Combinatorics for Certain Skew Tableaux, Dyck Paths, Triangulations, and Dissections

Su Ji Hong[†] and George D. Nasr[‡]

[†]*Department of Mathematics, Yale University*
Email: suji.hong@yale.edu

[‡]*Department of Mathematics, Augustana University*
Email: george.nasr@augie.edu

ABSTRACT: We present combinatorial bijections and identities between certain skew standard Young tableaux, Dyck paths, triangulations, and dissections.

1. Introduction

Bijections between Catalan objects are well understood. For instance, see [11]. There are many generalizations of these objects and the bijections between them. In [10], Stanley provides a bijection between certain standard Young tableaux and dissections of a polygon. In [6, Proposition 2.3], the authors provide a bijection between the same tableaux and certain Dyck paths. Meanwhile, various papers consider a certain collection of skew standard Young tableaux—which may be seen as a generalization of the aforementioned tableaux—which are used to compute formulas for the ordinary and equivariant Kazhdan–Lusztig polynomial for uniform, sparse paving, and paving matroids [2, 3, 7–9].[*]

The primary goal of this paper is to generalize the bijection in [6, Proposition 2.3], so that it involves the skew tableaux mentioned above, while simultaneously including bijections involving certain triangulations. As a result of these bijections, properties of the skew tableaux will have implications for the Dyck paths and triangulation objects of interest. Motivated by our findings, we then find a combinatorial bijection between the dissections in [10] and our triangulations.

In the next section, we will define relevant terminology for skew standard Young tableaux in Subsection 2.1, Dyck paths in Subsection 2.2, and then both dissections and triangulations in Subsection 2.3. Then in Subsection 2.4, we discuss the main results and findings of this paper in detail. In Sections 3 and 4, we provide the definitions for the maps involved in the main results. Finally, in the Appendix, we give some tables enumerating the objects in this paper.

---

[*]Kazhdan-Lusztig polynomials for matroids were first defined in [1].

2. Background and Main Results

2.1 skew standard Young tableaux

Definition 2.1. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be positive integers. We say that $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ is a partition of n if $\lambda_1 + \cdots + \lambda_k = n$. The Young diagram of shape λ is represented by boxes that are left justified so that the ith row has λ_i boxes. A standard Young tableau is achieved by filling the boxes with the numbers so that*

- *each row strictly increases from left to right;*

- *each column increases from top to bottom; and*

- *if there are n boxes, the numbers 1 through n are used exactly once.*

See Figure 1 below for an example of a Young diagram and a standard Young tableau.
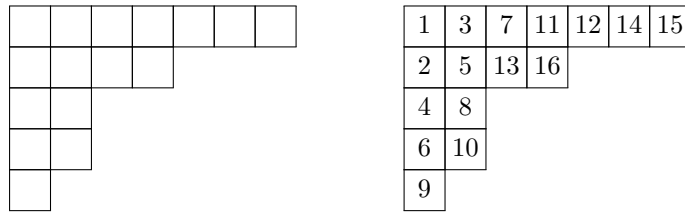
1	3	7	11	12	14	15
2	5	13	16			
4	8					
6	10					
9						

Figure 1: The Young diagram and a standard Young tableau of shape $[7, 4, 2, 2, 1]$.

Definition 2.2. *Given partitions $\mu = [\mu_1, \ldots, \mu_\ell]$ and $\lambda = [\lambda_1, \ldots, \lambda_k]$ so that $\mu_i \leq \lambda_i$ for all i, the skew Young diagram $\lambda \setminus \mu$ is the set of squares from the diagram for λ that are not in the diagram for μ. As before, we define a skew standard Young tableau to be a skew Young diagram filled with numbers following the same rules described for standard Young tableau.*

See Figure 2 for an example of a skew standard Young tableaux.
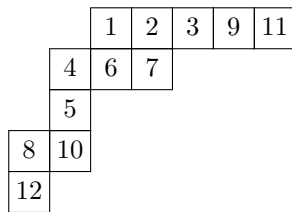
Figure 2: A skew standard Young tableaux of shape $\lambda \setminus \mu$ where $\lambda = [7, 4, 2, 2, 1]$ and $\mu = [2, 1, 1]$.

The authors in [9] introduce the notation $\mathrm{Skyt}(a, i, b)$ to denote the skew standard Young tableaux of shape $[(i+1)^b, 1^{a-2}]/[i^{b-2}]$, where we write x^t to denote $x, x, \ldots, x$, where x is written t times. These are precisely the skew tableaux we discussed in the introduction. The diagram for the tableaux in $\mathrm{Skyt}(a, i, b)$ is shown in Figure 3.

In [9], the authors provide the following enumeration for $\#\,\mathrm{Skyt}(a, i, b)$.

Lemma 2.1. *[9, Lemma 5]*

$$\#\,\mathrm{Skyt}(a, i, b) = \binom{a+i-2}{i}\binom{a+b+2i-2}{b+i-1}\sum_{k=0}^{b-2}\frac{\binom{b+i-k-3}{i-1}}{\binom{a+i+k}{k+1}}$$

This, in turn, gives a formula for the other objects in this paper which we show are in bijection with $\mathrm{Skyt}(a, i, b)$.
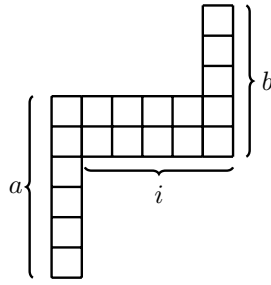
Figure 3: The diagram for $\mathrm{Skyt}(a,i,b)$.

2.2 Dyck Paths

Definition 2.3. *A Dyck path of semi-length n is a string in $\{U,D\}^{2n}$ so that*

1. *the string has the same number of U's and D's (that is, n of each); and*

2. *the number of U's is at least the number of D's in any initial segment of the word.*

We will also often represent such a path visually using $(1,1)$ segments for U and $(-1,1)$ segments for D, as in Figure 4.
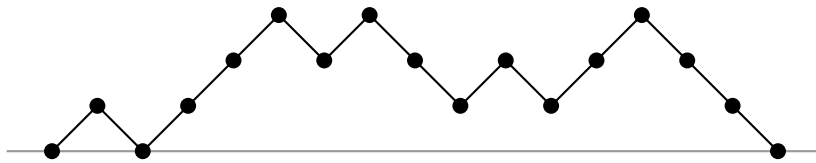

Figure 4: The visual representation of the path corresponding to $UDU^3DUD^2UDU^2D^3$.

Definition 2.4. *A long ascent is a maximal ascent of length at least 2. A singleton is a maximal ascent of length 1. Let $\mathrm{Dyck}(n,\ell,s)$ be the Dyck paths of semi-length n with ℓ long ascents and s singletons so that no singleton appears after the last long ascent.*

Thus, the Dyck path in Figure 4 is an element of $\mathrm{Dyck}(8,2,3)$.

2.3 Dissections and Triangulations

Throughout this section, we assume polygons with n vertices have their vertices labeled 1 through n in counterclockwise order.

Definition 2.5. *A dissection of a polygon P is a way of adding chords between non-adjacent vertices so that no two chords intersect in the interior of the polygon. Throughout, we let $\mathrm{Dis}(n,i)$ be the set of all dissections of an n-gon with i chords. Note that i in $\mathrm{Dis}(n,i)$ is at most $n-3$. The elements of $\mathrm{Dis}(n,n-3)$ are the triangulations of an n-gon.*

Given a vertex x in a triangulated polygon, a *fan at x* is a maximal collection of triangles all containing x. In this case, we call x the *origin* of the fan. A *singular* fan is a fan with only one triangle. Let e be a boundary edge of a fan F at x.

We are interested in being able to uniquely partition a triangulation into a collection of fans. This leads to the following definition.

Definition 2.6. *Let T be a triangulation. A fan decomposition is the pair of sequences $(\mathcal{F}(T),\delta(T))$, where $\mathcal{F}(T)$ and $\delta(T)$ are defined as follows:*

- *We let $\mathcal{F}(T)$ be a sequence of fans defined recursively as follows. Let F_1 be the fan at the vertex with the smallest label. Delete this vertex and all edges incident with it in T to obtain a sequence of triangulations $T_1, \ldots, T_k$, arranged in counter clock-wise order so that $T_i \cap T_{i+1}$ consists of a single vertex. If T is just an edge, then $\mathcal{F}(T)$ is the empty sequence, and otherwise $\mathcal{F}(T) := (F_1, F_2, \ldots, \mathcal{F}_k)$ where $F_i = F(T_{i-1})$.*

- *Let x_j be the label of the origin of F_j. We let $\delta(T) := (d_1, \ldots, d_{k-1})$, where $d_i := x_{i+1} - x_i$ and k is the number of fans in $\mathcal{F}(T)$.*

- *Given a fan F_i, let the size of F_i be the number of triangles in F_i.*

Remark 2.1. *Let $T = ((F_1, F_2, ..., F_\ell), (d_1, d_2, ..., d_\ell))$ be a triangulation. For every $k \leq \ell$, there is a relationship between $\sum_{j=1}^{k} d_i$ and*

$$\sum_{j=1}^{k} \text{number of triangles in } F_j$$

that will be useful later. One can think of d_i as the number of edges between the origins of F_i and F_{i+1} when traveling along the boundary of T counter-clockwise. Note that $\sum_{j=1}^{k} d_j$ is the distance between the origins of F_1 and F_k i.e., the distance between x_1 and x_k. Now, let T_k be the triangulation obtained by $((F_1, F_2, \ldots, F_k), (d_1, \ldots, d_{k-1}))$, that is, T_k is the part of T built from the first k fans in T. Let n_k be the number of triangles in T_k, i.e, T_k is a triangulation of $n_k + 2$-gon. Note that if F_{k+1} is attached to the vertex $n_k + 2$ in order to get T_{k+1}, then that fan is attached to the edge $(1, n_k + 2)$ and triangle from F_{k+1} with vertices 1, $n_k + 2$, and $n_{k+1} + 2$ will become part of F_1. Thus x_k cannot be the vertex labeled with $n_k + 2$. Thus, since $x_1 = 1$ and $x_k \leq n_k + 1$,

$$\sum_{j=1}^{k} d_i = x_k - x_1 \leq n_k = \sum_{j=1}^{k} \text{number of triangles in } F_j.$$

Example 2.1. *Consider the triangulation T in Figure 5. Observe that $\mathcal{F}(T) = (F_1, F_2, F_3, F_4, F_5)$ where F_1 is the size 1 fan at vertex 1, F_2 is the size 3 fan at vertex 2, F_3 is the size 1 fan at vertex 4, F_4 is the size 1 fan at vertex 5, and F_5 is the size 4 fan at vertex 7. Thus, $\delta(T) = (1, 2, 1, 2)$. Figure 5 shows the five fans, distinguishing them by thick boundary edges and different shades of orange in their interior. The white vertices correspond to the origins of the fans.*
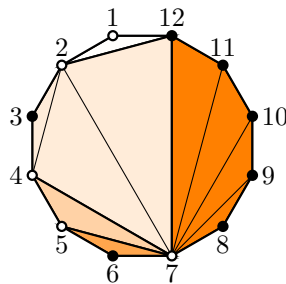
Figure 5: A triangulation and its partition into fans.

Remark 2.2. *Observe that a fan decomposition uniquely determines T. That is, knowing the order and size of each fan along with the distance between the origins of consecutive fans uniquely determines a triangulation.*

Let $\mathrm{Tri}(n, t, s)$ be the triangulations T of an n-gon so that $\mathcal{F}(T)$ has $s + t$ fans so that precisely s are singular and so that the last fan is not singular. Thus, the triangulation in Figure 5 is an element of $\mathrm{Tri}(12, 2, 3)$.

2.4 Main Results

We now may state the main results of this paper. First, let us state [6, Proposition 2.3], the result which we plan to generalize. We state the result by referencing the object $\mathrm{Skyt}(a, i, b)$ we defined above.

Proposition 2.1 ([6])**.** *The tableaux in* $\mathrm{Skyt}(a, i, 2)$ *are in bijection with Dyck paths of length* $2(a + 2i)$ *with* $i + 1$ *peaks and no singletons.*

In Section 3, we will provide explicit combinatorial maps which give us the following Theorem.

Theorem 2.1. *The following objects are in bijection:*

1. $\mathrm{Skyt}(a, i, b)$;

2. $\mathrm{Dyck}(a + b + 2i - 2, i + 1, b - 2)$; *and*

3. $\mathrm{Tri}(a + b + 2i, i + 1, b - 2)$.

Proof. The maps between the three objects are defined in section 3. For convenience, we identify maps according to where the map is from and to by using $\mathtt{S}$ for skew standard Young tableaux, $\mathtt{T}$ for Triangulations, and $\mathtt{D}$ for Dyck paths. For instance, $\mathtt{ST}$ represents the map from skew standard Young tableaux to triangulations, and $\mathtt{TD}$ represents a map from triangulations to Dyck paths. The following pairs of maps are mutual inverses:

- maps $\mathtt{SD}$ and $\mathtt{DS}$;

- maps $\mathtt{ST}$ and $\mathtt{TS}$; and

- maps $\mathtt{TD}$ and $\mathtt{DT}$.

$\square$

Note that this generalizes the result stated in Proposition 2.1, in addition to adding a triangulation interpretation.

With the original motivation for this paper in mind, we specialize Theorem 2.1 to $b = 2$. After incorporating the work of [10] which provides a combinatorial bijection between $\mathrm{Dis}(n + 2, i)$ and $\mathrm{Skyt}(n - i + 1, i, 2)$, we have the following.[†]

Corollary 2.1. *The following objects are in bijection.*

1. $\mathrm{Dis}(n + 2, i)$.

2. $\mathrm{Skyt}(n - i + 1, i, 2)$.

3. $\mathrm{Dyck}(n + i + 1, i + 1, 0)$.

4. $\mathrm{Tri}(n + i + 3, i + 1, 0)$.

By specializing the maps involved in Theorem 2.1, we already have combinatorial bijections between the standard Young tableaux, Dyck paths, and triangulations in this theorem. Note that the bijection between $\mathrm{Skyt}(n - i + 1, i, 2)$ and $\mathrm{Dyck}(n + i + 1, i + 1, 0)$ is precisely the proof of Proposition 2.1.

This leaves two pairs of objects with missing combinatorial bijections. In section 4 we demonstrate a bijection between the dissections and triangulations given in this corollary. Using our bijection between Dyck paths and Triangulations, one can extend our work in section 4 to give a bijection between the dissections and Dyck paths in Corollary 2.1, but we omit this interpretation from this paper.

After some rewriting, with Lemma 2.1 in mind, we have the following.

Corollary 2.2.
$$\# \mathrm{Dyck}(n, \ell, s) = \binom{s + 3\ell - n - 1}{\ell - 1} \binom{n}{s + \ell} \sum_{k=0}^{s} \frac{\binom{s + \ell - k - 2}{\ell - 2}}{\binom{s + 3\ell - n + k + 1}{k + 1}}$$

Corollary 2.3.
$$\# \mathrm{Tri}(n, t, s) = \binom{s + 3t - n - 3}{t - 1} \binom{n}{s + t} \sum_{k=0}^{s} \frac{\binom{s + t - k - 2}{t - 2}}{\binom{s + 3t - n + k - 1}{k + 1}}$$

[†]It is worth noting that this connection between $\mathrm{Skyt}(a, i, b)$ and dissections of polygons has resurfaced recently in the work of Kazhdan-Lusztig polynomials for Matroids [1]. Compare the comments in [5, Remark 5.3] with the representation theoretic result [4, Theorem 3.1] after setting $m = 1$ and considering dimensions.

Using these formulas, one can attain enumeration for each of these objects. We provide tables for some cases in the Appendix.

For our final result, we recall the following lemma in terms of Dyck paths and triangulations.

Lemma 2.2. *[8, Lemma 5] Let a, i, and b be nonnegative integers. Then*

$$\# \operatorname{Skyt}(a, i, b) = \# \operatorname{Skyt}(b, i, a).$$

One may apply Theorem 2.1 to this Lemma in order to get the following.

Corollary 2.4.

1. *Let n, ℓ, s be nonnegative integers. Then*

$$\# \operatorname{Dyck}(n, \ell, s) = \# \operatorname{Dyck}(n, \ell, n - s - 2\ell).$$

2. *Let n, t, s be nonnegative integers. Then*

$$\# \operatorname{Tri}(n, t, s) = \# \operatorname{Tri}(n, t, n - t - 2\ell - 2).$$

Although these equalities are naturally obtained with Theorem 2.1 and Lemma 2.2, there is no known direct combinatorial bijection describing these equalities. Hence we pose the following.

Problem 2.1. *Find a direct combinatorial proof of Corollary 2.4 which does not rely on using the skew tableaux or bijections given in this paper.*

3. Combinatorial Bijections

The following subsections describe maps going between any two of the objects given in Theorem 2.1. Recall that, we identify maps according to where they map from and to by using S for skew standard Young tableaux, T for Triangulations, and D for Dyck paths. Examples are used to alleviate any ambiguity with our maps.

Before proceeding, however, we will point out a handy reinterpretation of the tableaux in $\operatorname{Skyt}(a, i, b)$. Let $\lambda \in \operatorname{Skyt}(a, i, b)$. Let $X = \{x_1, x_2, \ldots, x_{i+b-1}\}$ be the set of values in the top $b - 1$ rows so that $x_1 < x_2 < \cdots < x_{i+b-1}$. If x_j is in row $b - 1$, define y_j to be the entry in the tableau directly below x_j. Then for $1 \leq j < i + b - 1$ let

$$A_j := \begin{cases} \{x_j\} & \text{if } x_j \text{ is in the first } b - 2 \text{ rows;} \\ \{x_j\} \cup ([y_j, y_k - 1] \setminus X) & \text{if } x_j \text{ is in row } b - 1 \text{ and } y_k \text{ is to the right of } y_j, \end{cases}$$

where $[y_j, y_k - 1] = \{y_j, y_j + 1, y_j + 2, \ldots, y_k - 1\}$. Let $A_{i+b-1} := \{x_j\} \cup ([x_j + 1, a + b + 2i - 2] \setminus X)$. Note that x_j is always the minimum of A_j. When $|A_j| > 1$, note the elements of A_j are precisely the entries in row $b - 1$ and b in column j along with all entries of column 1 which are between y_j and y_k. See Figure 6.

The sequence $(A_1, \ldots, A_{i+b-1})$ has enough information to reconstruct λ. Starting with $j = 1$, do the following.

1. If $A_j = \{x\}$, then place x in the highest possible position in the last column.

2. If $|A_j| > 1$, then let $x_j = \min A_j$ and $y_j = \min(A_j \setminus \{x_j\})$. Place x_j in row $b - 1$ column j and place y_j in row b column ℓ, where j is the ℓ-th number such that $|A_j| > 1$. Place all remaining entries from A_j—in increasing order—at the top most available position(s) in the first column.

3. Increase the value of j by 1. If $j < i + b - 1$, repeat these steps. Otherwise, λ is filled and we are done.

Let x_j and y_j be defined in step (2) of the preceding procedure. Pick an integer j' minimally so that $j < j'$ and $|A_{j'}| > 1$. Note that $(A_1, \ldots, A_{i+b-1})$ is an ordered partition of $[a + b + 2i - 2]$ so that $x_j < x_{j+1}$ and $y_j < y_{j'}$, whenever y_j and $y_{j'}$ exist. These conditions guarantee that the rows of λ increase left-to-right. In fact, these give rise to a particular type of partition defined in [6], which we restate here in our own notation.
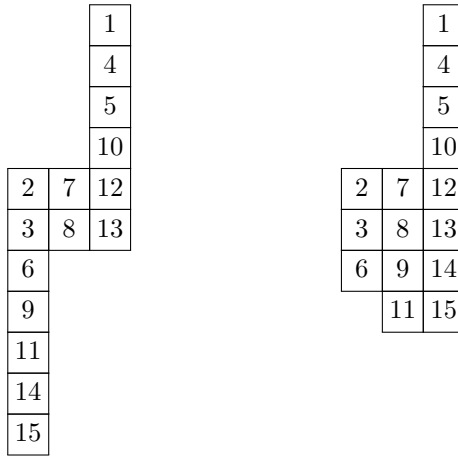
Figure 6: We "push" entries below row b as far right as possible while maintaining the property that columns increase from top to bottom. We have $A_1 = \{1\}$, $A_2 = \{2,3,6\}$, $A_3 = \{4\}$, $A_4 = \{5\}$, $A_5 = \{7,8,9,11\}$, $A_6 = \{10\}$, $A_7 = \{12,13,14,15\}$.

Definition 3.1. *Let A_1, A_2, ..., A_k be a partition of $[n]$ so that $x_1 < x_2 < \cdots < x_k$, where $x_i = \min A_i$. We say this partition is nomincreasing if the $\max(A_i) < \min(A_{i+1} \setminus \{x_{i+1}\})$, provided both $|A_i|$ and $|A_{i+1}|$ are larger than 1.*

We leave it to the reader to verify our sequence $(A_1, A_2, \ldots, A_{i+b-1})$ is indeed nomincreasing. In light of that, we define the following.

Definition 3.2. *Let $\lambda \in \mathrm{Skyt}(a,i,b)$ and define $A_1, \ldots, A_{i+b-1}$ for λ as above. We define $\mathrm{Nom}(\lambda)$ to be the nomincreasing partition $\mathrm{Nom}(\lambda) := (A_1, A_2, \ldots, A_{i+b-1})$.*

Remark 3.1. *One thing that will be useful to note for future reference is that for $\mathrm{Nom}(\lambda) = (A_1, \ldots, A_{i+b-1})$, we always have $|A_{i+b-1}| > 1$. This is preceisely because with the aforementioned choice of $x_1 < \cdots < x_{i+b-1}$, we always have that x_{i+b-1} is the entry in row $b-1$ column $i+1$ of λ. In particular, this means given a nomincreasing sequence $(A_1, \ldots, A_{i+b-1})$, even if $i+1$ of the A_j satisfy $|A_j| > 1$ and the remaining satisfy $|A_j| = 1$, there does not necessarily exists a $\lambda \in \mathrm{Skyt}(a,i,b)$ so that $\mathrm{Nom}(\lambda) = (A_1, \ldots, A_{i+b-1})$. Additionally, we need to have $|A_{i+b-1}| > 1$. Given this, though, such a λ must exist.*

3.1 Map SD

In this section, we define the map SD from $\mathrm{Skyt}(a,i,b)$ to $\mathrm{Dyck}(a+b+2i-2, i+1, b-2)$. For simplicity, let $n := a+b+2i-2$. Note that given $\lambda \in \mathrm{Skyt}(a,i,b)$, n is the number of entries in λ and t is the number of entries in the first $b-1$ rows of λ.

Definition 3.3. *Let $\lambda \in \mathrm{Skyt}(a,i,b)$. We define $\mathrm{SD}(\lambda)$, a certain lattice path, as follows.*

Let $\mathrm{Nom}(\lambda) = (A_1, \ldots, A_{i+b-1})$. Let x_j denote the minimum of A_j. Let $a_j := |A_j|$. Then let $\mathrm{SD}(\lambda)$ be the lattice path given by the following string in $\{\mathsf{U}, \mathsf{D}\}^{2n}$:

$$\mathsf{U}^{a_1} \mathsf{D}^{x_2 - x_1} \mathsf{U}^{a_2} \mathsf{D}^{x_3 - x_2} \cdots \mathsf{U}^{a_{t-1}} \mathsf{D}^{x_t - x_{t-1}} \mathsf{U}^{a_t} \mathsf{D}^{n - x_t + x_1}. \tag{1}$$

Lemma 3.1. *Given λ be a tableau in $\mathrm{Skyt}(a,i,b)$, $\mathrm{SD}(\lambda)$ is a Dyck path. In particular, the lattice path $\mathrm{SD}(\lambda)$ is an element of $\mathrm{Dyck}(n, i+1, b-2)$.*

Proof. Recall that given $\lambda \in \mathrm{Skyt}(a,i,b)$, we can have $\mathrm{Nom}(\lambda) = (A_1, \ldots, A_{i+b-1})$ where $|A_{i+b-1}| > 1$. Recall that $x_j = \min A_j$ is an entry in the top $b-1$ rows of λ.

In the string given in (1), U corresponds to an up step and D corresponds to a down step. Note that for any k, we have $[x_1, x_k] \subseteq A_1 \cup \cdots \cup A_k$. Otherwise, there exists a $w \in [x_1, x_k]$ so that $w \in A_j$ for some $j > k$. Then we have $w \geq x_j > x_k \geq w$, a contradiction.

Thus, for $k < t$

$$\sum_{j=1}^{k}(x_{j+1} - x_j) = x_{k+1} - x_1 \le \sum_{j=1}^{k} a_j.$$

Also,

$$\left[\sum_{j=1}^{t}(x_{j+1} - x_j)\right] + (n - x_t + x_1) = x_t - x_1 + n - x_t + x_1 = n.$$

Moreover, there are precisely $b - 2$ of the a_j so that $a_j = 1$, and there are $i + 1$ of the a_j so that $a_j > 1$. Also, observe that $a_{i+b-1} > 1$, so the last ascent in $\mathtt{SD}(\lambda)$ is not a singleton. Consequently, the constructed Dyck path is an element of $\mathrm{Dyck}(n, i + 1, b - 2)$. □

Example 3.1. *Recall the skew standard Young tableau from figure 6. Note that, $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 7, x_6 = 10$, and $x_7 = 12$. As x_2, x_5 and x_7 are the entries in row 5, y_i is defined for $i = 2, 5, 7$. We have $y_2 = 3$, $y_5 = 11$, and $y_7 = 13$. Thus, $A_1 = \{1\}, A_2 = \{2, 3, 6\}, A_3 = \{4\}, A_4 = \{5\}, A_5 = \{7, 8, 9, 11\}, A_6 = \{10\}$, and $A_7 = \{12, 13, 14, 15\}$.*

Thus $\mathtt{SD}(\lambda)$ is the Dyck path given by $UDU^3D^2UDUDDU^4D^3UDDU^4D^4$.

3.2 Map DS

This subsection gives the map DS from $\mathrm{Dyck}(n, \ell, s)$ to $\mathrm{Skyt}(n - s - 2\ell - 2, \ell - 1, s + 2)$, which is the inverse of SD. The reader can verify that they are indeed inverses.

Definition 3.4. *Let P be a Dyck path in $\mathrm{Dyck}(n, \ell, s)$. We define $\mathtt{DS}(P)$, a skew tableau, as follows.*

Given P, label the down-steps, left to right, in increasing order, from 1 to n. Next, use the label on the down-step at each peak as the label for the up-step at the same peak. Going through the ascents from left to right, greedily label the unlabeled up-steps from top to bottom using the smallest possible numbers from $[a + 2i + b - 2]$ not already appearing on any up-step.

Let A_j be the labels appearing on the jth ascent. Now construct $\mathtt{DS}(P)$ so that $\mathrm{Nom}\big(\mathtt{DS}(P)\big) = (A_1, \ldots, A_{\ell+s})$.

Lemma 3.2. *Given a Dyck path P in $\mathrm{Dyck}(n, \ell, s)$, the tableau $\mathtt{DS}(P)$ is a skew standard Young tableau in $\mathrm{Skyt}(n - s - 2\ell - 2, \ell - 1, s + 2)$.*

Proof. There are $\ell + s$ ascents in P. Thus, precisely ℓ of the A_j satisfy $|A_j| > 1$, and precisely s of the A_j satisfy $|A_j| = 1$. Next, notice that $(A_1, \ldots, A_{\ell+s})$ is a nomincreasing sequence due to the greedy labeling of up steps of P. Also, note that $|A_{s+\ell}| > 1$ since the last ascent in P is not a singleton. Since there are n up steps in P, we have $\mathtt{DS}(P) \in \mathrm{Skyt}(n - s - 2\ell - 2, \ell - 1, s + 2)$. □

Example 3.2. *Given a Dyck path in $\mathrm{Dyck}(15, 3, 4)$, we first label the down-steps, left to right, in increasing order.*
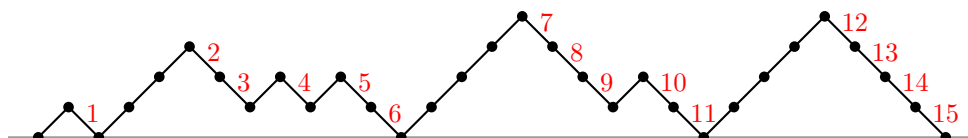
Then label the upstep of each peak:
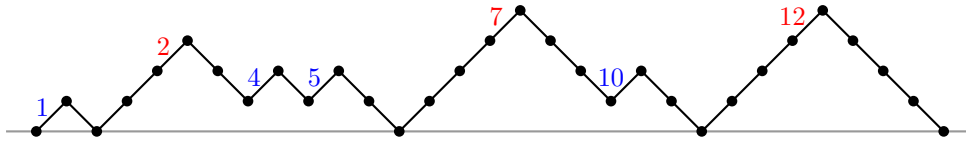
Now greedily label remaining up-steps.

Thus, we have $A_1 = \{1\}$, $A_2 = \{2, 3, 6\}$, $A_3 = \{4\}$, $A_4 = \{5\}$, $A_5 = \{7, 8, 9, 11\}$, $A_6 = \{10\}$, and $A_7 = \{12, 13, 14, 15\}$. The map DT *gives the tableau in Figure 7.*

Figure 7: The construction of the skew tableaux in the final steps of the DS map.
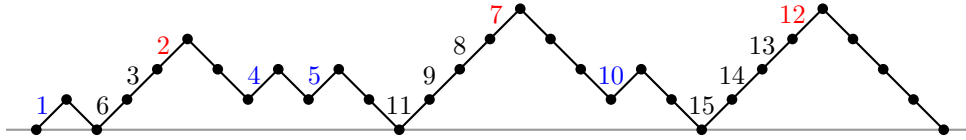
3.3 Map DT

The inspiration for the following map DT, a map from $\mathrm{Dyck}(n, \ell, s)$ to $\mathrm{Tri}(n+2, \ell, s)$, comes from [11, Proposition 6.2.1].

Definition 3.5. *Let P be a Dyck path in $\mathrm{Dyck}(n, \ell, s)$. We define* DT$(P)$ *as a certain type of triangulation, as follows.*

A Dyck path in $\mathrm{Dyck}(n, \ell, s)$ has the form

$$U^{u_1} D^{d_1} U^{u_2} D^{d_2} \cdots U^{u_{s+\ell}} D^{d_{s+\ell}},$$

where s is the number of singletons, ℓ is the number of long ascents, and the u_i and d_i are positive integers. Recall that the triangulation is determined by its fan decomposition. Let F_j be a fan with u_j triangles. Then DT(P) *is given by the fan decomposition $\big((F_1, \ldots, F_{s+\ell}), (d_1, \ldots, d_{s+\ell-1})\big)$. Lemma 3.3 proves that this pair of sequences is indeed a valid fan decomposition.*

Lemma 3.3. *If P is a Dyck path in $\mathrm{Dyck}(n, \ell, s)$, then* DT$(P)$ *is a triangulation in $\mathrm{Tri}(n+2, \ell, s)$.*

Proof. Given $P \in \mathrm{Dyck}(n, \ell, s)$, note that the number of triangles in DT(P) is given by sum of sizes of F_j:

$$\sum_{j=1}^{s+\ell} u_j = n.$$

As DT(P) has n triangles, the boundary must have $n + 2$ edges. Also note that since there is no singleton after the last long ascent in our Dyck path, the last fan $\mathcal{F}(T)$ will not be a singleton fan. Since ℓ of the u_j satisfy $u_j > 1$, our triangulation has ℓ non-singular fans. Similarly, since s of the u_j satisfy $u_j = 1$, our triangulation has s singular fans. Thus, we have constructed a triangulation in $\mathrm{Tri}(n+2, \ell, s)$, where the vertices are labeled as follows: label the origin of F_1 as 1, and then label the remaining vertices from 2 to n in clock-wise order by starting at 1 and traveling along the boundary of the triangulation. □

Example 3.3. *Consider the path below.*
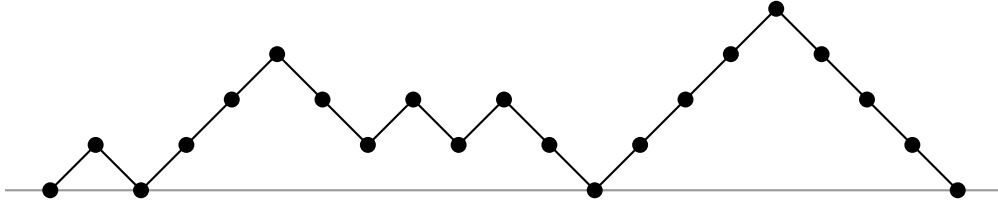
Associated to this are five fans, given below. We shade these fans differently so we may more easily keep track of them throughout.

F_1 F_2 F_3 F_4 F_5
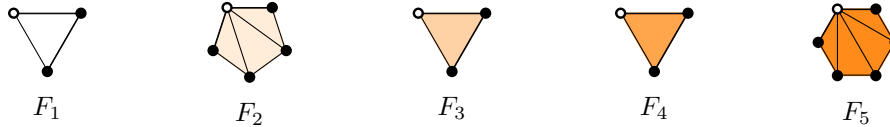
The sequence of the numbers of down steps in this Dyck path, omitting the last descent, is $(1, 2, 1, 2)$. Thus origins of F_1 and F_2 are one edge apart when attaching the fans. Below are the subsequent steps of attaching F_j.

Redrawn with vertex labels, we have the following, where the vertex labeled 1 is the origin of F_1, the vertex labeled 2 is the origin of F_2, and so on.
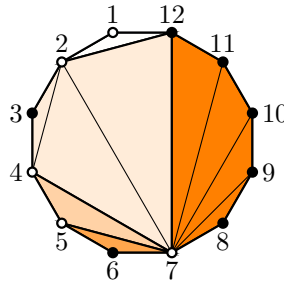
3.4 Map TD

In this section, we construct the map TD from $\mathrm{Tri}(n,t,s)$ to $\mathrm{Dyck}(n-2,t,s)$, which is the inverse to DT.

Definition 3.6. *Given $T \in \mathrm{Tri}(n,t,s)$, we define $\mathtt{TD}(T)$, a certain lattice path, as follows.*
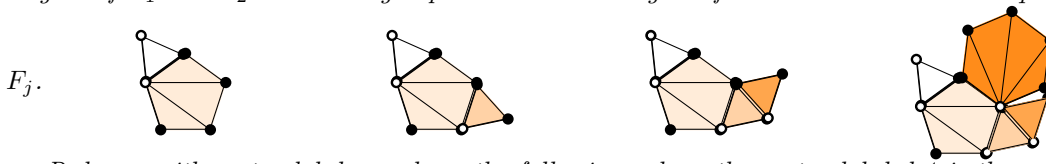*Consider the fan decomposition $(\mathcal{F}(T), \delta(T)) = ((F_1, F_2, \ldots, F_{t+s}), (d_1, \ldots, d_{t+s-1}))$. Let x_j denote the label of the origin of F_j in T and let d_{t+s} be the number of boundary edges from x_{t+s} to x_1 minus 2. Letting u_j be the number of triangles in F_j, define $\mathtt{TD}(T)$ to be*

$$U^{u_1} D^{d_1} U^{u_2} D^{d_2} \ldots U^{u_{t+s}} D^{d_{t+s}}.$$

Lemma 3.4. *If $T \in \mathrm{Tri}(n,t,s)$, then the string $\mathtt{TD}(T)$ is a Dyck path in $\mathrm{Dyck}(n-2,t,s)$.*

Proof. We claim $\mathtt{TD}(T)$ is a valid Dyck path. First, note that given any fan decomposition of a triangulation on n vertices, the largest label for an origin is $n-1$. Thus, the largest number of boundary edges between the first and last origin (traveling counter-clockwise) is $n-2$, which is precisely the number of triangles in the triangulation. By Remark 2.1, we see that

$$\sum_{j=1}^{k} d_j \leq \sum_{j=1}^{k} u_j.$$

Also, note that

$$\sum_{j=1}^{t+s} u_j = \text{number of triangles in } T = n-2 = \text{number of boundary edges of } T \text{ minus } 2 = \sum_{j=1}^{t+s} d_j.$$

Hence this path is a Dyck path with semi-length $n-2$. Also, note that the number of singletons in $\mathtt{TD}(T)$ is precisely the number of u_j which equals 1, which is the number of singleton fans in T, which is s. This also means that $\mathtt{TD}(T)$ has t long ascents. Finally, note that $u_{t+s} = \#F_{t+s} \geq 2$ since F_{t+s} is the last fan in $\mathcal{F}(T)$. Hence, we have constructed a path in $\mathrm{Dyck}(n-2,t,s)$. $\qquad\square$

Example 3.4. *Suppose we start with the following triangulation.*

Below, we identify the origins of the fans in $\mathcal{F} = (F_1, F_2, F_3, F_4, F_5)$ by using white dots for such vertices.

Hence, F_1, F_3, and F_4 are all fans of size 1, F_2 is a fan of size 3, and F_5 is a fan of size 4. The origins of these fans are identified in the image below. Consequently, $u_1 = u_3 = u_4 = 1$, $u_2 = 3$, and $u_5 = 4$. Also note that $d_1 = d_3$ and $d_2 = d_4 = 2$. Hence we have the Dyck path $UDU^3D^2UDUD^2U^4D^4$, which visualized gives the following path.

3.5 Map ST

In this section, we construct the map $\mathtt{ST}$ from $\mathrm{Skyt}(a,i,b)$ to $\mathrm{Tri}(a+b+2i, i+1, b-2)$. This map is an explicit interpretation of the map $\mathtt{SD}$ composed with $\mathtt{DT}$ without needing to bring up Dyck paths.

Definition 3.7. *Given a tableau $\lambda \in \mathrm{Skyt}(a,i,b)$, we define $\mathtt{ST}(\lambda)$, a triangulation, as follows.*

Let $d_j = x_{j+1} - x_j$, where $x_1, \ldots, x_{i+b-1}$ are the entries in the top $b-1$ rows of λ so that $x_1 < x_2 < \cdots < x_{i+b-1}$. Let $\mathrm{Nom}(\lambda) = (A_1, \ldots, A_{i+b-1})$. For $1 \leq j \leq i+b-1$, let $f_j = |A_j|$. Let F_j be a fan of size f_j. Then we define $\mathtt{ST}(\lambda)$ to be the triangulation whose fan decomposition is $(\mathcal{F}, \delta)$, where $\mathcal{F} = (F_1, \ldots, F_{i+b-1})$ and $\delta = (d_1, \ldots, d_{i+b-2})$.

Lemma 3.5. *Let $\lambda \in \mathrm{Skyt}(a,i,b)$. Then $\mathtt{ST}(\lambda) \in \mathrm{Tri}(a+b+2i, i+1, b-2)$.*

Proof. Recall triangulations are uniquely determined by their fan decomposition, and thus $\mathtt{ST}(\lambda)$ is guaranteed to be a triangulation. Note that the number of triangles in this triangulation is precisely the number of entries of λ, which is $a+b+2i-2$. Hence, the boundary of our constructed triangulation has $a+b+2i$ edges. Recall that among $A_1, \ldots, A_{i+b-1}$, precisely $i+1$ have cardinality larger than 1, and precisely $b-2$ have cardinality exactly 1. Thus, our proposed fan decomposition for $\mathtt{ST}(\lambda)$ has precisely $i+1$ nonsingular fans and $b-2$ singular

fans. Finally, note that by construction, $f_{i+b-1} > 1$ since $|A_{i+b-1}| > 1$ by construction of $\text{Nom}(\lambda)$. That is, the last fan appearing in $\mathcal{F}$ is not singular. All together, this verifies that we have constructed a triangulation in $\text{Tri}(a + b + 2i, i + 1, b - 2)$. □

Example 3.5. *Consider the following choice for λ.*

		1
		4
2	5	7
3	6	9
8		
10		

Hence, $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{4\}$, $A_4 = \{5, 6, 8\}$, and $A_5 = \{7, 9, 10\}$. We have $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 5$, and $x_5 = 7$. Thus, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, and $d_4 = 2$. Also $f_1 = 1$, $f_2 = 2$, $f_3 = 1$, $f_4 = 3$, and $f_5 = 3$. This constructs the following triangulation.
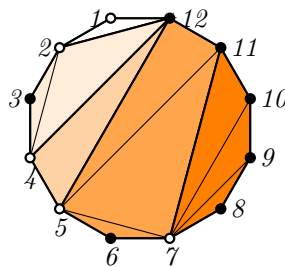

3.6 Map TS

In this section, we construct a map TD from $\text{Tri}(n, t, s)$ to $\text{Skyt}(n - 2 - s - 2t, t - 1, s + 2)$, which is the inverse to ST. This map is an explicit interpretation of the map TD composed with DS without needing to bring up Dyck paths.

Definition 3.8. *Let $T \in \text{Tri}(n, t, s)$. We define $\text{TS}(T)$, a skew tableau, as follows. We label the triangles in the triangulation in the following way: For each fan, label a triangle with the label of the fan's origin. Then, greedily label other triangles with the unused integers in $[1, n - 2]$ in the order that fans appear in $\mathcal{F}(T)$. (Triangles within a single fan need not be labeled in any particular order.) Let A_j be the labels appearing in the fan F_j. Let $\text{TS}(T)$ be the skew diagram so that $\text{Nom}\big(\text{TS}(T)\big) = (A_1, \ldots, A_{t+s})$.*

Lemma 3.6. *Given $T \in \text{Tri}(n, t, s)$, we have $\text{TS}(T) \in \text{Skyt}(n - 2 - s - 2t, t - 1, s + 2)$.*

Proof. Note that the number of A_j so that $|A_j| = 1$ is precisely the number of singleton fans in T, which is s. Also, the number of A_j so that $|A_j| > 1$ is t. Next, notice that $(A_1, \ldots, A_{\ell+s})$ is a nomincreasing sequence due to the greedy labeling of the triangles in T, and we also have that $|A_{\ell+s}| > 1$ since $F_{\ell+s}$, the last fan appearing in $\mathcal{F}(T)$, must contain more than 1 triangle. Finally, the number of triangles in T is $n - 2$. Thus, $\text{TS}(T) \in \text{Skyt}(n - 2 - s - 2t, t - 1, s + 2)$. □
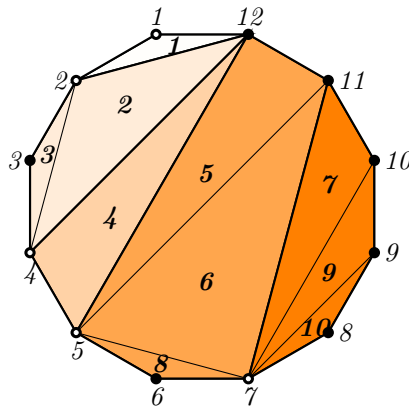
Example 3.6. *For example, given the triangulation in $\text{Tri}(12, 3, 2)$ below, label the vertices in counter-clockwise order.*
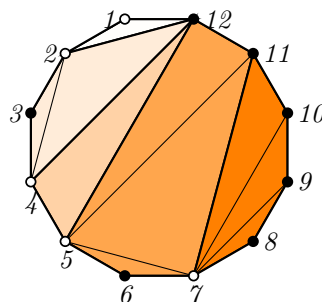
We now label a single triangle in each fan with the label of the corresponding origin.
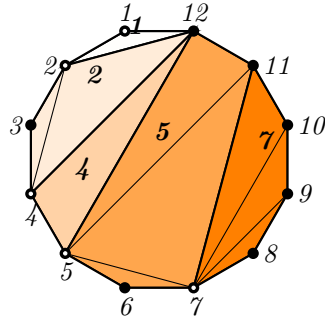
We now label the remaining triangles greedily in the order that fans appear in $\mathcal{F}(T)$.

Hence, $A_1 = \{1\}$, $A_2 = \{2,3\}$, $A_3 = \{4\}$, $A_4 = \{5,6,8\}$, and $A_5 = \{7,9,10\}$. This gives the following tableau.

		1
		4
2	5	7
3	6	9
8		
10		

4. Dissections and Triangulations

In this section, we construct a combinatorial bijection between $\mathrm{Dis}(n+2,i)$ and $\mathrm{Tri}(n+i+3,i+1,0)$ as mentioned in the discussion following Corollary 2.1.

First, we define the map from $\mathrm{Tri}(n+i+3,i+1,0)$ to $\mathrm{Dis}(n+2,i)$.

Definition 4.1. *Let T be a triangulation of an $(n+i+3)$-gon with $i+1$ non-singular fans and no singular fans. We know $\mathcal{F}(T)$ is of the form $\mathcal{F}(T) = (F_1, \ldots, F_{i+1})$. Remove the internal diagonals of F_j in T for each j, leaving us with exactly i diagonals in T. Let x_j be the origin of F_j. For each vertex x_j, let y_j be the immediate vertex that follows x_j counterclockwise. Note that it is possible to have $y_j = x_{j+1}$. Also, we always have that the y_j is a vertex in F_j, since (x_j, y_j) must bound a triangle, and by the definitions, this triangle is a part of F_j. Contract each edge (x_j, y_j), creating an $(n+2)$-gon. Note that the vertex labeled 1 will always be the origin of F_1, so consequently we always contract $(1,2)$. Let 1 be the label of the new vertex after contracting this edge. Relabel the vertices in increasing counterclockwise order, starting at the original vertex 1. Since no fan of T was singular, it must be that the contractions preserved all i diagonals, giving us a dissection in $\mathrm{Dis}(n+2,i)$.*

Now we define the map inverse to the one given above in Definition 4.1, which is a map from $\mathrm{Dis}(n+2,i)$ to $\mathrm{Tri}(n+i+3,i+1,0)$.

Definition 4.2. *For the inverse map, let D be a dissection of an $(n+2)$-gon with i chords, say $c_1, c_2, \ldots, c_i$. We assume, as with triangulations, that the vertices of D are already labeled with the numbers 1 through $n+2$. We will describe a process that allows us to add new vertices and edges to D. Let $1'$ be a new vertex so that $(1', 1)$ is an edge and $(1', 2)$ is an edge. Delete the edge $(1, 2)$. Thus the new polygon has $n+3$ vertices. If 1 was incident to more than one chord, shift all chords that do not form a triangle with the edge $(1, n+2)$ so that they are incident with $1'$ instead of 1. Now, let x be the next vertex counterclockwise to 1 incident to a chord. (Note it may be that $x = 1'$.) Proceed with the following procedure.*

1. *Let $c_{j_1}, c_{j_2}, \ldots, c_{j_k}$ be the list of chords incident with x.*

2. *Let z be the vertex immediately counterclockwise of x. Remove the edge (x, z) and add a new vertex x' along with edges (x, x') and (x', z).*

3. *If x is incident to exactly one chord, continue to step (5). Otherwise, let y be the vertex immediately clockwise to x. The edge (y, x) bounds a closed region which contains exactly one chord c_{j_ℓ}. For each c_{j_m} with $m \neq \ell$, change its incidence with x to an incidence with x'. That is, if c_{j_m} was of the form (x, w), for some vertex w, replace this chord with (x', w).*

4. *If, after doing the prior step, (x, x') becomes a boundary edge to a region that we have already added a boundary edge to, undo the prior step and continue to the next step.*

5. *Move to the next vertex counterclockwise to x incident to some chord, calling this new vertex x. (Note this new vertex may be the vertex x' constructed in step (2).) If x is a vertex we have already visited before, terminate the procedure. Otherwise, restart at step (1).*

After doing this, observe that no region is a triangle. Also, observe that we added a single edge to the boundary for each region (hence the importance of step (4)), and so we now have an $(n+i+3)$-gon. Relabel the vertices, starting at the vertex labeled 1 and continuing counterclockwise. We can decompose our new polygon into $i+1$ regions, labeled $P_1, P_2, \ldots, P_{i+1}$. Make each of these fans so that the origin of P_j is the vertex with the minimum label of P_j. This gives us a triangulation of an $(n+i+3)$-gon with $i+1$ non-singular fans and no singular fans.

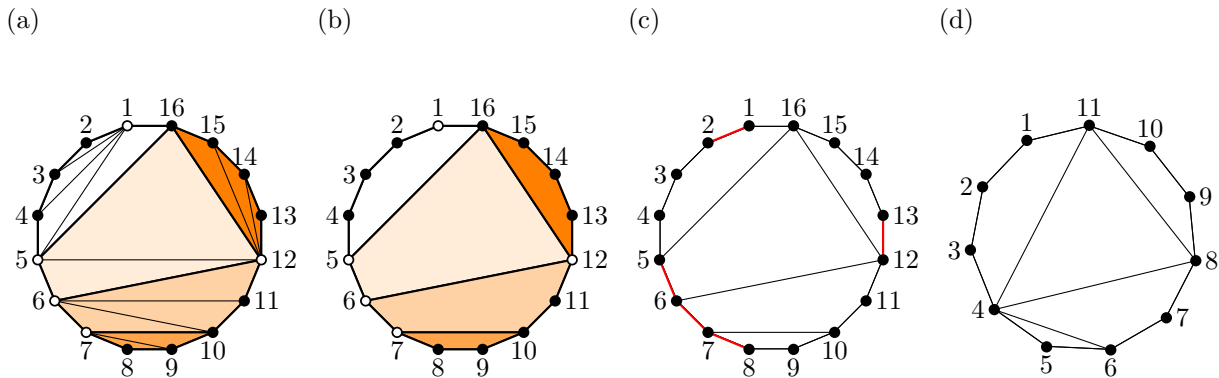(a)   (b)   (c)   (d)

Figure 8: The steps transforming an element of $\mathrm{Tri}(16, 5, 0)$, Figure 8(a), to a dissection of a 11-gon with 4 chords, Figure 8(d). The fans of (a) are colored different shades of gray, and we omit this shading after the fans no longer become relevant in part (c). In (c), the red boundary edges (1,2), (5,6), (6,7), (7,8), and (12,13) are the boundary edges that get contracted. The vertex labeled 1 in (d) is the vertex labeled 2 in the other parts.

Remark 4.1. *There are a couple of things to keep in mind that may help justify why the maps given in Definitions 4.1 to 4.2 are mutual inverses.*

1. *The edges we contract going from a triangulation to a dissection are exactly the edges we add back going from a dissection to a triangulation. This is because the origins of fans in triangulations are always chosen by the smallest vertex appearing in a fan, which appears sooner traveling counterclockwise around the polygons than vertices with larger labels. The regions in a dissection are ultimately what become our*

fans for a triangulation, so we consequently always add an edge on the boundary of a dissection right after the vertex that would end up being the origin for a fan.

2. *The chords of a dissection should be viewed as the parts of the triangulation that ultimately form the boundaries of the fans (along with the actual boundary of the polygon). Hence, we can not expect two such chords to remain incident in the triangulation, as this would alter the number of fans.*

See Figure 8 below to see an illustration of the map from a triangulation to a dissection and Figure 9 to see an illustration of the map of the other direction.
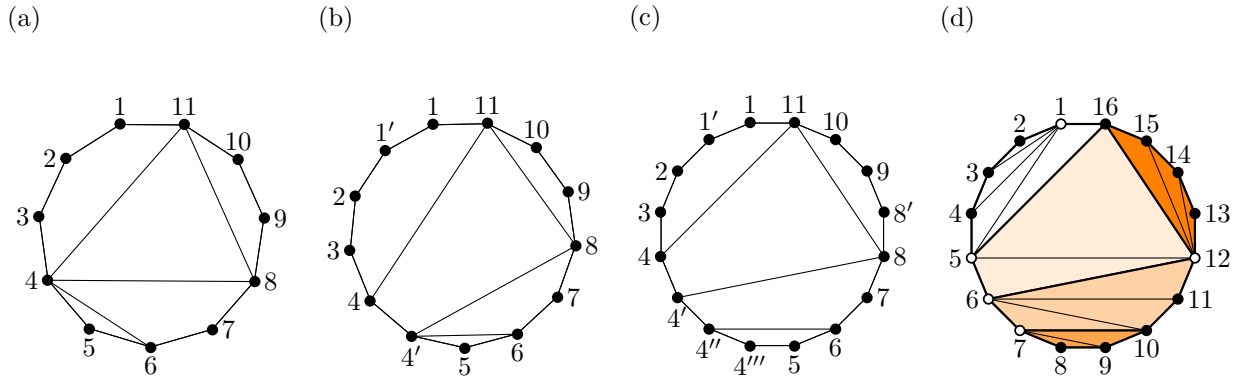
(a)   (b)   (c)   (d)

Figure 9: The steps transforming a dissection of a 11-gon with 4 chords, Figure 9(a), to an element of $\mathrm{Tri}(16, 5, 0)$, Figure 9(d). In (b), we see the initial step of adding vertex $1'$ and an application of step (3) from the description of the combinatorial bijection for Corollary 2.1 (1)-(4), requiring us to change an adjacency of a chord. In (c), we then do this a second time to vertex $4'$ (as it is still adjacent to multiple chords) and add the remaining vertices as described in the map. Adding chords to the vertex with the minimum label in each part gives us the triangulation in (d).

Acknowledgments

References

[1] B. Elias, N. Proudfoot, and M. Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*, Adv. Math. 299 (2016), 36–70.

[2] L. Ferroni, G. D. Nasr, and L. Vecchi, *Stressed Hyperplanes and Kazhdan–Lusztig Gamma-Positivity for Matroids*, Int. Math. Res. Not. IMRN (24) (2023), 20883–20942.

[3] A. L. L. Gao, M. H. Y. Xie, and A. L. B. Yang, *The equivariant inverse Kazhdan-Lusztig polynomials of uniform matroids*, SIAM J. Discrete Math. 36(4) (2022), 2553–2569.

[4] K. Gedeon, N. Proudfoot, and B. Young, *The equivariant Kazhdan-Lusztig polynomial of a matroid*, J. Combin. Theory Ser. A 150 (2017), 267–294.

[5] K. Gedeon, N. Proudfoot, and B. Young *Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures*, Sém. Lothar. Combin. 78B (2017), Article 80.

[6] J. B. Gil, P. R. W. McNamara, J. O. Tirrell, and M. D. Weiner, *From Dyck paths to standard Young tableaux*, Ann. Comb. 24(1) (2020), 69–93.

[7] T. Karn, G. D. Nasr, N. Proudfoot, and L. Vecchi, *Equivariant Kazhdan-Lusztig theory of paving matroids*, Algebr. Comb. 6(3) (2023), 677–688.

[8] K. Lee, G. D. Nasr, and J. Radcliffe, *A combinatorial formula for Kazhdan-Lusztig polynomials of ρ-removed uniform matroids*, Electron. J. Combin. 27(4) (2020), Paper No. 4.7.

[9] K. Lee, G. D. Nasr, and J. Radcliffe, *A combinatorial formula for Kazhdan-Lusztig polynomials of sparse paving matroids*, Electron. J. Combin. 28(4) (2021), Paper No. 4.44.

[10] R. P. Stanley, *Polygon dissections and standard Young tableaux*, J. Combin. Theory Ser. A 76(1) (1996),175–177.

[11] R. P. Stanley, *Enumerative combinatorics*, Volume 2, Volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

Appendix

Here we provide some tables for the cardinalities of $\#\operatorname{Skyt}(a,i,b)$, $\#\operatorname{Tri}(n,t,s)$, and $\#\operatorname{Dyck}(n,\ell,s)$ for various values of inputs, made possible through computer computation and Lemma 2.1.

First, we give tables for $\#\operatorname{Skyt}(a,i,b)$, where $i=1$, $i=2$, and $i=3$.

$a\backslash b$	2	3	4
2	2	5	9
3	5	14	28
4	9	28	62
5	14	48	117
6	20	75	200
7	27	110	319

$a\backslash b$	2	3	4
2	5	21	56
3	21	98	288
4	56	288	927
5	120	675	2365
6	225	1375	5214
7	385	2541	10374

$a\backslash b$	2	3	4
2	14	84	300
3	84	552	2145
4	300	2145	9020
5	825	6380	28886
6	1925	16016	77714
7	4004	35672	184730

Figure 10: Values for $\#\operatorname{Skyt}(a,1,b)$, $\#\operatorname{Skyt}(a,1,b)$, and $\#\operatorname{Skyt}(a,1,b)$, from left-to-right.

Next, we give similar tables for $\#\operatorname{Dyck}(n,\ell,s)$, where $s=0$, $s=1$, and $s=2$. We will not provide the same tables for $\#\operatorname{Tri}(n,t,s)$, since the translation between parameters is more direct: $\#\operatorname{Tri}(n,t,s) = \#\operatorname{Dyck}(n-2,t,s)$.

$n\backslash \ell$	2	3	4
5	5	0	0
6	14	0	0
7	28	21	0
8	48	98	0
9	75	288	84
10	110	675	552
11	154	1375	2145

$n\backslash \ell$	2	3	4
6	9	0	0
7	28	0	0
8	62	56	0
9	117	288	0
10	200	927	300
11	319	2365	2145
12	483	5214	9020

$n\backslash \ell$	2	3	4
7	14	0	0
8	48	0	0
9	117	120	0
10	242	675	0
11	451	2365	825
12	780	6534	6380
13	1274	15522	28886

Figure 11: Values for $\#\operatorname{Dyck}(n,\ell,1)$, $\#\operatorname{Dyck}(n,\ell,2)$, and $\#\operatorname{Dyck}(n,\ell,3)$, from left-to-right.