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A Symmetric Chain Decomposition of $L(5, n)$

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ABSTRACT: We provide a constructive proof that Young's lattice $L(5, n)$ can be partitioned into saturated symmetric chains.

Keywords: Poset; Symmetric chain decomposition; Young's lattice

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1. Introduction

For positive integers m and n, *Young's lattice* $L(m, n)$ refers to the partially ordered set (*poset*) of m-tuples $(a_1, a_2, \cdots, a_m)$, where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n$, with the ordered relation $(a_1, a_2, \cdots, a_m) \leq (b_1, b_2, \cdots, b_m)$ if $a_i \leq b_i$ for all $i = 1, 2, \cdots, m$.

The *rank* of $\overrightarrow{a} = (a_1, a_2, \cdots, a_m)$ is defined as $rank(\overrightarrow{a}) = \sum_{i=1}^{m} a_i$, and a chain $\overrightarrow{v_1} < \overrightarrow{v_2} < \cdots < \overrightarrow{v_k}$ in $L(m, n)$ is called *saturated* if it skips no ranks and is called *symmetric* if $rank(\overrightarrow{v_1}) + rank(\overrightarrow{v_k}) = mn$. A symmetric chain decomposition (*SCD*) of a poset is a way to express it as a disjoint union of saturated symmetric chains.

Stanley [3] conjectured that Young's lattice $L(m, n)$ has an SCD, and various SCDs for $L(4, n)$ and $L(3, n)$ have been constructed ([5], [1] and [4]). One of the major problems in order theory is the explicit construction of SCDs for Young's Lattice for all m and n. In 1989, Kathy O'Hara ([2], see also [6]) constructed SCDs for the trivial extension of $L(m, n)$, in which all partitions of one rank are related to the next. However, the problem remains wide open for Young's lattice itself.

There are more than 30 types of symmetric chains found in the SCD of $L(5, n)$. To present the information neatly and clearly, we focus on nine types of parallel chains in Section 2. In Section 3, we illustrate how to obtain symmetric chains from these parallel chains, and finally, we provide a computer proof in Section 4.

2. A list of disjoint parallel chains of L(5, n)

Here are nine types of disjoint chains of $L(5, n)$ from which an SCD of $L(5, n)$ could be derived. We refer to these chains as *parallel* chains because, with varying parameters p and q while keeping other parameters fixed, they can form rectangles (Section 3).

Parameters i, j, k, u, p, q, and w in these tables are all non-negative integers. One line of vertical dots in these tables represents an abbreviated saturated chain with only one entry increasing. Two lines of vertical dots (in tables C_3, C_4, C_5, and C_6) represent an abbreviated saturated chain with a zigzag path where two entries increase by 1 alternatively. There are two helper rows with an extra parameter t, which serves as an increasing iterator, in the zigzag path. For example, in chain C_3, the first and third entries increase by 1 alternatively at the beginning. It is also worth mentioning that when $n = 2u + 7 + 6j + 4k + 3i$, it is a special case of table 6 that does not include the first segment of the table. For $n = 2u + 7 + 6j + 4k + 3i$, parallel chains start from the darkened row of table 6.

$2+2i+2j+3k \le n$				
$(p,$	$k,$	$j+k,$	$1+i+j+2k,$	$1+i+2j+2k)$
		$\vdots$		
$(p,$	$k,$	$1+i+j+k+p,$	$1+i+j+2k,$	$1+i+2j+2k)$
	$\vdots$			
$(p,$	$1+i+j+k,$	$1+i+j+k+p,$	$1+i+j+2k,$	$1+i+2j+2k)$
$\vdots$				
$(1+i+k,$	$1+i+j+k,$	$1+i+j+k+p,$	$1+i+j+2k,$	$1+i+2j+2k)$
				$\vdots$
$(1+i+k,$	$1+i+j+k,$	$1+i+j+k+p,$	$1+i+j+2k,$	$-k+n+p)$
			$\vdots$	
$(1+i+k,$	$1+i+j+k,$	$1+i+j+k+p,$	$-k+n,$	$-k+n+p)$
		$\vdots$		
$(1+i+k,$	$1+i+j+k,$	$-j-k+n,$	$-k+n,$	$-k+n+p)$
	$\vdots$			
$(1+i+k,$	$-1-i-j-2k+n,$	$-j-k+n,$	$-k+n,$	$-k+n+p)$
$\vdots$				
$(-1-i-2j-2k+n,$	$-1-i-j-2k+n,$	$-j-k+n,$	$-k+n,$	$-k+n+p)$

Table 1: Parallel chains C_1, $0 \le p \le k$

$3+2i+2j+3k \le n$				
$(k-p,$	$k,$	$j+k,$	$1+i+j+2k,$	$2+i+2j+2k)$
				$\vdots$
$(k-p,$	$k,$	$j+k,$	$1+i+j+2k,$	$-1-i-k+n)$
			$\vdots$	
$(k-p,$	$k,$	$j+k,$	$-1-i-j-k+n,$	$-1-i-k+n)$
		$\vdots$		
$(k-p,$	$k,$	$-1-i-j-k+n-p,$	$-1-i-j-k+n,$	$-1-i-k+n)$
	$\vdots$			
$(k-p,$	$-1-i-j-2k+n,$	$-1-i-j-k+n-p,$	$-1-i-j-k+n,$	$-1-i-k+n)$
$\vdots$				
$(-1-i-2j-2k+n,$	$-1-i-j-2k+n,$	$-1-i-j-k+n-p,$	$-1-i-j-k+n,$	$-1-i-k+n)$
				$\vdots$
$(-1-i-2j-2k+n,$	$-1-i-j-2k+n,$	$-1-i-j-k+n-p,$	$-1-i-j-k+n,$	$n-p)$
			$\vdots$	
$(-1-i-2j-2k+n,$	$-1-i-j-2k+n,$	$-1-i-j-k+n-p,$	$-k+n,$	$n-p)$
		$\vdots$		
$(-1-i-2j-2k+n,$	$-1-i-j-2k+n,$	$-1-j-k+n,$	$-k+n,$	$n-p)$

Table 2: Parallel chains C_2, $0 \le p \le k$

		$0 \le u \le 1$ $2u + 1 + 6j + 4k + 3i \le n$ $2 \le t \le i - 1$		
$(2j,$	$i + 2j + p,$	$i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$(1 + 2j,$	$i + 2j + p,$	$i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$(1 + 2j,$	$i + 2j + p,$	$1 + i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$\vdots$		$\vdots$		
$(t + 2j,$	$i + 2j + p,$	$t - 1 + i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$(t + 2j,$	$i + 2j + p,$	$t + i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$\vdots$		$\vdots$		
$(i + 2j,$	$i + 2j + p,$	$2i - 1 + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
$(i + 2j,$	$i + 2j + p,$	$2i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$2i + 4j + 4k + 2u)$
				$\vdots$
$(i + 2j,$	$i + 2j + p,$	$2i + 2j + 2k + u,$	$2i + 4j + 2k + u,$	$-2j + n)$
			$\vdots$	
$(i + 2j,$	$i + 2j + p,$	$2i + 2j + 2k + u,$	$-i - 2j - 2k + n + p - u,$	$-2j + n)$
		$\vdots$		
$(i + 2j,$	$i + 2j + p,$	$-i - 2j - 2k + n - u,$	$-i - 2j - 2k + n + p - u,$	$-2j + n)$
	$\vdots$			
$(i + 2j,$	$-2i - 4j - 2k + n - u,$	$-i - 2j - 2k + n - u,$	$-i - 2j - 2k + n + p - u,$	$-2j + n)$
$\vdots$				
$(-2i - 4j - 4k + n - 2u,$	$-2i - 4j - 2k + n - u,$	$-i - 2j - 2k + n - u,$	$-i - 2j - 2k + n + p - u,$	$-2j + n)$

Table 3: Parallel chains C_3, $0 \le p \le 2k + u$

		$0 \le u \le 1$ $2u + 4 + 6j + 4k + 3i \le n$ $2 \le t \le i - 1$		
$(1 + 2j,$	$1 + i + 2j + p,$	$1 + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$(1 + 2j,$	$1 + i + 2j + p,$	$2 + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$\vdots$		$\vdots$		
$(t + 2j,$	$1 + i + 2j + p,$	$t + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$(t + 2j,$	$1 + i + 2j + p,$	$1 + t + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$\vdots$		$\vdots$		
$(i + 2j,$	$1 + i + 2j + p,$	$2i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$(i + 2j,$	$1 + i + 2j + p,$	$1 + 2i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
$(1 + i + 2j,$	$1 + i + 2j + p,$	$1 + 2i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
				$\vdots$
$(1 + i + 2j,$	$1 + i + 2j + p,$	$1 + 2i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$-1 - 2j + n)$
			$\vdots$	
$(1 + i + 2j,$	$1 + i + 2j + p,$	$1 + 2i + 2j + 2k + u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$
		$\vdots$		
$(1 + i + 2j,$	$1 + i + 2j + p,$	$-1 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$
	$\vdots$			
$(1 + i + 2j,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$
$\vdots$				
$(-3 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$

Table 4: Parallel chains C_4, $0 \le p \le 2k + u$

		$0 \le u \le 1$ $2u + 4 + 6j + 4k + 3i \le n$ $1 \le t \le i - 2$		
$(2j,$	$1 + i + 2j + p,$	$1 + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$3 + 2i + 4j + 4k + 2u)$
				$\vdots$
$(2j,$	$1 + i + 2j + p,$	$1 + i + 2j + 2k + u,$	$2 + 2i + 4j + 2k + u,$	$-1 - i - 2j + n)$
			$\vdots$	
$(2j,$	$1 + i + 2j + p,$	$1 + i + 2j + 2k + u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
		$\vdots$		
$(2j,$	$1 + i + 2j + p,$	$-2 - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
	$\vdots$			
$(2j,$	$-2 - 2i - 4j - 2k + n - u,$	$-2 - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
$\vdots$				
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-2 - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-i - 2j + n)$
		$\vdots$		$\vdots$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 + t - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 + t - i - 2j + n)$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 + t - 2i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$t - i - 2j + n)$
		$\vdots$		$\vdots$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-2 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-2 - 2j + n)$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-2 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$
$(-2 - 2i - 4j - 4k + n - 2u,$	$-2 - 2i - 4j - 2k + n - u,$	$-1 - i - 2j - 2k + n - u,$	$-1 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$

Table 5: Parallel chains C_5, $0 \le p \le 2k + u$

Additional conditions			$0 \le u \le 1$ $2u + 7 + 6j + 4k + 3i \le n$ $1 \le t \le i - 2$		
$2u + 7 + 6j + 4k + 3i < n$	$(1 + 2j,$	$2 + i + 2j + p,$	$2 + i + 2j + 2k + u,$	$4 + 2i + 4j + 2k + u,$	$6 + 2i + 4j + 4k + 2u)$
					$\vdots$
$2u + 7 + 6j + 4k + 3i < n$	$(1 + 2j,$	$2 + i + 2j + p,$	$2 + i + 2j + 2k + u,$	$4 + 2i + 4j + 2k + u,$	$-2 - i - 2j + n)$
Chains start from this row if $2u + 7 + 6j + 4k + 3i = n.$	$(1 + 2j,$	$2 + i + 2j + p,$	$2 + i + 2j + 2k + u,$	$5 + 2i + 4j + 2k + u,$	$-2 - i - 2j + n)$
				$\vdots$	
	$(1 + 2j,$	$2 + i + 2j + p,$	$2 + i + 2j + 2k + u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - i - 2j + n)$
			$\vdots$		
	$(1 + 2j,$	$2 + i + 2j + p,$	$-3 - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - i - 2j + n)$
		$\vdots$			
	$(1 + 2j,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - i - 2j + n)$
	$\vdots$				
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - i - 2j + n)$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-2 - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-1 - i - 2j + n)$
			$\vdots$		$\vdots$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 + t - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-1 + t - i - 2j + n)$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-2 + t - 2i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-1 + t - i - 2j + n)$
			$\vdots$		$\vdots$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-4 - i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - 2j + n)$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 - i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-2 - 2j + n)$
	$(-5 - 2i - 4j - 4k + n - 2u,$	$-4 - 2i - 4j - 2k + n - u,$	$-3 - i - 2j - 2k + n - u,$	$-2 - i - 2j - 2k + n + p - u,$	$-1 - 2j + n)$

Table 6: Parallel chains C_6, $0 \le p \le 2k + u$

		$u = n\,\mathbf{mod}2,\ u \in \{0,1\}$ $6+3u+6w+6i+2k=n$		
$(1+i,$	$2+2i+p+u+2w,$	$3+3i+k+u+2w,$	$4+4i+k+q+2u+4w,$	$4+4i+2k+2u+4w)$
$\vdots$				
$(1+i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+u+2w,$	$4+4i+k+q+2u+4w,$	$4+4i+2k+2u+4w)$
		$\vdots$		
$(1+i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+2u+4w,$	$4+4i+k+q+2u+4w,$	$4+4i+2k+2u+4w)$
$(1+i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+2u+4w,$	$4+4i+k+q+2u+4w,$	$5+4i+2k+2u+4w)$
$\vdots$				
$(1+2i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+2u+4w,$	$4+4i+k+q+2u+4w,$	$5+4i+2k+2u+4w)$
				$\vdots$
$(1+2i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+2u+4w,$	$4+4i+k+q+2u+4w,$	$5+5i+2k+3u+6w)$
$(2+2i+u+2w,$	$2+2i+p+u+2w,$	$3+3i+k+2u+4w,$	$4+4i+k+q+2u+4w,$	$5+5i+2k+3u+6w)$

Table 7: Parallel chains C_7, $0 \le p \le k, 0 \le q \le p$

		$u = n\,\mathbf{mod}2,\ u \in \{0,1\}$ $12-3u+6w+6i+2k=n$		
$(1+i,$	$4+2i+p-u+2w,$	$5+3i+k-u+2w,$	$8+4i+k+q-2u+4w,$	$9+4i+2k-2u+4w)$
$\vdots$				
$(3+2i-u+2w,$	$4+2i+p-u+2w,$	$5+3i+k-u+2w,$	$8+4i+k+q-2u+4w,$	$9+4i+2k-2u+4w)$
				$\vdots$
$(3+2i-u+2w,$	$4+2i+p-u+2w,$	$5+3i+k-u+2w,$	$8+4i+k+q-2u+4w,$	$9+5i+2k-2u+4w)$
$(4+2i-u+2w,$	$4+2i+p-u+2w,$	$5+3i+k-u+2w,$	$8+4i+k+q-2u+4w,$	$9+5i+2k-2u+4w)$
		$\vdots$		
$(4+2i-u+2w,$	$4+2i+p-u+2w,$	$7+3i+k-2u+4w,$	$8+4i+k+q-2u+4w,$	$9+5i+2k-2u+4w)$
				$\vdots$
$(4+2i-u+2w,$	$4+2i+p-u+2w,$	$7+3i+k-2u+4w,$	$8+4i+k+q-2u+4w,$	$10+5i+2k-3u+6w)$

Table 8: Parallel chains C_8, $0 \le p \le k, 0 \le q \le p$

		$u = n\,\mathbf{mod}2,\ u \in \{0,1\}$ $2k+3u+6w=n$		
$(0,$	$p+u+2w,$	$k+u+2w,$	$k+q+2u+4w,$	$2k+2u+4w)$
$\vdots$				
$(u+2w,$	$p+u+2w,$	$k+u+2w,$	$k+q+2u+4w,$	$2k+2u+4w)$
		$\vdots$		
$(u+2w,$	$p+u+2w,$	$k+2u+4w,$	$k+q+2u+4w,$	$2k+2u+4w)$
				$\vdots$
$(u+2w,$	$p+u+2w,$	$k+2u+4w,$	$k+q+2u+4w,$	$2k+3u+6w)$

Table 9: Parallel chains C_9, $0 \le p \le k, 0 \le q \le p$

3. Getting symmetric chains from parallel chains

For fixed parameters i, j, and k, the chains of C_1 with parameters $p = 0, 1, \cdots, k$ form a rectangle illustrated in the following figure.

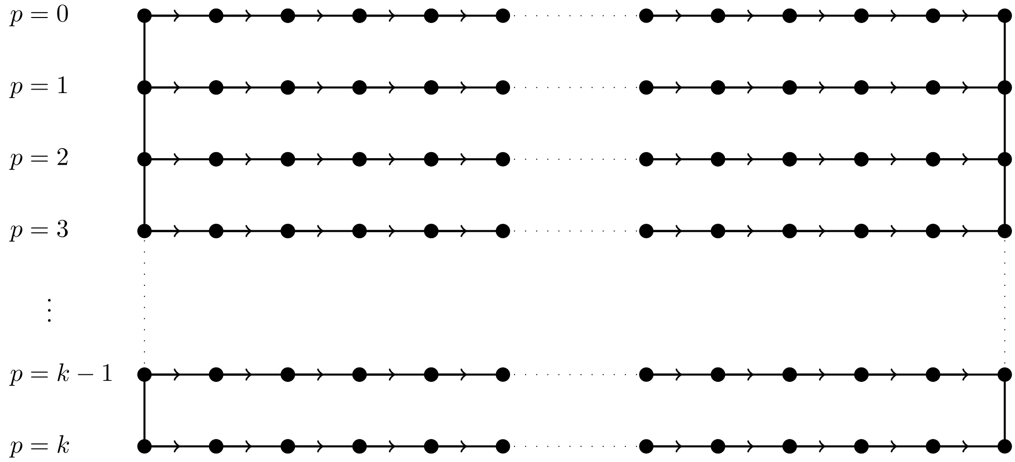

Figure 1: A rectangle formed by parallel chains in C_1 for fixed i, j, k.

The perimeter of the rectangle contains up to 2 symmetric chains from the left-upper corner (the lattice point with the lowest coordinates) to the right-bottom corner (the lattice point with the highest coordinates). The two corner lattice points should be in the same chain, and in our cases, they could be in either of the 2 border chains. All symmetric chains are obtained by taking perimeters off these rectangles recursively, as illustrated in the following figure.

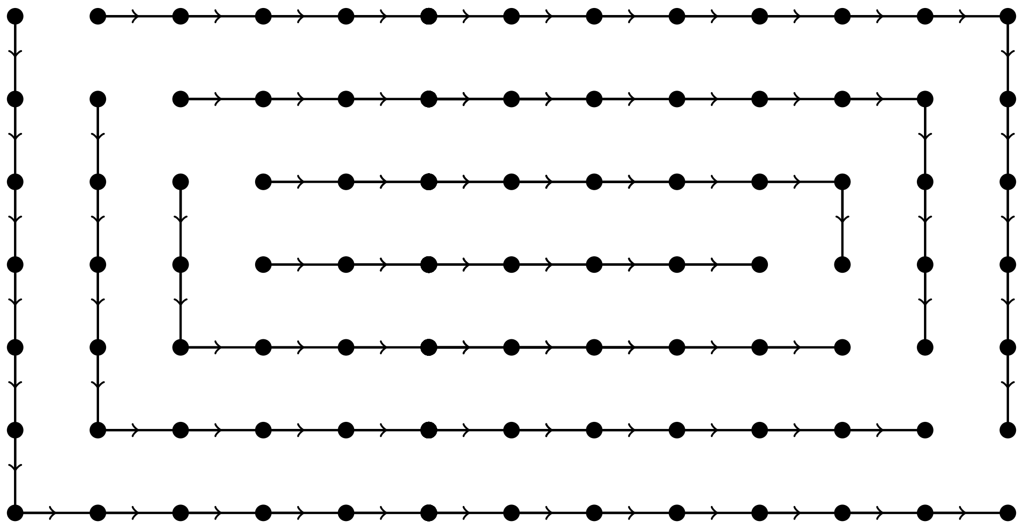

Figure 2: Getting symmetric chains from parallel chains

We could get symmetric chains from parallel chains $C_2, C_3, \cdots, C_6$ similarly.

Parameters (q, p) with conditions $0 \leq p \leq k$, $0 \leq q \leq p$ in C_7 (C_8, C_9) are actually elements in L(2, k) which have an SCD

$$(t, t) \to (t, t+1) \to (t, t+2) \to \cdots \to (t, k-t) \to (t+1, k-t) \to \cdots \to (k-t, k-t)$$

where $0 \leq t \leq \lfloor \frac{k}{2} \rfloor$.

Each element in a symmetric chain of $L(2, k)$ corresponds to an $L(5, n)$ parallel chain in C_7 (C_8, C_9), and these chains together also form a rectangle illustrated in the following figure. Symmetric chains could be obtained by taking perimeters off these rectangles recursively.

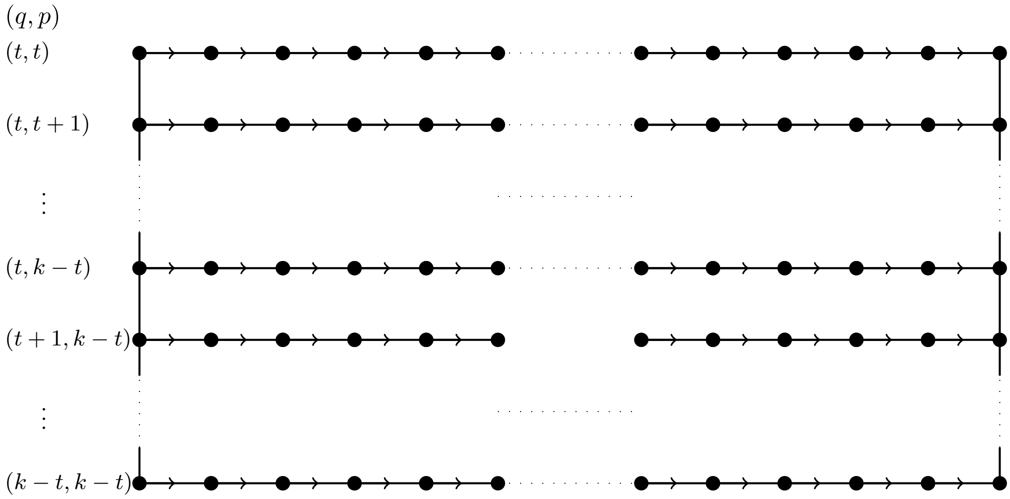

Figure 3: A rectangle formed by parallel chains in C_7 (C_8, C_9)
for fixed u, w, i, k, $0 \leq t \leq \lfloor \frac{k}{2} \rfloor$

4. Proof

It is trivial to check that the chains obtained in section 3 are saturated and symmetric.

The **weight** of an element $\overrightarrow{a} = (a_1, a_2, \cdots, a_m)$ in $L(m, n)$ with commutative variables $x_0, x_1, x_2, \cdots, x_m$ is defined as follows:

$$w(\overrightarrow{a}) = (x_0)^{n-a_m}(x_1)^{a_m-a_{m-1}}(x_2)^{a_{m-1}-a_{m-2}} \cdots (x_{m-1})^{a_2-a_1}(x_m)^{a_1}$$

For a fixed m, it is easy to see that the total weight of all the elements in $L(m, n)$

$$\sum_{n=0}^{\infty} \sum_{\overrightarrow{a} \in L(m,n)} w(\overrightarrow{a})$$

constitutes an ordinary multivariate generating function:

$$F(x_0, x_1, \cdots, x_m) = \frac{1}{(1 - x_0)(1 - x_1)(1 - x_2) \cdots (1 - x_m)}$$

Each term in the expanded power series of $F(x_0, x_1, \cdots, x_m)$ has a coefficient of 1 and corresponds to a unique element in $L(m, n)$. On the other hand, for each element in $L(m, n)$, there is a unique corresponding term in the power series of $F(x_0, x_1, \cdots, x_m)$. Therefore, proving that each element in $L(5, n)$ appears only once in the SCD is equivalent to demonstrating that the total weights of all the elements in the chains are equal to $F(x_0, x_1, x_2, x_3, x_4, x_5)$.

As our symmetric chains are derived from parallel chains, we only need to prove that:

$$\sum_{n=0}^{\infty} \sum_{i=1}^{9} \sum_{\overrightarrow{a} \in C_i} w(\overrightarrow{a}) = \frac{1}{(1 - x_0)(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5)}$$

where $C_1, C_2, \cdots, C_9$ are the parallel chains in Section 2.

This is done by a computer program that sums up all the weights of items in chains C_1, C_2, $\cdots$, C_9. The proof in Wolfram Language is available at
`https://www.wolframcloud.com/obj/xwen/Published/SCDL5nProof.nb`.

The function $SCDL5n[n]$, which provides an SCD of $L(5, n)$ written in Wolfram Language, is available at `https://www.wolframcloud.com/obj/xwen/Published/SCDL5n.nb`.

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