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k-Non-crossing Trees and Edge Statistics Modulo k

Helmut Prodinger[†]

[†] *Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa and NITheCS (National Institute for Theoretical and Computational Sciences), South Africa*
Email: hproding@sun.ac.za

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Abstract: Instead of k-Dyck paths we consider the equivalent concept of k-non-crossing trees. This is our preferred approach relative to down-step statistics modulo k (first studied by Heuberger, Selkirk, and Wagner by different methods). One symmetry argument about subtrees is needed and the rest goes along the lines of a paper by Flajolet and Noy.

Keywords: Butterflies; k-Dyck paths; Non-crossing trees
2020 Mathematics Subject Classification: 05A15

1. Non-crossing trees revisited

Assume that the nodes $1, \ldots, n$ are arranged in a circle, call node 1 the root, and draw a tree using line segments such that no crossings occur. These objects are called *non-crossing trees*. We only cite [3] and our own [8], but there is much more literature that is not difficult to find. Every node except for the root has two types of successors: left ones and right ones. See [3, 8]. Sometimes this is drawn as two trees that share a root node ('butterfly'); corresponding drawings are found in many papers on the subject.

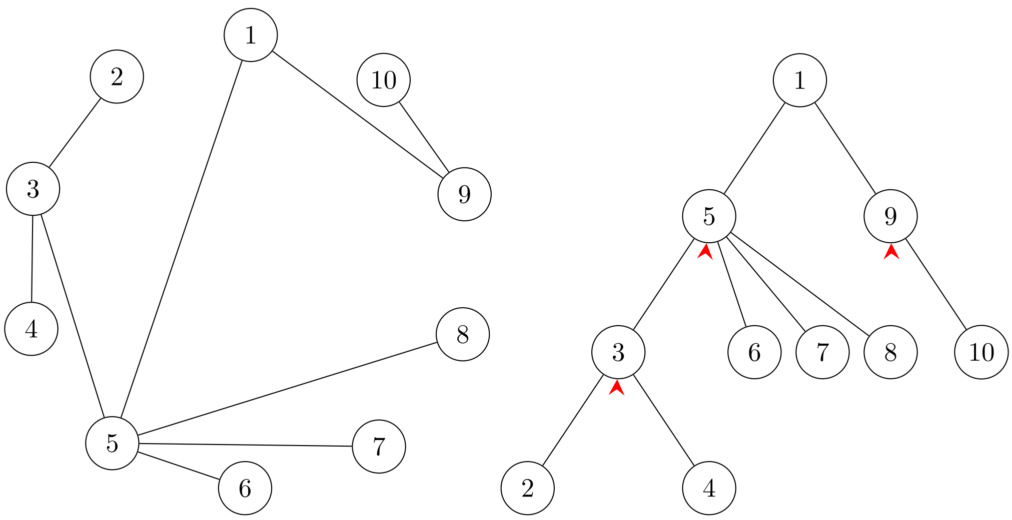


Figure 1: A non-crossing tree with 10 nodes and separators indicating where the non-root nodes split into the left part and the right part.

It is interesting to note that *even* trees as in [1] are a similar concept to non-crossing trees.

We enumerate non-crossing trees with n nodes and j left and $n-1-j$ right edges. Clearly, the total number of edges is $n-1$. As one can see, the distribution isn't fair, as the root has all these right edges as successors. We will use several variables: z for the number of nodes and ℓ and r for the two types of edges. We will use the butterfly decomposition due to Flajolet and Noy [3].

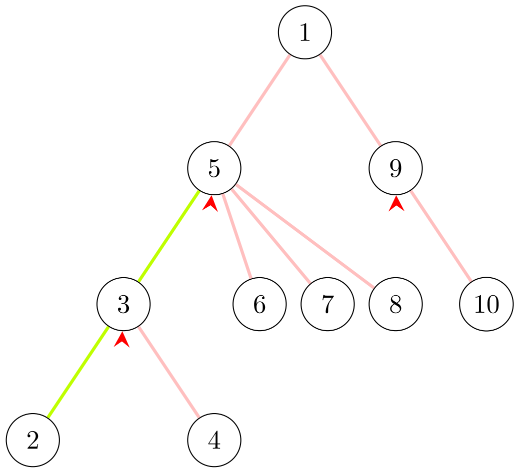


Figure 2: Left resp. right edges are depicted in different colors; by design, the edges emanating from the root are all right edges.

$$T = \frac{z}{1-B}, \quad B = \frac{T^2}{z};$$

T stands for tree and B, which is only an auxiliary quantity, for butterfly. We prefer to use the letter F instead of T. However, because of the anomaly of the root, we temporarily make B the center of interest:

$$B = \frac{F^2}{z} = \frac{z}{(1-B)^2}.$$

Using the substitution $z = v(1-v)^2$, this can be solved, and the relevant solution is just $B = v$, and further $\frac{F}{z} = \frac{1}{1-v}$. This can be extended with our extra variables ℓ and r:

$$B = \frac{z}{(1-\ell B)(1-rB)},$$

then $z = v(1-\ell v)(1-rv)$ and $B = v$ and $\frac{F}{z} = \frac{1}{1-rB} = \frac{1}{1-rv}$. Now we read off coefficients:

$$\begin{aligned}
[z^n]F &= [z^{n-1}]\frac{F}{z} = [z^{n-1}]\frac{1}{1-rv} = \frac{1}{n-1}[z^{n-2}]\frac{d}{dz}\frac{1}{1-rv} \\
&= \frac{1}{n-1}[z^{n-2}]\frac{dv}{dz}\frac{d}{dv}\frac{1}{1-rv} \\
&= \frac{1}{n-1}[z^{n-2}]\frac{1}{1-2\ell v-2rv+3\ell rv^2}\frac{r}{(1-rv)^2} \\
&= \frac{1}{n-1}\frac{1}{2\pi i}\oint\frac{dz}{z^{n-1}}\frac{1}{1-2\ell v-2rv+3\ell rv^2}\frac{r}{(1-rv)^2} \\
&= \frac{1}{n-1}\frac{1}{2\pi i}\oint\frac{dv}{v^{n-1}(1-\ell v)^{n-1}(1-rv)^{n-1}}\frac{r}{(1-rv)^2} \\
&= \frac{r}{n-1}[v^{n-2}]\frac{1}{(1-\ell v)^{n-1}(1-rv)^{n+1}}.
\end{aligned}$$

As we can see, it is unnecessary to explicitly compute $\frac{dv}{dz}$ as it cancels out anyway. This will be very beneficial in the following sections. Furthermore,

$$\begin{aligned}
[z^n\ell^j r^{n-1-j}]F &= [v^{n-2}\ell^j r^{n-2-j}]\frac{1}{n-1}\frac{1}{(1-\ell v)^{n-1}(1-rv)^{n+1}} \\
&= \frac{1}{n-1}\binom{n-2-j}{j}\binom{2n-2-j}{n-2-j}.
\end{aligned}$$

This is the number of non-crossing trees with n nodes, j left edges, and $n-1-j$ right edges.

Recently, I detected a paper [7] with a similar title to ours; otherwise, there were not too many similarities.

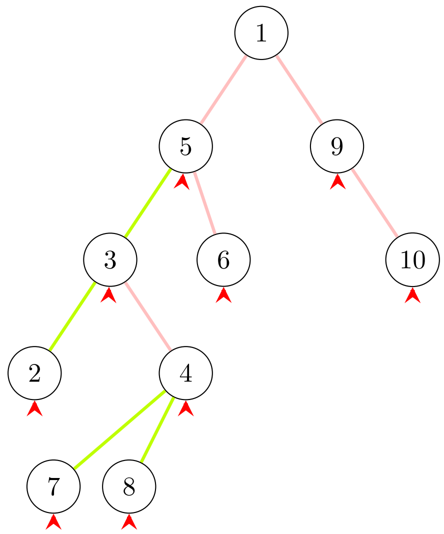
Another paper of interest was pointed out by a referee: [6], which has a functional equation for a trivariate generating function related to descents in non-crossing trees, as well as some bijections.

2. An application

Lattice paths and certain types of trees are intimately related, and sometimes it is easier to analyze the trees instead of the paths, an example being [2, 9]. This will also happen here, as we will use the analysis of non-crossing trees from the introductory section to lattice paths.

We transform non-crossing trees into so-called 2-Dyck paths: Up-steps $(1, 1)$ are as usual, but there are down-steps $(1, -2)$ of two units. Otherwise, the path must be non-negative and eventually return to the x-axis. For this transformation, we walk around the tree and translate down-steps into up-steps and vice versa. However, we need extra up-steps to keep the balance. For that, we use the separators, and also draw them for end-nodes, so that there are $n - 1$ such separator markers present. Then, whenever we meet one, we also make an up-step.

In the example, we get the path in Figure 3.



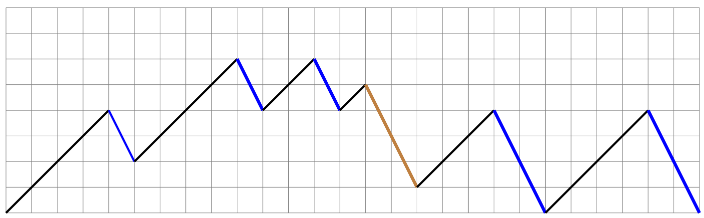


Figure 3: A non-crossing tree and the corresponding 2-Dyck path.

Note that n nodes of the tree correspond to $n - 1$ down-steps. More such considerations can be found in [4, 10, 11].

The goal is to match the brown down-steps to the left edges, say. In particular, the interest is, on which level modulo k they land (or, equivalently, start). First, the tree needs to be modified. The reason is this decomposition in Figure 5. Indeed, T_1 "sits" on level 1 (odd) but a subtree of T_1 "sits" on level 2 (even). So we need to swap subtrees in such a case. The next section will provide more details. Physically, it is not necessary to swap subtrees, all that needs to be controlled is how the formal variables ℓ, m, r are attached to the subtrees.

3. Generalization

Instead of down-steps of two units and one separator, this works as well for down-steps $(1, -k)$ and $k - 1$ separators. Here (Figure 6) is a 3-Dyck path: 6 down-steps land on level 0 (mod 3), 1 on level 1 (mod 3), and 3 on level 2 (mod 3). The butterfly equation is

$$B = \frac{F^k}{z} = \frac{z}{(1 - r_1 B) \ldots (1 - r_k B)}$$

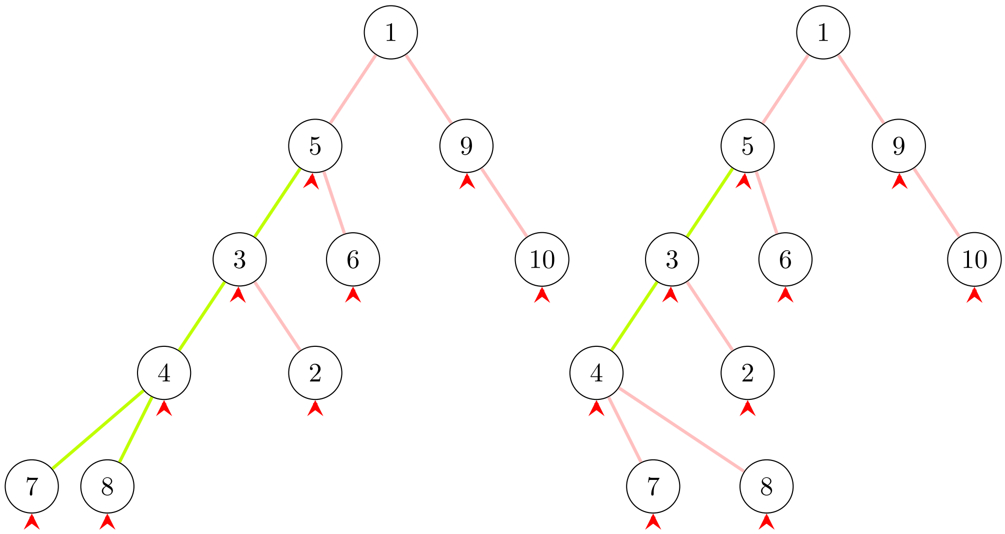


Figure 4: Transforming the tree. The number of green edges corresponds to the brown down-steps.

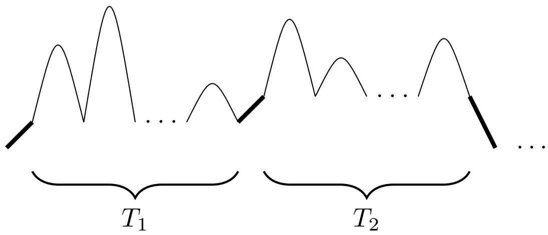


Figure 5: The decomposition of a 2-Dyck path.

with variables $r_1, \dots, r_k$ to count the down-steps ending (or beginning) on a level $\equiv i \pmod k$. Eventually $F = \frac{z}{1-r_k B}$. For the solution, the substitution $z = v(1-r_1 v)\dots(1-r_k v)$ works, and $B = v$, and thus $\frac{F}{z} = \frac{1}{1-r_k v}$.

Reading off coefficients is similar to the previous case $k = 2$:

$$\begin{aligned}
[z^n]F &= [z^{n-1}]\frac{F}{z} = \frac{1}{n-1}[z^{n-2}]\frac{d}{dz}\frac{F}{z} \\
&= \frac{1}{n-1}[z^{n-2}]\frac{dv}{dz}\frac{d}{dv}\frac{F}{z} = \frac{1}{n-1}[z^{n-2}]\frac{dv}{dz}\frac{r_k}{(1-r_k v)^2} \\
&= \frac{1}{n-1}\frac{1}{2\pi i}\oint \frac{dz}{z^{n-1}}\frac{dv}{dz}\frac{r_k}{(1-r_k v)^2} \\
&= \frac{1}{n-1}\frac{1}{2\pi i}\oint \frac{dv}{v^{n-1}(1-r_1 v)^{n-1}\dots(1-r_k v)^{n-1}}\frac{r_k}{(1-r_k v)^2} \\
&= \frac{1}{n-1}[v^{n-2}]\frac{r_k}{(1-r_1 v)^{n-1}\dots(1-r_{k-1} v)^{n-1}(1-r_k v)^{n+1}}.
\end{aligned}$$

Furthermore (with $a_1 + \dots + a_k = n-1$)

$$\begin{aligned}
[z^n r_1^{a_1}\dots r_k^{a_k}]F &= \frac{1}{n-1}[v^{n-2} r_1^{a_1}\dots r_k^{a_k}]\frac{r_k}{(1-r_1 v)^{n-1}\dots(1-r_{k-1} v)^{n-1}(1-r_k v)^{n+1}} \\
&= \frac{1}{n-1}[v^{n-2} r_1^{a_1}\dots r_k^{a_k-1}]\frac{1}{(1-r_1 v)^{n-1}\dots(1-r_{k-1} v)^{n-1}(1-r_k v)^{n+1}} \\
&= \frac{1}{n-1}\binom{n-2+a_1}{a_1}\dots\binom{n-2+a_{k-1}}{a_{k-1}}\binom{n-1+a_k}{a_k-1}.
\end{aligned}$$

This is the formula in Theorem 6 in [5], for $t = 0$, and $n \to n+1$. For more general $-k < -t \le 0$ ($\Leftrightarrow 0 \le t < k$), we will work this out in the next section.

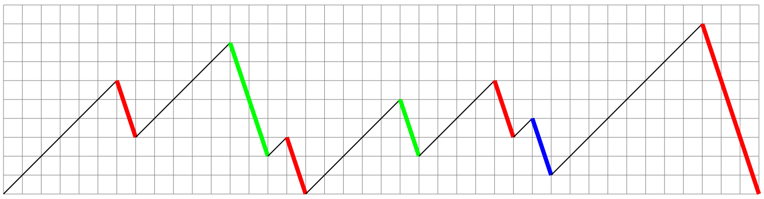
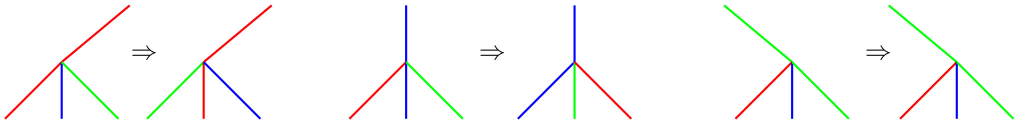


Figure 6: A 3-Dyck path. 6 down-steps land on level 0 (mod 3), 1 on level 1 (mod 3), and 3 on level 2 (mod 3)

The modification of the tree (rotation of the subtrees, from top to bottom, so that the down-step enumeration matches the edge enumeration) is as follows (for $k = 3$)



It is easy to figure out how this works more generally for k successors instead of 3. It is always a cyclic shift, by $k-1, k-2, \dots, 0$ positions, depending on the edge we are considering.

The following differentiation will be used in the sequel. It is just the differentiation of a product, as usual.

$$\frac{d}{dv}\prod_{j=k-t}^{k}\frac{1}{1-r_jv} = \prod_{j=k-t}^{k}\frac{1}{1-r_jv}\cdot\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}.$$

Our strategy is to bijectively map k-Dyck paths bounded below by $y = -t$ into $t+1$ k-non-crossing trees of altogether $n-t-1$ edges and the special symbol attached to the root varies from r_k, r_{k-1} ... to r_{k-t}. Figure 7 shows a small example and more examples are in [10]. Then

$$\begin{aligned}
[z^n]F &= [z^{n-t-1}]\frac{F}{z^{t+1}} = [z^{n-t-1}]\prod_{j=k-t}^{k}\frac{1}{1-r_jv} = \frac{1}{n-t-1}[z^{n-t-2}]\frac{d}{dz}\prod_{j=k-t}^{k}\frac{1}{1-r_jv}\\
&= \frac{1}{n-t-1}[z^{n-t-2}]\frac{dv}{dz}\prod_{j=k-t}^{k}\frac{1}{1-r_jv}\cdot\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}\\
&= \frac{1}{n-t-1}\frac{1}{2\pi i}\oint\frac{dz}{z^{n-t-1}}\frac{dv}{dz}\prod_{j=k-t}^{k}\frac{1}{1-r_jv}\cdot\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}\\
&= \frac{1}{n-t-1}\frac{1}{2\pi i}\oint\frac{dv}{v^{n-t-1}(1-r_1v)^{n-t-1}\dots(1-r_kv)^{n-t-1}}\prod_{j=k-t}^{k}\frac{1}{1-r_jv}\cdot\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}\\
&= \frac{1}{n-t-1}\frac{1}{2\pi i}\oint\frac{dv}{v^{n-t-1}\prod_{h=1}^{k-t-1}(1-r_hv)^{n-t-1}\prod_{\ell=k-t}^{k}(1-r_\ell v)^{n-t}}\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}\\
&= \frac{1}{n-t-1}[v^{n-t-2}]\frac{1}{\prod_{h=1}^{k-t-1}(1-r_hv)^{n-t-1}\prod_{\ell=k-t}^{k}(1-r_\ell v)^{n-t}}\sum_{i=k-t}^{k}\frac{r_i}{1-r_iv}.
\end{aligned}$$

Furthermore ($a_1 + \dots + a_k = n-t-1$)

$$\begin{aligned}
&[z^nr_1^{a_1}\dots r_k^{a_k}]F\\
&\quad= \frac{1}{n-t-1}[v^{n-t-2}r_1^{a_1}\dots r_k^{a_k}]\sum_{i=k-t}^{k}\frac{r_i}{\prod_{h=1}^{k-t-1}(1-r_hv)^{n-t-1}\prod_{\ell=k-t}^{k}(1-r_\ell v)^{n+[i=\ell]-t}}\\
&\quad= \frac{1}{n-t-1}\sum_{i=k-t}^{k}\prod_{h=1}^{k-t-1}\binom{n-t-2+a_h}{a_h}\prod_{\ell=k-t}^{k}\binom{n+[i=\ell]-t-1+a_\ell-[i=\ell]}{a_\ell-[i=\ell]}\\
&\quad= \frac{1}{n-t-1}\sum_{i=k-t}^{k}\prod_{h=1}^{k-t-1}\binom{n-t-2+a_h}{a_h}\prod_{\ell=k-t}^{k}\binom{n-t-1+a_\ell}{a_\ell-[i=\ell]}\\
&\quad= \frac{1}{n-t-1}\sum_{i=k-t}^{k}\frac{a_i}{n-t}\prod_{h=1}^{k-t-1}\binom{n-t-2+a_h}{a_h}\prod_{\ell=k-t}^{k}\binom{n-t-1+a_\ell}{a_\ell}.
\end{aligned}$$

The formula looks better when $n-t-1=N$; then it compares and matches with the formula from [5]

$$\frac{a_{k-t}+\cdots+a_k}{N(N+1)}\prod_{h=1}^{k-t-1}\binom{N-1+a_h}{a_h}\prod_{\ell=k-t}^{k}\binom{N+a_\ell}{a_\ell}.$$

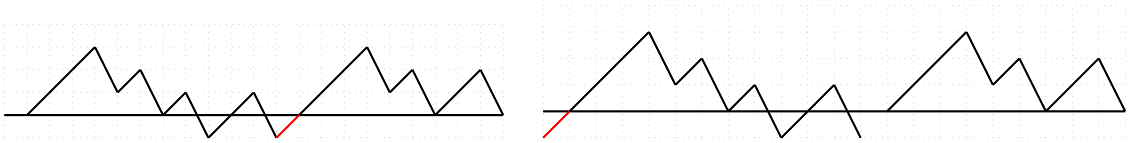


Figure 7: Decomposition of paths bounded by the line $y=-1$ into two paths.

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