

where $P_n(x)$'s are generated by $P_{n+1}(x) = (1+x^2)P_n'(x)$, $n = 0, 1, \ldots$, with $P_0(x) = x$.

Denote by $\{Q_n(x)\}$ the other family of derivative polynomials generated by $Q_{n+1}(x) = xQ_n(x) + (1+x^2)Q_n'(x)$, $n = 1, 2, \ldots$, with $Q_1(x) = x$.

Ma and Yeh [4] conjectured that

$$\widehat{B}_n(x) = (1-x)^n Q_n\left(\frac{1+x}{1-x}\right),$$

which was later given a generating function proof by Ma *et al.* [3]. A combinatorial proof was given recently by Pan [5]. It would be interesting to approach the interlacing/alternating properties of the real and imaginary parts of zeros of $\widehat{B}_n(x)$ based on this connection with $Q_n(x)$.

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