ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **4:4** (2024) Article #S2R27
https://doi.org/10.54550/ECA2024V4S3R27

Counting Finite Topologies

Eldar Fischer and Johann A. Makowsky

[†]*Department of Computer Science, Mathematics, Technion-IIT, Haifa, Israel*
Email: eldar@cs.technion.ac.il, janos@cs.technion.ac.il

Received: December 31, 2023, **Accepted**: May 20, 2024, **Published**: May 31, 2024

Abstract: A finite topology $\mathcal{T} = (A, T)$ consists of a finite set A together with a family T of subsets of A, the open sets, satisfying the axioms of a topology. $\mathcal{T}_\phi(n)$ is the number of distinct topolgies T of subsets of $[n]$ which satisfy ϕ, where ϕ is a property of topologies expressible in TMSOL, topological monadic second order logic.

A sequence $s(n)$ of integers is C-finite if it satisfies a linear recurrence relation with constant coefficients. It is MC-finite of for every modulus m the sequence $s^m(n) = s(n) \pmod{m}$ satisfies a linear recurrence relation with constant coefficients depending on m. In general $\mathcal{T}_\phi(n)$ is not C-finite, because it grows too fast.

In this paper we show that $\mathcal{T}_\phi(n)$ is MC-finite for every $\phi \in$ TMSOL. We also show that this is still true for TCMSOL, the extensions of TMSOL with modular counting quantifiers.

Keywords: Counting finite topologies; MC-finiteness; Specker-Blatter theorem
2020 Mathematics Subject Classification: 05; 05A15; 06A17

1. Introduction

In the last decade finite topologies have received renewed attention due to their role in image analysis and data science. There is a vast literature testifying to this. We just mention two references as typical examples, [9,26]. For mathematical applications of finite topologies, see [1]. Finite metric spaces are studied in [6,27]. The model theory of topological spaces was studied in the late 1970s, see [19,22,28,35].

In [7] A. Broder introduced the restricted r-Stirling numbers and r-Bell numbers. They have found various applications in enumerative combinatorics, e.g. see [2]. Inspired by this we study in this paper the number of finite topologies on a finite set subject to various restrictions.

Assume you are given the set $[r + n] = \{1, 2, 3, \dots, r + n\}$ and you want to count the number of finite topologies on the set $[r + n]$ such that

(i) Each of the elements in $[r]$ are in a different component, and

(ii) Each connected component is of odd size.

Or think of variations thereof, where the topologies on the elements of $[r]$ satisfy some prescribed topological configuration, such as all the singletons in $[r]$ being closed sets, or being pairwise separable by an open set. In the applications from [9,26] counting finite topologies with such or similar restrictions might be interesting.

Counting finite topologies is a difficult problem. Even for the case without restrictions, no explicit formula is known. There are some asymptotic results, but the best results known so far are congruences modulo a fixed integer m. Computing $T(n)$ for $n = 1, 2, 3, 4$ can be done by hand. There are two papers giving values for $T(5)$: In [11] it says that $T(5) = 6942$. In [33] it says that $T(5) = 7181$. E. Specker in 1980 got interested in modular counting of finite topologies in order to prove which one of these claims must be false.

A sequence $s(n)$ of integers is *C-finite*, if the sequence satisfies a linear recurrence relation with constant coefficients. A sequence $s(n)$ of integers is *MC-finite, modularily C-finite*, if for every integer m the sequence $s^m(n) = s(n) \pmod{m}$ is an ultimately periodic sequence of positive integers. C. Blatter and E. Specker showed in [4,5] the following theorem:

Theorem 1.1. *(i) The number of finite topologies $T(n)$ on $[n]$ is not C-finite, but it is MC-finite.*

(ii) For every $m \in \mathbb{N}^+$ there is a polynomial time algorithm which computes $T(n)$ modulo m.

(iii) $T(n) = 2 \mod 5$, *hence* $T(5) \neq 7181$.

The proof of his theorem uses both logic and advanced combinatorics. In [34] Specker gave a very readable expository version of its proof.

The purpose of this paper is to show similar results for the number of topologies with restrictions as suggested above. In the presence of the restrictions we have in mind, the method underlying the Specker-Blatter theorem cannot be applied directly, but it can be applied using our recent results together with a suitable definition of a logic from [16, 17].

We will use logic to make the framework of the restrictions precise. Let $\mathcal{T} = (X, \mathcal{U})$ be a finite topological space on the finite set X and $\mathcal{U}$ the family of open sets in X. We associate with $\mathcal{T}$ a two sorted first-order structure $\mathcal{T}' = (X, \mathcal{U}, E)$ where $E \subseteq X \times \mathcal{U}$ and $E(x, U)$ says that x is an element of U. If $X = [r+n]$ we use constant symbols $a_1, \ldots, a_r$ which have a fixed interpretation: a_i is interpreted by $i \in [r]$. We say that the constant symbols a_i are *hard-wired*. The topological restrictions are now described by first-order formulas $\phi(a_1, \ldots, a_r)$ over the structure $([r+n], \mathcal{U}, \in, a_1, \ldots, a_r)$. We denote by $T_{\phi,r}(n)$ the number of topologies on the set $[r+n]$ which satisfy $\phi(a_1, \ldots, a_r)$. For a positive integer m we denote by $T^m_{\phi,r}(n)$ the sequence $T_{\phi,r}(n)$ modulo m.

1.1 Main result

Our main result is stated here for topological first-order logic TFOL:

Theorem 1.2. *(i) For every formula ϕ of TFOL and every positive integer m, the sequence $T^m_{\phi,r}(n)$ is ultimately periodic modulo m. In other words $T^m_{\phi,r}(n)$ is MC-finite.*

*(ii) Given ϕ and m, the sequence $T^m_{\phi,r}(n)$ is Fixed Parameter Tractable (**FPT**) where the parameters depend on ϕ, r and m.*

The proof uses recent results on extensions of the Specker-Blatter theorem due to the authors, [16, 17]. It also uses model-theoretic methods as described in [35]. The same method was applied to prove congruences for restricted Bell and Stirling numbers in [13].

One of our main contributions lies in identifying the logic TCMSOL, a topological version of Monadic Second Order Logic with modular counting. This allows us to prove Theorem 5.4 in Section 5, which is like Theorem 1.2 but stated for TCMSOL instead of TFOL.

2. Background

2.1 C-finite and MC-finite sequences of integers

Some of the background material also appears in [13] and [16, 17].

A sequence of integers $s(n)$ is *C-finite*[*] if there are constants $p, q \in \mathbb{N}$ and $c_i \in \mathbb{Z}, 0 \leq i \leq p-1$ such that for all $n \geq q$ the linear recurrence relation

$$s(n+p) = \sum_{i=0}^{p-1} c_i s(n+i)$$

holds for $s(n)$. C-finite sequences have limited growth, see e.g. [12, 23]:

Proposition 2.1. *Let s_n be a C-finite sequence of integers. Then there exists $c \in \mathbb{N}^+$ such that for all $n \in \mathbb{N}$, $a_n \leq 2^{cn}$.*

Actually, a lot more can be said, see [18], but we do not need it for our purposes.

To prove that a sequence $s(n)$ of integers is not C-finite, we can use Proposition 2.1. To prove that a sequence $s(n)$ of integers is C-finite, there are several methods: One can try to find an explicit recurrence relation, one can exhibit a rational generating function, or one can use a method based on model theory as described in [14, 15].

A sequence of integers $s(n)$ is modular C-finite, abbreviated as *MC-finite*, if for every $m \in \mathbb{N}$ there are constants $p_m, q_m \in \mathbb{N}^+$ such that for every $n \geq q_m$ there is a linear recurrence relation

$$s(n+p_m) \equiv \sum_{i=0}^{p_m-1} c_{i,m} s(n+i) \bmod m$$

with constant coefficients $c_{i,m} \in \mathbb{Z}$. Note that the coefficients $c_{i,m}$ and both p_m and q_m generally do depend on m.

We denote by $s^m(n)$ the sequence $s(n) \bmod m$.

[*] These are also called constant-recursive sequences or linear-recursive sequences in the literature.

Proposition 2.2. *The sequence $s(n)$ is MC-finite iff $s^m(n)$ is ultimately periodic for every m.*

Proof. MC-finiteness clearly implies periodicity. The converse is from [30]. □

Clearly, if a sequence $s(n)$ is C-finite it is also MC-finite with $r_m = r$ and $c_{i,m} = c_i$ for all m. The converse is not true, there are uncountably many MC-finite sequences, but only countably many C-finite sequences with integer coefficients, see Proposition 2.3 below.

Examples 2.1.

(i) The Fibonacci sequence is C-finite.

(ii) If $s(n)$ is C-finite it has at most simple exponential growth, by Proposition 2.1.

(iii) The Bell numbers $B(n)$ are not C-finite*, but are* MC-finite.

(iv) Let $f(n)$ be any integer sequence. The sequence $s_1(n) = 2 \cdot f(n)$ is ultimately periodic modulo 2, but not necessarily MC-finite.

(v) Let $g(n)$ be any integer sequence. The sequence $s_2(n) = n! \cdot g(n)$ is MC-finite.

(vi) The sequence $s_3(n) = \frac{1}{2}\binom{2n}{n}$ is not MC-finite: $s_3(n)$ is odd if and only if n is a power of 2, and otherwise it is even (Lucas, 1878). A proof may be found in [21, Exercise 5.61] or in [34].

(vii) The Catalan numbers $C(n) = \frac{1}{n+1}\binom{2n}{n}$ are not MC-finite, since $C(n)$ is odd iff n is a Mersenne number, i.e., $n = 2^m - 1$ for some m, see [25, Chapter 13].

(viii) Let p be a prime and $f(n)$ be monotone increasing. The sequence $s(n) = p \cdot f(n) + z(n)$, where $z(n)$ is defined to equal 1 if n is a power of p and to equal 0 for any other n, is monotone increasing but not ultimately periodic modulo p, hence not MC-finite.

Proposition 2.3. *There are uncountably many monotone increasing sequences that are MC-finite, and uncountably many which are not MC-finite.*

Proof. Use examples 2.1 (v) and (viii). □

Although we are mostly interested in MC-finite sequences $s(n)$, it would be natural to check in each example whether the sequence $s(n)$ is also C-finite. In most concrete examples the answer is negative, which can be seen by a growth argument. Proposition 3.2 in Section 3 gives a model-theoretic tool for finite topological structures. However, we will not elaborate on this further.

Actually, in the following subsection, we show that almost all bounded integer sequences are not MC-finite.

2.2 Normal sequences

Let $s(n)$ be an integer sequence, and $b \in \mathbb{N}^+$. The sequence $s^b(n) = s(n) \bmod b$ is normal, if, when we chunk it into substrings of length $\ell \geq 1$, each of the b^ℓ possible strings of $[b]^\ell$ appears in $s^b(n)$ with equal limiting frequency. It is *absolutely normal* if it is normal for every b. The sequence $s^b(n) = s(n) \bmod b$ can be viewed as a real number r_b written in base b. A classical theorem from 1922 by E. Borel says that almost all reals are absolutely normal, [12]. The proposition below shows that MC-finite integer sequences are very rare.

Let PR_b be the set of integer sequences $s^b(n)$ with $s^b(n) = s(n) \bmod b$ for some integer sequence $s(n)$. PR_b is the projection of all integer sequences to sequences over $\mathbb{Z}_b$. We think of PR_b as a set of reals with the usual topology and its Lebesgue measure. Let $UP_b \subseteq PR_b$ be the set of sequences $s^b(n) \in PR_b$ which are ultimately periodic.

Proposition 2.4. *(i) Almost all reals are absolutely normal by their Lebesgue measure.*

(ii) $s(n)$ is MC-finite iff for every $b \in \mathbb{N}^+$ the sequence $s^b(n)$ is ultimately periodic.

(iii) If $s^b(n)$ is normal for some b, then $s(n)$ is not MC-finite.

(iv) $UP_b \subseteq PR_b$ has measure 0.

Proving that a specific sequence is normal is usually very difficult. It has been an elusive goal to prove the normalcy of numbers that are not artificially constructed. While $\sqrt{2}$, π, $ln(2)$ and e are strongly conjectured to be normal, it is still not known whether they are normal or not. It has not even been proven that all digits actually occur infinitely many times in the decimal expansions of those constants. It has also been conjectured that every irrational algebraic number is absolutely normal, and no counterexamples are known in any base. However, not even a single irrational algebraic number has been proven to be normal in any base.

2.3 Counting finite topologies

Here we follow the presentation from [29]. Let $\mathcal{T} = (X, \mathcal{U})$ be a finite topological space on the finite labeled set X and let $\mathcal{U}$ be the family of open sets in X. Counting topologies on X is defined as counting the number of distinct families $\mathcal{U}$ of subsets of X. By Alexandroff's Theorem 2.2(i) and 5.1 this is equivalent to counting the number of labeled finite quasi-orders. Let $T(n)$ and $T_0(n)$ be the number of topologies and T_0-topologies respectively on a the set $[n] = \{1, \ldots, n\}$. Recall that a topology on $[n]$ is T_0 if for all $a, b \in [n]$, there is some open set containing one but not both of them. No explicit formulas for $T(n)$ and $T_0(n)$ are known.

The following is known:

Theorem 2.2.

(i) $T(n) = Q(n)$, *where* $Q(n)$ *is the number of pre-orders on* $[n]$, *[29]. It is* A000798 *in the Online Encyclopedia of Integer Sequences, [20].*

(ii) $T_0(n) = P(n)$, *where* $P(n)$ *is the number of partial orders on* $[n]$, *[29]. It is* A001035 *in the Online Encyclopedia of Integer Sequences.*

(iii) $Q(n) = \sum_{k=0}^{n} S(n,k) \cdot P(k)$, *where* $S(n,k)$ *is the Stirling number of the second kind, [8].*

(iv) $B(n) \leq P(n) \leq Q(n)$, *where* $B(n)$ *are the Bell numbers, which count the number of equivalence relations on* $[n]$. *Furthermore, see [3, 10],*

$$\left(\frac{n}{e \ln n}\right)^n \leq B(n) \leq \left(\frac{n}{e^{1-\epsilon} \ln n}\right)^n,$$

(v) The logarithm with base 2 *of both* $T(n)$ *and* $T_0(n)$ *goes asymptotically to* $\frac{n^2}{4}$ *as* n *goes to infinity, [24].*

Theorem 2.3. *(i)* $T(n)$ *and* $T_0(n)$ *are not C-finite.*

(ii) $T(n)$ *and* $T_0(n)$ *are MC-finite.*

Proof. (i) follows from Theorem 2.2(v).
(ii) follows from Theorem 2.2(i) and (ii) and the Specker-Blatter Theorem 4.1. ☐

3. Topologies as relational structures

3.1 The logics TCMSOL and CMSOL

Let $\mathcal{T} = (X, \mathcal{U}, \in)$ be a finite topological space on the finite set X and let $\mathcal{U}$ be the family of open sets in X. The relation $\in$ is the natural membership relation between elements of X and sets in $\mathcal{U}$. We associate with a finite topological space $(X, \mathcal{U})$ a two-sorted relational structure $\mathcal{T} = (X, \mathcal{U}, E)$, where $E \subseteq X \times \mathcal{U}$ is a binary relation and $E(x, U)$ says that x is an element of U. E is required to satisfy the extensionality axiom

$$\forall U, V \in \mathcal{U}(u = v \leftrightarrow \forall x \in X(E(x, U) \leftrightarrow E(x, v))).$$

There is a natural bijection between finite topological spaces and their associated first-order structures. Given $\mathcal{T} = (X, \mathcal{U}, \in)$, a finite topological space, we define the first- order structure $t((X, \mathcal{U}, E))$ by setting $\mathcal{U} = \mathcal{U}$ and $E(x, U)$ iff $x \in U$. Conversely, given $\mathcal{T} = (X, \mathcal{U}, E)$ which satisfies the extensionality axiom, we define $t^{-1}((X, \mathcal{U}, E))$ by setting for $U \in \mathcal{U}$

$$x \in U \text{ iff } E(x, U).$$

Proposition 3.1. *t is a bijection between finite topological spaces and their associated first-order structures. Furthermore, t^{-1} is its inverse.*

We denote by TCMSOL the monadic second-order logic for structures of this form possibly augmented by constant symbols. We allow quantification over *subsets of* X, but only quantification over *elements of* $\mathcal{U}$. Furthermore, we have a modular counting quantifier $C_{m,a} x \phi(x)$ which says that modulo m there are a elements satisfying $\phi(x)$. TFOL is the logic without second-order quantification and without modular counting. Similarly, CMSOL is defined as TCMSOL for one-sorted structures with one binary relation.

For *finite structures* of the form $\mathcal{T}$ the following are TCMSOL-definable.

(i) $\mathcal{U}$ is a topology for the finite set A: (i) $\emptyset \in \mathcal{U}$, $A \in \mathcal{U}$. (ii) $\mathcal{U}$ is closed under unions. (iii) $\mathcal{U}$ is closed under intersections.

(ii) $\mathcal{U}$ is T_0: $\forall a, b \in A \exists U \in \mathcal{U}(E(a, U) \wedge \neg E(b, U) \vee (\neg E(a, U) \wedge E(b, U))$.

(iii) $\mathcal{U}$ is T_1 if every complement of a singelton set is open: $\forall a \in A(A - a \in \mathcal{U})$.

(iv) X is connected: There are no two non-empty disjoint open sets U_1, U_2 with $U_1 \cup U_2 = X$.

(v) The TFOL-formula $\phi_{U_x}(x, U)$ says that U is the smallest open set containing x:
$(U \in \mathcal{U}) \wedge (E(x, U)) \wedge (\forall V \in \mathcal{U}(E(x, V) \to V \subseteq U))$

(vi) A typical formula which is in TCMSOL would be: There is a set of points of even cardinality which is not an open set.

3.2 Hard-wired constant symbols

Let $\bar{a} = (a_1, \ldots, a_k)$ be k constant symbols. For each of them, there are n possible interpretations in the set $[n]$. However, we say that $(a_1, \ldots, a_r)$, for $r \leq k$ are *hard-wired* on $[n]$, if a_i is interpreted by $i \in [n]$. In the presence of constant symbols (hard-wired or not) $a_1, \ldots, a_k$ we can say:

(i) $\{a_1, \ldots, a_k\}$ is a minimal non-empty open set:

$$\begin{gathered}\exists U \in \mathcal{U}((\bigwedge_{i=1}^{r} E(a_i, U)) \\ \wedge(\forall x(\bigwedge_{i=1}^{r}(x \neq a_i) \to \neg E(x, U))) \\ \wedge(\forall V \in \mathcal{U}((V \subseteq U) \to ((V = U) \vee (V = \emptyset)))))\end{gathered}$$

(ii) There are pairwise disjoint open sets $U_1, \ldots, U_r$ such that a_i in U_i.

(iii) The elements denoted by a_i are all in different connected components.

In analogy to Broder's r-Stirling numbers, we also count finite topologies restricted by TCMSOL-formulas with hard-wired constant symbols.

3.3 Topological model theory

In his retiring presidential address presented at the annual meeting of the Association for Symbolic Logic in Dallas, January 1973 Abraham Robinson suggested, among other topics, to develop a model theory for topological structures, [31]. This led to several approaches described in [19,22,28,35,36]. M. Ziegler introduced the logic L_t, which is a fragment of TFOL with the additional property that it is *basis-invariant* in the following sense:

Let ϕ be a formula of L_t and $\mathcal{T} = (X, \mathcal{U})$ be a (not necessarily finite) topological structure, and let $\mathcal{B}$ be any basis for $\mathcal{U}$. Then

$$\mathcal{T} = (X, \mathcal{U}) \models \phi \text{ iff } \mathcal{T} = (X, \mathcal{B}) \models \phi.$$

In fact in [19,35,36] X can be replaced by arbitrary first order structures. L_t now shares most model-theoretic characterstics of first-order logic, like compactness, Löwenheim-Skolem theorems, preservation theorems, etc. However, if we restrict the topological structures for L_t to be finite, this is not true anymore. A topological structure $\mathcal{T} = (X, \mathcal{U})$ is an *open substructure* of $\mathcal{T}' = (Y, \mathcal{V})$ if $X \subseteq Y$, $\mathcal{U} = \{A \subset X : A = A' \cap X, A' \in \mathcal{V}\}$ and $X \in \mathcal{V}$. We write $\mathcal{T} \subseteq_o \mathcal{T}'$. A formula $\phi \in L_t$ is *preserved under open extensions* if for every pair of topological structures $\mathcal{T} \subseteq_o \mathcal{T}'$ we have

$$\mathcal{T} \models \phi \text{ implies } \mathcal{T}' \models \phi.$$

In [19] there is also a syntactical characterization of the formulas preserved under open extensions. However, it fails if restricted to finite structures. Nevertheless, we can use this to show (here without proof):

Proposition 3.2. *Let $\phi \in L_t$ with r constant symbols (hard-wired or not) which has arbitrarily large finite models and is preserved under open extensions. Then $T_{\phi,r}(n)$ is not C-finite.*

4. Most general Specker-Blatter Theorem

In order to prove Theorem 1.2 for TCMSOL, we now state the Specker-Blatter Theorem for CMSOL with hard-wired constants, [16,17]. This generalizes substantially the original Specker-Blatter Theorem from 1981, but is still formulated for one-sorted relational structures for binary and unary relation symbols. Note that in [16,17] it is also shown that Theorem 4.1 does not hold for ternary relations.

Let τ be a finite set of binary and unary relation symbols. Let $Sp_{\phi,r}(n)$ be the number of labeled τ-structures on the set $[r+n]$ where the elements of $[r]$ are hard-wired and which satisfy the formula ϕ of CMSOL.

Theorem 4.1 (Most general Specker-Blatter Theorem)**.** *(i) $Sp_{\phi,r}(n)$ is MC-finite.*

(ii) For every $m \in \mathbb{N}^+$ the sequence $Sp^m_{\phi,r}(n) = Sp_{\phi,r}(n) \bmod m$ is computable in polynomial time. In fact it is in FPT *(Fixed Parameter Tractable) with parameters ϕ, r and m.*

For the technical details and the history of this theorem, the logically inclined reader should consult [16,17].

The theorem holds for a fixed number of hard-wired constants. It does not hold for of a hard-wired relation. A relation $R \subseteq [n]^s$ is hard-wired if R has a unique interpretation on $[n]^s$. We can view the natural order $NO \subseteq [n]^2$ as hard-wired. The number of equivalence relations (set partitions) on $[n]$ is given by the Bell numbers $B(n)$ which are MC-finite. They satisfy the hypothesis of the Specker-Blatter Theorem, as an equivalence relation is definable in FOL.

Proposition 4.1. *The Specker-Blatter Theorem does not hold in the presence a hard-wired linear order on $[n]$.*

Proof. Let A and B be two blocks of a partition of $[n]$. A and B are *crossing* if there are elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$ or $b_1 < a_1 < b_2 < a_2$. The number $B(n)^{nc}$ of non-crossing set partitions on $[n]$ is one of the interpretations of the Catalan numbers $C(n)$, [32], hence $C(n) = B(n)^{nc}$. Non-crossing set partitions are definable in FOL in the presence of the hard-wired natural order on $[n]$. But the Catalan numbers are not MC-finite. □

5. Proof of Theorem 1.2

5.1 Alexandroff's Theorem

Our next task is now to show how to reduce the theorem for TCMSOL to Theorem 4.1 of Section 4. Besides using Theorem 4.1 we use several classical facts about finite topologies, taken from [29].

First, we state a bijection theorem.

Theorem 5.1 (Alexandroff, 1931)**.** *There are bijections α and α' between finite topologies and finite quasi-orders and vice versa. Furthermore, they are the inverses of each other.*

Proof. Let $\mathcal{T} = (X, \mathcal{U}, E)$ be a finite topology. For $x \in X$ define U_x to be the intersection of all open sets which contain x. The sets U_x form a basis for $\mathcal{U}$. Define a relation $\leq_{\mathcal{U}}$ on the set X by $x \leq_{\mathcal{U}} y$ if $x \in U_y$ or, equivalently, $U_x \subseteq U_y$. Write $x <_{\mathcal{U}} y$ if the inclusion is proper. The relation $\leq_{\mathcal{U}}$ is transitive and reflexive, hence it defines for each $\mathcal{U}$ a unique quasi-order $\leq_{\mathcal{U}}$. We define $\alpha(\mathcal{T}) = (X, \leq_{\mathcal{U}})$.

Conversely, let $(X, \leq)$ be a quasi-order. The sets $U_x = \{y \in X : y \leq x\}$ form a basis for a topology $\mathcal{U}_{\leq}$. We define $\alpha'((X, \leq)) = (X, \mathcal{U}_{\leq})$. □

5.2 Translation schemes

The maps α and α' from Theorem 5.1 are actually definable in MSOL and TMSOL respectively by translation schemes as defined below. This is needed to construct an algorithm that translates a formula θ of TCMSOL into a formula $\theta^\sharp$ of CMSOL over quasi-orders such that for every finite quasi-order $\mathfrak{Q}$

$$\alpha'(\mathfrak{Q}) \models \theta \text{ iff } \mathfrak{Q} \models \theta^\sharp.$$

Conclusion 1. *The number of finite topologies on a set with n elements which satisfy θ equals the number of finite quasi-orders on a set with n elements which satisfy $\theta^\sharp$.*

We first show the definability of α. Let $\mathcal{T} = (X, \mathcal{U}, E)$ be a finite topology. We want to define inside $\mathcal{T}$ a quasi-order $\mathcal{Q} = (X, \leq)$. For this we exhibit two formulas $\phi(x)$ and $\phi_{\leq}(x, y)$ in TMSOL which form a translation scheme $\Phi = (\phi(x), \phi_{\leq}(x, y))$.

With Φ we associate two maps. $\Phi^\star$ and $\Phi^\sharp$. $\Phi^\star$ maps finite topological spaces into quasi-orders, and $\Phi^\sharp$ maps formulas of CMSOL into formulas of TCMSOL. Let $\mathcal{T}$ a finite topology and θ a formula of CMSOL for quasi-orders. These two maps satisfy the following:

$$\Phi^\star(\mathcal{T}) \models \theta \text{ iff } \mathcal{T} \models \Phi^\sharp(\theta)$$

and

$$\alpha(\mathcal{T}) = \Phi^\star(\mathcal{T}).$$

For α' we proceed similarly. Let $\mathcal{Q}$ be a quasi-order. We want to define inside $\mathcal{Q}$ a finite topology using formulas of CMSOL given by

$$\Psi = (\psi(x), \psi_{open}(U), U(x))$$

- $\psi(x) := x = x$ defines the universe of the topology.
- $\psi_{basic}(x, U) := (U(x) \wedge (\forall y(U(y) \leftrightarrow y \leq x))$ defines the basic open sets.
- $\psi_{open}(U) := \forall z(U(z) \leftrightarrow \exists V \subseteq U \exists y(\psi_{basic}(y, V) \wedge V(z))$ says that U is a union of basic open sets.

With Ψ we associate two maps. $\Psi^\star$ and $\Psi^\sharp$. $\Psi^\star$ maps finite quasi-orders into finite topologies, and $\Psi^\sharp$ maps formulas of TCMSOL into formulas of CMSOL. Let $\mathcal{Q}$ a finite quasi-order and θ a formula of TCMSOL. These two maps satisfy the following:

$$\Psi^\star(\mathcal{T}) \models \theta \text{ iff } \mathcal{T} \models \Psi^\sharp(\theta)$$

and

$$\alpha'(\mathcal{Q}) = \Psi^\star(\mathcal{Q}).$$

$\Psi^\sharp$ is defined inductively.

The atomic formulas of TCMSOL are $E(x, U)$, which is translated as $\psi_{open}(U) \wedge U(x)$, and $U(x)$, which is translated by $U(x)$. The translations commute with the boolean operations and quantifications of TCMSOL.

Let U_x be the intersection of all open sets U which contain x.

This can be expressed in TFOL by the formula $\phi_{U_x}(x, U)$ from the previous section. Now $x \leq y$ can be defined by $U_x \subseteq U_y$, which be expressed as

$$\phi_{\leq}(x, y) : \forall z \left[\exists U(E(z, U) \wedge \phi_{U_x}(x, U)) \to \exists V(E(z, V) \wedge \phi_{U_x}(y, V))\right]$$

The translation scheme $\Phi = (x = x, \phi_{\leq}(x, y))$ consists of two formulas. The first is a tautology in the free variable x and defines the new universe, which in this case is also X. The second formula, $\phi_{\leq}(x, y))$ defines the quasi-order.

Φ induces to maps, $\Phi^\star$ which maps finite topologies onto quasi-orders over the same universe, and $\Phi^\sharp$, which maps CMSOL-formulas into TCMSOL-formulas, by replacing each occurrence of $x_1 \leq x_2$ by $\phi_{\leq}(x_1, x_2)$.

Lemma 5.1. $\alpha(\mathcal{T}) = \Phi^\star(\mathcal{T})$.

In the other direction, let $\mathcal{Q} = (X, \leq)$ be a finite quasi-order. We want to define inside $\mathcal{Q}$ a topology $\mathcal{T} = (X, \mathcal{U}, E)$. We actually define a structure

$$\mathcal{T}' = (X, P(X), \mathcal{U}, \mathcal{B}, E, E_{top}, E_{basis})$$

where $P(A)$ is the powerset of A, and $\mathcal{B}$ is a minimal basis for the topolgy $\mathcal{U}$. E, E_{top}, E_{basis} are the membership relations for elements of X and $P(A), \mathcal{U}, \mathcal{B}$ respectively.

Again X can be defined by $x = x$, and $P(A)$ can be defined by $\phi_{set}(X) : \forall x(X(x) \leftrightarrow X(x))$, or, for that matter, by any appropriate tautology in one free variable.

The non-empty basic sets are defined by

$$\mathcal{B} = \{B \in P(X) : \exists x \forall y(y \in B \leftrightarrow (y \leq x)\}.$$

Hence we put

$$\phi_{basis}(B) : \exists x \forall y(E(y, B) \leftrightarrow (y \leq x).$$

Then the non-empty open sets are defined by

$$\mathcal{U} = \{U \in P(X) : \forall x(x \in U \leftrightarrow (\exists B \in \mathcal{B}(x \in B \wedge B \subseteq U)))\}.$$

Hence we put

$$\phi_{top}(U) : \forall x(E(x, U) \leftrightarrow (\exists B \in \mathcal{B}(x \in B \wedge B \subseteq U))).$$

The translation scheme is now defined by

$$\Psi = (x = x, \phi_{set}(X), \phi_{top}(U), \phi_{basis}(B), X(x)).$$

Ψ induces to maps, $\Psi^\star$, which maps finite quasi-order onto topologies over the same underlying set. and $\Psi^\sharp$, which transforms TCMSOL-formulas by replacing each occurrence of $E(x, X), E(x, U), E(x, B)$ by its definitions.

Lemma 5.2. $\alpha'(\mathcal{Q}) = \Psi(\mathcal{Q})$.

Theorem 5.2. *The translation schemes Φ and Ψ satisfy the following:*

(i) $\Phi^\star(\mathcal{T}) = \alpha(\mathcal{T})$ *and* $\Psi^\star(\mathcal{Q}) = \alpha'(\mathcal{Q})$*;*

(ii) for every $\theta \in \mathrm{CMSOL}$ *and every finite quasi-order* $\mathcal{Q}$*,*

$$\alpha(\mathcal{T}) = \mathcal{Q} \models \theta \text{ iff } \mathcal{T} \models \Phi^{\sharp}(\theta).$$

(iii) for every $\sigma \in \mathrm{TCMSOL}$ *and every finite topology* $\mathcal{T}$*,*

$$\alpha'(\mathcal{Q}) = \mathcal{T} \models \sigma \text{ iff } \mathcal{Q} \models \Psi^{\sharp}(\sigma).$$

In [16, 17] the following is shown:

Theorem 5.3. *Let* $\theta(a_1, \ldots, a_r)$ *be a sentence in* CMSOL *with* r *constant symbols, and let* $S(n) = S_{\theta(a_1,\ldots,a_r)}(n)$ *be the number of relations* $R \subseteq [n]^2$*, such that*

$$([n], R, (a_1, \ldots, a_r)) \models \theta(a_1, \ldots, a_r).$$

In both cases, where the constant symbols are hard-wired or not, S *is MC-finite.*

Let $\sigma(a_1, \ldots, a_r)$ be a sentence in TCMSOL with r constant symbols, and let $S^t(n) = S^t_{\sigma(a_1,\ldots,a_r)}(n)$ be the number of topologies on $[n]$ such that

$$([n], \mathcal{U},\ (a_1, \ldots, a_r)) \models \sigma(a_1, \ldots, a_r).$$

and let $\hat{S}^t(n) = S^t_{\neg\sigma(a_1,\ldots,a_r)}(n)$ be the number of topologies on $[n]$ such that

$$([n], \mathcal{U},\ (a_1, \ldots, a_r)) \models \neg\sigma(a_1, \ldots, a_r).$$

Theorem 5.4. *In both cases, whether the constant symbols are hard-wired or not,*

(i) $S^t(n)$ *and* $\hat{S}^t(n)$ *are MC-finite, but*

(ii) at least one of them is not C-finite.

6. Conclusions

We have shown that counting finite topologies $T_{\phi,r}(n)$ on a set of n elements subject to restrictions with a fixed finite number r of (hard-wired) constants expressed by $\phi \in \mathrm{TCMSOL}$ is an MC-finite sequence. The underlying set X of the topology can be equipped with unary and binary relations, but counting now also counts the number of their interpretations. As we have seen in Section 4 the Specker-Blatter Theorem does not work for hard-wired relations.

No explicit formulas for $T(n)$ and $T_0(n)$ are known, but we have

$$T(n) = \sum_{k=0}^{n} S(n,k) \cdot T_0(k),$$

where $S(n, k)$ is the Stirling number of the second kind.

Problem 1. *Find better descriptions of* $T(n)$ *and* $T_0(n)$*.*

The logic TCMSOL can express many topological properties. Here are some possibly challenging test problems.

In analogy to counting various set partitions on $[n]$ as described in [13], one can look at topological set partitions of finite topological spaces on $[n]$ such that each block is a connected component, an open, or a closed set. Note that the topology in a topological set partition is not hard-wired. We count the set partitions of X and the topologies separately. Given $[n]$ the Bell numbers $B(n)$ count the partitions of $[n]$ and for each such partition we count the number of topologies such that the blocks are connected, open or closed. The Stirling numbers of the second kind $S(n, k)$ count the partitions of $[n]$ into k blocks, and again we can require that the blocks are connected, open or closed.

Similarly, we can look at topological spaces on $[r + n]$ such that the hard-wired constants of $[r]$ are all in different blocks. Like with the Bell and Stirling numbers, the number of non-empty blocks may be arbitrary or fixed to k blocks. If the topology is assumed to be discrete, we get the restricted Bell numbers $B_r(n)$ and Stirling numbers of the second kind $S_r(n, k)$ from [7]. Otherwise, these give different topological versions of $B_r(n)$ and $S_r(n, k)$.

Problem 2. *What can we say beyond that the number of such topological set partitions form an MC-finite sequence? Are some of them C-finite?*

References

[1] J. A. Barmak, *Algebraic topology of finite topological spaces and applications*, Volume 2032, Springer, 2011.

[2] B. Bényi, M. Méndez, J. L. Ramírez, and T. Wakhare, *Restricted r-Stirling numbers and their combinatorial applications*, Appl. Math. Comput. 348 (2019), 186–205.

[3] D. Berend and T. Tassa, *Improved bounds on Bell numbers and on moments of sums of random variables*, Prob. Math. Stat. 30:2 (2010), 185–205.

[4] C. Blatter and E. Specker, *Le nombre de structures finies d'une théorie á charactère fini*, Sciences Mathématiques, Fonds Nationale de la recherche Scientifique, Bruxelles, 1981, 41–44.

[5] C. Blatter and E. Specker, *Recurrence relations for the number of labeled structures on a finite set*, In E. Börger and G. Hasenjaeger and D. Rödding, editors, In Logic and Machines: Decision Problems and Complexity, Lecture Notes in Computer Science 171, Springer, 1984, 43–61.

[6] J. Bourgain, V. Milman, and H. Wolfson, *On type of metric spaces*, Trans. Amer. Math. Soc. 294(1) (1986), 295–317.

[7] A. Z. Broder, *The r-Stirling numbers*, Discrete Math. 49(3) (1984), 241–259.

[8] K. K.-H. Butler and G. Markowsky, *Enumeration of finite topologies*, In Proc. 4th SE Conf. Combin., Graph Theory, Computing, Congress. Numer, Volume 8, 1973, 169–184.

[9] F. Chazal and B. Michel, *An introduction to topological data analysis: fundamental and practical aspects for data scientists*, Frontiers in artificial intelligence 4 (2021), 667963.

[10] N. G. De Bruijn, *Asymptotic methods in analysis*, Colume 4. Courier Corporation, 1981.

[11] W. H. F. Evans and M. S. Lynn, *On the computer enumeration of finite topologies*, Communications of the ACM 10 (1967), 295–297.

[12] G. Everest, A. J. van der Poorten, I. Shparlinski, T. Ward, et al., *Recurrence sequences*, Volume 104, American Mathematical Society Providence, RI, 2003.

[13] Y. Filmus, E. Fischer, J. A. Makowsky, and V. Rakita, *MC-finiteness of restricted set partitions*, J. Int. Seq. 26 (2023), Article 23.7.4.

[14] E. Fischer, T. Kotek, and J. A. Makowsky, *Application of logic to combinatorial sequences and their recurrence relations*, Model Theoretic Methods in Finite Combinatorics 558 (2011), 1–42.

[15] E. Fischer and J. A. Makowsky, *Linear recurrence relations for graph polynomials*, In Pillars of computer science, Springer, 2008, 266–279.

[16] E. Fischer and J. A. Makowsky, *Extensions and limits of the Specker-Blatter theorem*, J. Symb. Logic 89 (2024).

[17] E. Fischer and J. A. Makowsky, *Extensions and limits of the Specker-Blatter theorem*, In 32nd EACSL Annual Conference on Computer Science Logic (CSL 2024), Schloss-Dagstuhl-Leibniz Zentrum für Informatik, 2024.

[18] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University press, 2009.

[19] J. Flum and M. Ziegler, *Topological model theory*, Volume 769 of Lecture Notes in Mathematics, Springer, 1980.

[20] OEIS Foundation, *The on-line encyclopedia of integer sequences*, `https://oeis.org/`.

[21] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: a foundation for computer science*, Addison-Wesley, 1989.

[22] J. A. Makowsky and M. Ziegler, *Topological model theory with an interior operator: Consistency properties and back—and forth arguments*, Archiv für mathematische Logik und Grundlagenforschung 21(1) (1981), 37–54.

[23] M. Kauers and Paule, *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates*, Springer, 2011.

[24] D. Kleitman and B. Rothschild, *The number of finite topologies*, Proc. Amer. Math. Soc. 25(2) (1970), 276–282.

[25] T. Koshy, *Catalan numbers with applications*, Oxford University Press, 2008.

[26] V. A. Kovalevsky, *Finite topology as applied to image analysis*, Computer vision, graphics, and image processing 46(2) (1989), 141–161.

[27] N. Linial, *Finite metric spaces–combinatorics, geometry and algorithms*, Proceedings of the eighteenth annual symposium on Computational geometry, 2002, 63. See also arXiv preprint math/0304466.

[28] J. A. Makowsky, *Topological model theory*, In P. Mangani, editor, Model Theory and Applications, Edizione Cremonese, Rome, 1975, 121–150.

[29] J. P. May, *Finite topological spaces*, Notes for REU, 2003.

[30] J. A. Reeds and N. J. A. Sloane, *Shift register synthesis (modulo m)*, SIAM Journal on Computing 14(3) (1985), 505–513.

[31] A. Robinson, *Metamathematical problems*, J. Symb. Logic 38(3) (1973), 500–516.

[32] S. Roman et al., *An introduction to Catalan numbers*, Springer, 2015.

[33] A. Shafaat, *On the number of topologies definable for a finite set*, J. Australian Math. Soc. 8 (1968), 194–198.

[34] E. Specker, *Application of logic and combinatorics to enumeration problems*, In E. Börger, editor, Trends in Theoretical Computer Science, 141–169, Computer Science Press, 1988. Reprinted in: Ernst Specker, Selecta, Birkhäuser 1990, 324–350.

[35] M. Ziegler, *Topological model theory*, In Model-Theoretic Logics, Perspectives in Mathematical Logic, Chapter 15, Springer Verlag, 1985.

[36] M. Ziegler, *A language for topological structures which satisfies a lindström-theorem*, Bull. Amer. Math. Soc. (1976), 568–570.