

The Kostka Semigroup and its Hilbert Basis

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ABSTRACT: The *Kostka semigroup* consists of partitions (λ, μ) with at most r parts that have positive Kostka coefficient. For this semigroup, Hilbert basis membership is an NP-complete problem. We introduce *KGR graphs* and *conservative subtrees*, through the Gale-Ryser theorem on contingency tables, as a criterion for membership. Our main application shows that if (λ, μ) is in the Hilbert basis then $\lambda_1 \leq r$. We also classify the extremal rays of the associated polyhedral cone; these rays correspond to a (strict) subset of the Hilbert basis. In an appendix, J. Kiers and G. Orelowitz show that a natural extension of our main result on the Kostka semigroup cannot be extended to the Littlewood-Richardson semigroup. This furthermore gives a counterexample to a recent speculation of P. Belkale concerning the semigroup controlling nonvanishing conformal blocks.

Keywords: Kostka coefficients; semigroup; Hilbert basis; Gale-Ryser theorem

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1. Introduction

1.1 Background

Fix r and n . Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$ and $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$ be integer partitions of n with at most r nonzero parts; let $\text{Par}_r(n)$ be the set of such partitions. A *semistandard tableau* of shape λ and content μ is a filling of λ (depicted as a Young diagram) with entries $1, 2, \dots, r$ such that:

- the rows weakly increase from left to right;
- the columns strictly increase from top to bottom; and
- there are μ_i many i 's.

The *Kostka coefficient* $K_{\lambda, \mu}$ is the number of such tableaux. It is a basic notion in symmetric functions; see *e.g.*, [14].

Example 1.1. $K_{(4,2,1), (3,2,1,1)} = 4$, as witnessed by the following semistandard tableaux:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & & \\ \hline 4 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 4 & & \\ \hline 3 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & & \\ \hline 4 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array}.$$

Define *dominance order* on $\text{Par}_r(n)$ by setting

$$\lambda \geq_{\text{Dom}} \mu \text{ if } \sum_{i=1}^t \lambda_i \geq \sum_{i=1}^t \mu_i, \text{ for } 1 \leq t \leq r. \tag{1}$$

It is textbook [14, Proposition 7.10.5 and Exercise 7.12] that

$$K_{\lambda, \mu} > 0 \iff \lambda \geq_{\text{Dom}} \mu. \tag{2}$$

Define the *Kostka cone*

$$\text{Kostka}_r = \left\{ (\lambda, \mu) \in \mathbb{R}^{2r} : \begin{array}{l} \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0 \\ \mu_1 \geq \mu_2 \geq \dots \geq \mu_r \geq 0 \\ \sum_{i=1}^t \lambda_i \geq \sum_{i=1}^t \mu_i, \text{ for } 1 \leq t \leq r-1 \\ \sum_{i=1}^r \lambda_i = \sum_{i=1}^r \mu_i \end{array} \right\} \subseteq \mathbb{R}^{2r}.$$

The lattice points of Kostka_r , namely,

$$\text{Kostka}_r^{\mathbb{Z}} := \text{Kostka}_r \cap \mathbb{Z}^{2r},$$

form a semigroup. That is,

$$K_{\alpha, \beta} > 0, K_{\lambda, \mu} > 0 \implies K_{\alpha + \lambda, \beta + \mu} > 0.$$

From the general theory, $\text{Kostka}_r^{\mathbb{Z}}$ has a unique minimal generating set as we now recall.

Let C be any rational polyhedral cone inside \mathbb{R}^m . A finite set $\{a_1, \dots, a_k\} \subset C \cap \mathbb{Z}^m$ is called a *Hilbert basis* of C if every integral point in C can be realized as a nonnegative integral combination of a_1, \dots, a_k . Hilbert bases are known to exist for every rational polyhedral cone C ; moreover, when C is pointed, *i.e.*, $C \cap -C = \{\mathbf{0}\}$, there exists a unique minimal (with respect to set inclusion) Hilbert basis, see [13, Theorem 16.4]. We refer to that minimal Hilbert basis as *the* Hilbert basis of C .

The uniqueness proof cited above characterizes the Hilbert basis H of C as follows:

$$H = \{a \in C \cap \mathbb{Z}^m \mid a \neq 0; a \text{ is not the sum of two nonzero elements } b, c \in C \cap \mathbb{Z}^m\}.$$

In other words, the Hilbert basis consists of the irreducible elements of the semigroup $C \cap \mathbb{Z}^m$.

Since Kostka_r is determined by rational inequalities, it is a rational polyhedral cone. Moreover, Kostka_r is pointed. Therefore, $\text{Kostka}_r^{\mathbb{Z}}$ is finitely generated and has a (unique minimal) Hilbert basis. See Table 1 for the case $r = 4$.

The general problem, of deciding if a lattice point in a pointed cone is in the Hilbert basis, is NP-complete [8, Theorem 3.1]. This formal indication of difficulty *specifically* holds for the Kostka cone (with a similar argument). Define a decision problem KostkaHilbert as follows. The input is $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ described by their *columns* (*i.e.*, we input the conjugate shapes (λ', μ')). The output is whether (λ, μ) is *reducible*, that is, if there exist nontrivial $(\lambda^\bullet, \mu^\bullet), (\lambda^\circ, \mu^\circ) \in \text{Kostka}_r^{\mathbb{Z}}$ such that

$$(\lambda, \mu) = (\lambda^\bullet, \mu^\bullet) + (\lambda^\circ, \mu^\circ). \tag{3}$$

$(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ is *irreducible* if it is not reducible. This means that KostkaHilbert returns **True** if (λ, μ) is *not* in the Hilbert basis of $\text{Kostka}_r^{\mathbb{Z}}$. Measure the input (λ, μ) using bits, so the length of the input is $\sum \log \lambda'_i + \sum \log \mu'_i$, and assume arithmetic operations take constant time. This is proved in Appendix A:

Table 1: The Hilbert basis for $\text{Kostka}_4^{\mathbb{Z}}$. The basis elements that are also extremal rays are in black.

Theorem 1.1 (cf. [8, Theorem 3.1]). *KostkaHilbert is NP-complete.*

Hence, no polynomial-time algorithm exists for KostkaHilbert unless $P = NP$. This seems to rule out any simple, explicit classification of the Hilbert basis of $\text{Kostka}_r^{\mathbb{Z}}$.

1.2 Results

To each $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ we introduce the *Kostka-Gale-Ryser (KGR) graph* $G(\lambda, \mu)$ and the concept of a *conservative subtree** of $G(\lambda, \mu)$.

Theorem 1.2 (see Theorem 2.1). *Let $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$. If $G(\lambda, \mu)$ has a conservative subtree, then (λ, μ) is not a Hilbert basis element of $\text{Kostka}_r^{\mathbb{Z}}$.*

The converse is false (see Theorem 3.2 and discussion thereafter about Example 2.6). We apply Theorem 1.2 to prove a bound on the first coordinate of a Hilbert basis element. This is our main result:

Theorem 1.3 (Width Bound). *Suppose (λ, μ) is a Hilbert basis element of $\text{Kostka}_r^{\mathbb{Z}}$. Then $\lambda_1 \leq r$. Moreover, if $\lambda_1 = r$ then λ and μ are both rectangles.*

Theorem 1.3’s bound is sharp since $((t), (1^t)) \in \text{Kostka}_r^{\mathbb{Z}}$ is in the basis, for $1 \leq t \leq r$. The second sentence implies $\mu_1 \leq r - 1$; this bound is also sharp since $((r - 1) \times r, r \times (r - 1))$ is in the basis. Let $\ell(\lambda)$ be the number of nonzero parts of λ .

Corollary 1.1. *Suppose λ and μ are partitions of the same size, and $\lambda \geq_{\text{Dom}} \mu$. If $\ell(\mu) < \lambda_1$ then (λ, μ) is not in the Hilbert basis of $\text{Kostka}_r^{\mathbb{Z}}$ for any $r \geq \ell(\mu)$.*

Proof. The Hilbert basis of $\text{Kostka}_{r'}^{\mathbb{Z}}$ includes into that of $\text{Kostka}_r^{\mathbb{Z}}$ whenever $r' \leq r$. □

An *extremal ray* is a one-dimensional face of the cone Kostka_r . Every element along such a ray is not the sum of two nonparallel elements of Kostka_r ; hence the first lattice point along such a ray is not the sum of two nontrivial elements of $\text{Kostka}_r^{\mathbb{Z}}$. Thus, we have an injection from the set of extremal rays to the Hilbert basis by taking this first lattice point. We classify the extremal rays of Kostka_r , thereby describing this subset of the Hilbert basis.

Theorem 1.4. *Suppose $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$. (λ, μ) lies on an extremal ray of Kostka_r if and only if it is a positive rational multiple of one of the following:*

$$\left(\underbrace{a, \dots, a}_{b+\ell}, 0, \dots, 0; \underbrace{a, \dots, a}_{\ell}, \underbrace{b, \dots, b}_a, 0, \dots, 0 \right)$$

where a, b, ℓ are integers such that $r \geq a + \ell \geq a \geq b > 0$.

Counting possibilities for b, a , and $a + \ell$ above immediately gives:

Corollary 1.2. *The number of extremal rays of Kostka_r is $\binom{r}{3} + \binom{r}{2} + \binom{r}{1}$.*

r	# extremal rays	# Hilbert basis	r	# extremal rays	# Hilbert basis
1	1	1	10	175	3093
2	3	3	11	231	6876
3	7	8	12	298	14133
4	14	19	13	377	29788
5	25	50	14	469	59935
6	41	111	15	575	118893
7	63	281	16	696	232972
8	92	635	17	833	457982
9	129	1443			

Table 2: Counting extremal rays and Hilbert basis elements of $\text{Kostka}_r^{\mathbb{Z}}$. When $r = 3$, the lone Hilbert basis element that is not an extremal ray is the pair $(2, 1), (1, 1, 1)$

In contrast, see Table 2. For any $s \leq r$ and $|\lambda| = s$, $(\lambda, 1^s)$ is in the Hilbert basis. So the number of Hilbert basis elements of $\text{Kostka}_r^{\mathbb{Z}}$ is bounded below by the sum of partition functions (<http://oeis.org/A000070>). What is the actual asymptotic growth?

Kostka coefficients appear in the representation theory of $G = SL_r$. For each partition λ with at most r parts, there is a finite-dimensional representation V_λ of G , and $K_{\lambda, \mu}$ records the dimension of the largest subspace of V_λ on which a fixed maximal torus $T \subset G$ acts with character described by μ . For any reductive Lie group G , one defines the weight space dimensions $K_{\lambda, \mu}^G$, and one may study the cone of solutions (λ, μ) to the problem $K_{\lambda, \mu}^G > 0$. During preparation of this article, M. Besson, S. Jeralds, and J. Kiers [3] proved a generalization of Theorem 1.4. Those methods involve analysis of dominant weight polytopes in arbitrary Lie type. Our argument uses no Lie theory.

1.3 Organization

Section 2 introduces the key constructions of this paper. There we use the *Gale-Ryser theorem*, and specifically, the *canonical matrix* $A(\lambda, \mu)$ which is a $\{0, 1\}$ -matrix with prescribed row and column marginals. We define the auxiliary matrix $A^*(\lambda, \mu)$ from $A(\lambda, \mu)$. This permits us to define the *KGR graph* $G(\lambda, \mu)$ and *conservative subtrees*. Existence of such a subtree rules out (λ, μ) being a Hilbert basis element; this is Theorem 1.2 (see Theorem 2.1). Our application to the proof of the Width Bound (Theorem 1.3) is in Section 3. In that section, we also prove that existence of a conservative subtree is equivalent to $A(\lambda, \mu)$ being reducible (Theorem 3.2). Section 4 proves our result on extremal rays of Kostka_r (Theorem 1.4). Section 5 discusses some conjectures that were posed in an earlier version of this preprint. All of these conjectures have since been resolved (two of which by J. Kim [9]) and one, which concerns an extension to Littlewood-Richardson coefficients, has been proved false (Theorem B.1). Appendix A proves Theorem 1.1, by reducing the *Subset Sum problem* to *KostkaHilbert*.

2. The Gale-Ryser theorem, contingency tables, and directed graphs

2.1 KGR graphs, and conservative subtrees

Let Par_t be the set of partitions with at most t nonzero parts. Suppose $\alpha \in \text{Par}_r$ and $\beta \in \text{Par}_s$. Let $\text{GR}(\alpha, \beta)$ be the set of $\{0, 1\}$ -matrices (contingency tables) $(A_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ of dimension $r \times s$ such that

$$\sum_{j=1}^s a_{ij} = \alpha_i, \quad 1 \leq i \leq r; \quad \sum_{i=1}^r a_{ij} = \beta_j, \quad 1 \leq j \leq s.$$

The *Gale-Ryser theorem* states that

$$\text{GR}(\alpha, \beta) \neq \emptyset \iff \beta \leq_{\text{Dom}} \alpha'. \tag{4}$$

One reference is R. Brualdi's [4], who recounts that (4) was independently obtained in 1957, by D. Gale [7] and H. J. Ryser [12]. The former used "his supply-demand theorem for network flows" while the latter constructs a matrix in $\text{GR}(\alpha, \beta)$, as described below.

Proposition 2.1. $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ if and only if $\text{GR}(\mu, \lambda') \neq \emptyset$.

*Akin to "conservative vector field" and path independence.

Proof. By (2), and that \leq_{Dom} is an anti-automorphism under conjugation [14, Section 7.2],

$$(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}} \iff \mu \leq_{\text{Dom}} \lambda \iff \lambda' \leq_{\text{Dom}} \mu'.$$

Hence by (4), this is equivalent to $\text{GR}(\mu, \lambda') \neq \emptyset$. □

We also follow R. Brualdi [5]. Ryser gave an algorithm that takes $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ as an input, and outputs a $\{0, 1\}$ -matrix $A(\lambda, \mu)$ of dimension $r \times \lambda_1$ that exhibits $\text{GR}(\mu, \lambda') \neq \emptyset$, as guaranteed by Proposition 2.1. Call this matrix the *canonical matrix* for (λ, μ) .

We need to recall Ryser’s algorithm. First define $A(\mu)$ to be the $r \times \lambda_1$ dimension $\{0, 1\}$ -matrix whose i -th row has μ_i many 1’s placed flush-left. Let $\beta = \lambda'$ and $s = \lambda_1$. Shift β_s rightmost 1’s in each row of $A(\mu)$ to column s , by choosing 1’s in rows with the largest sum, and breaking ties by choosing 1’s in rows further south. Let $A'(\mu)$ be the submatrix of $A(\mu)$ using the first $s - 1$ columns. Repeat the shifting process, now to column $s - 1$ so that there are β_{s-1} many 1’s there. Continuing, Ryser’s algorithm eventually arrives at $A(\lambda, \mu)$. We refer the reader to [4] and the original [12] for additional details.

Example 2.1. If $\lambda = (8, 7, 7, 7, 3, 2)$ and $\mu = (7, 7, 4, 4, 4, 4)$, then

$$A(\mu) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, A(\lambda, \mu) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Here $\lambda' = (6, 6, 5, 4, 4, 4, 1)$, and the steps of Ryser’s algorithm are

$$\begin{aligned} A(\mu) &\mapsto \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \mapsto \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \mapsto \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \\ &\mapsto \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \mapsto \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right] \mapsto \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array} \right] \\ &\mapsto \left[\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \mapsto A(\lambda, \mu). \end{aligned}$$

If μ is a partition then $\mu_i - \mu_{i+1} \geq 0$ for all i . Define the column vector $\mu^* \in \mathbb{Z}_{\geq 0}^r$ by

$$\mu_i^* = \mu_i - \mu_{i+1},$$

where $\mu_{r+1} := 0$. Similarly, define $A^*(\lambda, \mu)$ to be the matrix with the same dimensions as $A(\lambda, \mu)$, with

$$A^*(\lambda, \mu)_{i,j} = A(\lambda, \mu)_{i,j} - A(\lambda, \mu)_{i+1,j}, \tag{5}$$

where $A(\lambda, \mu)_{r+1,j} := 0$.

Example 2.2. Given the same choices for λ and μ as before,

$$A^*(\lambda, \mu) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Definition 2.1. The Kostka-Gale-Ryser (KGR) graph $G(\lambda, \mu)$ is a directed graph obtained from $A^*(\lambda, \mu)$ by drawing an arc from each

- (I) -1 to the rightmost 1 to the left of it in the same row
- (II) 1 to the -1 in the same column (if it exists).

The well-definedness of (I) is established by Lemma 2.5.

Example 2.3. Below $G(\lambda, \mu)$ is drawn in blue, overlaid on top of $A^*(\lambda, \mu)$.

$$A^*(\lambda, \mu) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Define a *source* of a directed graph to be a vertex y with no arcs of the form $x \rightarrow y$. Similarly, a *sink* is a vertex x without arcs $x \rightarrow y$. Hence x may be both a source and sink if (and only if) it is an isolated vertex. In addition, *connected* will mean with respect to the underlying *undirected* graph structure. A *subtree* of a directed graph means a connected subgraph with no undirected cycles.

Definition 2.2. A connected nontrivial subgraph $G' \subset G(\lambda, \mu)$ is a conservative subtree if whenever a vertical arc appears in G' , all vertical arcs of $G(\lambda, \mu)$ in the same column do, and if one of the following is true:

- (C.1) For each column of $G(\lambda, \mu)$, either all of the vertical arcs of $G(\lambda, \mu)$ in that column are in G' , or none of them are.
- (C.2) One of the following is true:
 - (C.2.1) G' is a connected component of $G(\lambda, \mu)$ (with respect to the underlying undirected graph structure).
 - (C.2.2) The unique sink of G' is a -1 in $A^*(\lambda, \mu)$, one source of G' is a 1 in $A^*(\lambda, \mu)$ in the same row as the sink, and all other sources of G' are sources of $G(\lambda, \mu)$.

Example 2.4. We have marked a conservative subtree of $G(\lambda, \mu)$ in red.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

This is our main result about KGR graphs and conservative subtrees:

Theorem 2.1. Let $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$.

The undirected graph structure of $G(\lambda, \mu)$ is a forest that is planar (with respect to the specified embedding). Therefore, any conservative subtree of $G(\lambda, \mu)$ is in fact a tree.

If $G(\lambda, \mu)$ has a conservative subtree, then (λ, μ) is not in the Hilbert basis of $\text{Kostka}_r^{\mathbb{Z}}$.

The proof, given in Section 2.2, proceeds by building a number of additional properties of the graphs and $A^*(\lambda, \mu)$, which we also use later.

2.2 Proof of the main result on KGR graphs (Theorem 2.1)

Let

$$A^{(0)} = A(\mu), A^{(1)}, A^{(2)}, \dots, A^{(\lambda_1)} = A(\lambda, \mu)$$

be the sequence of matrices in Ryser's algorithm.

Lemma 2.1. *Each column of $A(\lambda, \mu)$ is one of the following:*

1. a single run of 1's, starting in the top row;
2. a single run of 1's, not starting in the top row; or
3. two runs of 1's, one starting in the top row.

Furthermore, the leftmost column must fall into case (1).

Proof. We prove that the stated properties hold for every column of $A^{(i)}$. Clearly, this is true for $i = 0$. Now assume $i > 0$ and the assertion holds for $A^{(i-1)}$. By construction, columns $\lambda_1 - i + 2, \lambda_1 - i + 3, \dots, \lambda_1$ are the same as those of $A^{(i-1)}$. Hence by induction, those columns satisfy the claim. In $A^{(i-1)}$ the 1's in columns $1, 2, \dots, \lambda_1 - i$ form a Young diagram θ . Consider the region R in those columns and strictly south of row λ'_{λ_1-i+1} ; the 1's there also form a sub-Young diagram $\underline{\theta}$ of θ . Ryser's rule $A^{(i-1)} \mapsto A^{(i)}$ is defined precisely to ensure that the 1's removed from $\underline{\theta}$ leave a partition. Hence the claim holds for columns $1, 2, \dots, \lambda_1 - i$ of $A^{(i)}$. Finally, it only remains to check column $\lambda_1 - i + 1$ of $A^{(i)}$. The point is that the only way to fall into cases (2) or (3) is if one uses the tiebreak rule of Ryser's algorithm; this can only occur once. If no tiebreak occurs, case (1) occurs. \square

Definition 2.3. $A(\lambda, \mu)$ is reducible if there exists some nontrivial subset S of the columns such that the sum of the columns in S forms a partition in Par_r , and the sum of the columns not in S forms a partition in Par_r .

Proposition 2.2. *If $A(\lambda, \mu)$ is reducible, then (λ, μ) is reducible.*

Proof. Let S be as in Definition 2.3. Let μ^\bullet be the vector obtained by adding all of the columns in S , and let $\mu^\circ = \mu - \mu^\bullet$ be the sum of all other columns. By the hypothesis that $A(\lambda, \mu)$ is reducible, $\mu^\bullet, \mu^\circ \in \text{Par}_r$. Similarly, let λ^\bullet and λ° be the partitions whose column multisets are $\{\lambda'_i : i \in S\}$, and $\{\lambda'_i : i \notin S\}$ respectively.

Hence, $\mu = \mu^\bullet + \mu^\circ$ and $\lambda = \lambda^\bullet + \lambda^\circ$, and these are both partitions. Consider the submatrix of $A(\lambda, \mu)$ consisting of only the columns in S . This is a $\{0, 1\}$ -matrix such that the sum of the entries in row i is μ^\bullet_i , and the sum of the entries in column j is $(\lambda^\bullet)_j$. Therefore, by Proposition 2.1, $(\lambda^\bullet, \mu^\bullet) \in \text{Kostka}_r^{\mathbb{Z}}$. By the same logic $(\lambda^\circ, \mu^\circ) \in \text{Kostka}_r^{\mathbb{Z}}$, so (λ, μ) is reducible. \square

Example 2.5. *Continuing Example 2.1, $A(\lambda, \mu)$ is reducible by letting $S = \{2, 3, 4, 8\}$. This corresponds to*

$$\lambda^\bullet = (4, 3, 3, 3, 2, 1), \mu^\bullet = (3, 3, 2, 2, 2, 2), \lambda^\circ = (4, 4, 4, 4, 1, 1), \text{ and } \mu^\circ = (4, 4, 2, 2, 2, 2).$$

Example 2.6. *The converse of Proposition 2.2 is not true. Let $\lambda = (3, 2, 1)$, $\mu = (2, 2, 1, 1)$. We have the decomposition*

$$\lambda^\circ = (1, 1), \mu^\circ = (1, 1), \lambda^\bullet = (2, 1, 1), \mu^\bullet = (1, 1, 1, 1).$$

However, one can verify that

$$A(\lambda, \mu) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is not reducible.

Lemma 2.2. *Each column of $A^*(\lambda, \mu)$ is one of the following:*

1. all 0's, except for a single 1;
2. all 0's, except for a single -1 and a single 1 below it; or
3. all 0's, except for a single -1 and with a single 1 above and a single 1 below it.

Moreover, the leftmost column is in the first case. The bottom row can not contain any -1 's.

Proof. This holds by the corresponding statements in Lemma 2.1, together with (5). \square

Let

$$\mu = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(i)} \supset \cdots \supset \mu^{(\lambda_1)} = \emptyset \quad (6)$$

denote the partitions of row sum vectors for the $r \times (\lambda_1 - i)$ leftmost submatrix of $A(\lambda, \mu)$. In our running example, the partitions are

$$\begin{aligned} (7, 7, 4, 4, 4, 4, 4) \supset (7, 6, 4, 4, 4, 4, 4) \supset (6, 5, 4, 4, 4, 3, 3) \supset (5, 4, 4, 3, 3, 3, 3) \\ \supset (4, 3, 3, 3, 3, 2) \supset (3, 3, 3, 2, 2, 2) \supset (2, 2, 2, 2, 1, 1) \supset (1, 1, 1, 1, 1, 0) \supset \emptyset \end{aligned}$$

Lemma 2.3. *At step i , $\mu^{(i-1)}$ transforms to $\mu^{(i)}$ in precisely one of the following three ways:*

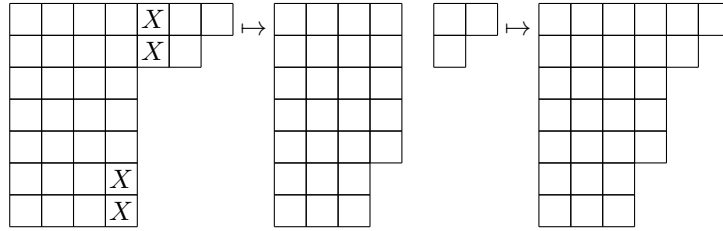
1. *One column of $\mu^{(i-1)}$ is deleted (and the result is left-justified).*
2. *The rightmost column of $\mu^{(i-1)}$ is shortened.*
3. *One column of $\mu^{(i-1)}$ is shortened, and the column C to its immediate right (equivalently, any column of the same length as C) is deleted, and the result is left-justified. The deleted column is strictly shorter than the shortened column's new length.*

Moreover, if a column of $\mu^{(i-1)}$ is shortened, the new length of that column is strictly larger than the length of any column of $\mu^{(i)}$ to its right.

Example 2.7. *In our running example, when $i = 2$, the transition*

$$\mu^{(1)} = (7, 6, 4, 4, 4, 4, 4) \mapsto \mu^{(2)} = (6, 5, 4, 4, 4, 3, 3)$$

is obtained by applying (3): shortening column 4 and deleting column 5 (or column 6).



Proof. Suppose we are executing the transition $A^{(i-1)} \mapsto A^{(i)}$ for $i \geq 1$. We are moving (or using “in place”) some 1’s from the $r \times (\lambda_1 - i + 1)$ left submatrix of $A^{(i-1)}$ and having them terminate in column $\lambda_1 - i + 1$. If no tie-breaking occurs, then in fact the rows of the 1’s used form an initial interval $[1, \ell]$ and there must exist a column of length ℓ in $\mu^{(i-1)}$ (specifically, the one that contains the southmost 1 moved), and we fall into case (1).

Now suppose tie-breaking occurs. Suppose that $\mu^{(i-1)}$ has largest part m , which occurs t many times. If the number of 1’s moved is strictly fewer than t , we are in case (2). Finally, suppose the number of 1’s moved is greater than t . Since tie-breaking is assumed, there is some part size m' , occurring t' many times in $\mu^{(i-1)}$, for which Ryser’s rule chooses some southmost segment of 1’s to move (as in the example $i = 2$ above where $m' = 4$ and $t' = 5$). This means that Ryser’s rule chooses 1 from every row north of this block of m' . This puts us in case (3).

The final sentence is clearly true if we are in case (2) since we are shortening the (weakly) shortest column. If instead we are in case (3) and the shortening made the column equal to some other column of $\mu^{(i)}$ it is straightforward to see that would imply no tie-breaking actually occurred, a contradiction of the analysis above. \square

Lemma 2.4. *The (exhaustive) cases (1), (2), (3) of Lemma 2.3 in the transition $\mu^{(i-1)} \mapsto \mu^{(i)}$ respectively correspond to the following ± 1 values of $A^*(\lambda, \mu)$ in column $\lambda_1 - i + 1$:*

- (1) $A^*(\lambda, \mu)_{\ell, \lambda_1 - i + 1} = 1$ if the deleted column is of length ℓ ;
- (2) $A^*(\lambda, \mu)_{\ell', \lambda_1 - i + 1} = -1, A^*(\lambda, \mu)_{\ell, \lambda_1 - i + 1} = 1$ if the shortened column was of length ℓ and becomes of length $\ell' < \ell$;
- (3) $A^*(\lambda, \mu)_{\ell'', \lambda_1 - i + 1} = 1, A^*(\lambda, \mu)_{\ell', \lambda_1 - i + 1} = -1, A^*(\lambda, \mu)_{\ell, \lambda_1 - i + 1} = 1$ if the shortened column was of length ℓ and becomes of length $\ell' < \ell$, and the deleted column is of length $\ell'' (< \ell')$.

Proof. This follows from the definition of $A^*(\lambda, \mu)$ and by Ryser’s algorithm. \square

Lemma 2.5. *Every -1 in $A^*(\lambda, \mu)$ must have a 1 to its left in the same row. Furthermore, all entries between this -1 and the rightmost such 1 are 0 's.*

Proof. Suppose $A^*(\lambda, \mu)_{\ell', \lambda_1 - i + 1} = -1$. By Lemma 2.4 a column C in $\mu^{(i-1)}$ of length $\ell > \ell'$ was shortened to length ℓ' at the step $A^{(i-1)} \mapsto A^{(i)}$ of Ryser's algorithm. The final sentence of Lemma 2.3 says that at each later stage no column of $\mu^{(i+k)}$ (for $k \geq 0$) can shorten to length ℓ' until column C is further shortened or deleted. Hence, by cases (2) and (3) of Lemma 2.4 no -1 will appear left of our -1 until column C is shortened/deleted. On the other hand, every column is eventually shortened/deleted. When this occurs, by Lemma 2.4(1), the desired 1 will appear. \square

Definition 2.4. *$A^*(\lambda, \mu)$ is $*$ -reducible if there exists a nontrivial subset S of the columns whose sum v^* satisfies $0 \leq v_i^* \leq \mu_i^*$ for all i .*

Proposition 2.3. *$A^*(\lambda, \mu)$ is $*$ -reducible if and only if $A(\lambda, \mu)$ is reducible.*

Proof. Suppose $A^*(\lambda, \mu)$ is $*$ -reducible; let S be the witnessing subset. Let v be the sum of the columns of S in $A(\lambda, \mu)$. Now

$$v^* \in \mathbb{Z}_{\geq 0}^r \iff v \in \text{Par}_r.$$

Similarly let v^c be the sum of the columns of S^c in $A(\lambda, \mu)$. We have

$$\mu^* - v^* \in \mathbb{Z}_{\geq 0}^r \iff v^c \in \text{Par}_r.$$

Thus, S witnesses $A(\lambda, \mu)$ being reducible. The converse proof reverses this argument. \square

Proposition 2.4. *$G(\lambda, \mu)$ has exactly μ_i^* sources in row i , and they are all 1 's in $A^*(\lambda, \mu)$.*

Proof. By Lemma 2.5 and the definition of $G(\lambda, \mu)$ any source must correspond to a 1 . Now since the sum of the entries in row i of $A^*(\lambda, \mu)$ is μ_i^* . Each -1 in that row can be paired with the 1 closest to its left (which exists by Lemma 2.5). There is a surplus of μ_i^* -many unpaired 1 's. These are the μ_i^* -many sources. \square

Proposition 2.5. *The undirected graph structure of $G(\lambda, \mu)$ is a forest that is planar (with respect to the specified embedding).*

Proof. For the forest claim, we show there is no undirected cycle. Assume otherwise and such a cycle \mathcal{C} exists in $G(\lambda, \mu)$. No horizontal arc of $G(\lambda, \mu)$ points to the right. Therefore, it follows that some vertex v of \mathcal{C} has two distinct arcs pointing away from v . But, by construction, no vertex of $G(\lambda, \mu)$ has this property, a contradiction.

Assume for the sake of contradiction that two edges of $G(\lambda, \mu)$ cross over each other. Locally, the two possible configurations look like:

$$A^*(\lambda, \mu) = \left[\begin{array}{cccccc} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 1 & \dots & \dots & \dots \\ \dots & \dots & \vdots & \dots & \dots & \dots \\ \dots & 1 & \leftarrow & -1 & \dots & \dots \\ \dots & \dots & \vdots & \dots & \dots & \dots \\ \dots & \dots & -1 & \dots & \dots & \dots \end{array} \right], \quad \left[\begin{array}{cccccc} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -1 & \dots & \dots & \dots \\ \dots & \dots & \vdots & \dots & \dots & \dots \\ \dots & 1 & \leftarrow & -1 & \dots & \dots \\ \dots & \dots & \vdots & \dots & \dots & \dots \\ \dots & \dots & 1 & \dots & \dots & \dots \end{array} \right] \begin{array}{l} a \\ \ell' \\ b \end{array}$$

$\lambda_1 - j + 1 \qquad \qquad \qquad \lambda_1 - j + 1$

In either case, the horizontal edge is in row ℓ' and the vertical edge is in column $\lambda_1 - j + 1$. The horizontal edge implies, by Lemma 2.4 that $\mu^{(j-1)}$ has a column of length ℓ' . However, the vertical edge implies, respectively that in the transition from $\mu^{(j-1)}$ to $\mu^{(j)}$,

1. a column of length $b' > b$ is shortened to length $b > \ell'$, and a column of length $a < \ell'$ is deleted; or
2. a column of length $b > \ell'$ is shortened to length $a < \ell'$.

By Lemma 2.3, neither of these is possible, a contradiction. Hence $G(\lambda, \mu)$ is planar. \square

Lemma 2.6. *Let u, v be vertices of $G(\lambda, \mu)$ with $\text{col}(v) < \text{col}(u)$. Let $u = v_0, v_1, \dots, v_N = v$ be distinct vertices of an undirected path P from u to v . No step of this path moves right (i.e., "backwards") on a horizontal arc $v_i \leftarrow v_{i+1}$ where $\text{col}(v_{i+1}) > \text{col}(v)$.*

Proof. Suppose a path P exists with this backward step $v_i \leftarrow v_{i+1}$. Then since $\text{col}(v_{i+1}) > \text{col}(v)$, there must exist a later step that uses a leftward (i.e., "forward") move along a horizontal arc $v_{j+1} \leftarrow v_j$ (for some $j > i$). Since this orientation change occurs, for some $i < k \leq j$, P uses a backward step $v_{k-1} \leftarrow v_k$ followed by a forward step $v_k \rightarrow v_{k+1}$ (possibly on a vertical arc). Since all the vertices v_i are distinct, we conclude v_k has an out-degree of at least two. However no vertex in $G(\lambda, \mu)$ has this property, a contradiction. \square

Lemma 2.7. $G(\lambda, \mu)$ is connected if and only if there is a horizontal arc emanating from every column in $G(\lambda, \mu)$ except the leftmost one.

Proof. (\Leftarrow) Suppose there is a horizontal arc beginning in every column C in $G(\lambda, \mu)$ that is not the leftmost one. This means, for every vertex $v \in C$ there is a directed path from v to some vertex u such that $\text{col}(u) < \text{col}(v)$. By iterating this, we obtain a directed path from v to s , the unique vertex in column 1. Our choice of v was arbitrary, so there is a directed path from any vertex to s , and so s is in the same connected component as all other vertices. Hence $G(\lambda, \mu)$ is connected.

(\Rightarrow) Suppose there is some (not leftmost) column C such that C has no horizontal edge emanating from it. We wish to show $G(\lambda, \mu)$ is not connected. Assume otherwise. Let $u \in C$, and say that v is a vertex with $\text{col}(v) < \text{col}(u)$. Since $G(\lambda, \mu)$ is connected, there exists an undirected path P from u to v . Since column C has no horizontal edges coming out of it, and all horizontal edges point left, the first horizontal step in P is to the right. Since $\text{col}(v) < \text{col}(u)$, Lemma 2.6 says P does not exist, a contradiction. \square

Lemma 2.8. If $G(\lambda, \mu)$ is not connected, any connected component G' is a conservative subtree.

Proof. In view of (C.2.1), it suffices to prove that whenever a vertical arc is in G' , all vertical arcs of $G(\lambda, \mu)$, in the same column, also appear. This follows from Lemma 2.2, the connectedness of G' and $G(\lambda, \mu)$'s definition. \square

Proposition 2.6. If $G(\lambda, \mu)$ has a conservative subtree then $A^*(\lambda, \mu)$ is $*$ -reducible.

Proof. Let G' be a conservative subtree of $G(\lambda, \mu)$, and let S be the set of columns containing vertices of G' . Suppose u is a vertex that is not a sink or source. Since G' is by definition connected, by construction of $G(\lambda, \mu)$, u is incident to one horizontal arc. Let u' be the other vertex of this arc. In any directed path of G' , the vertices alternate from -1 to 1 . Therefore, the entries of $A^*(\lambda, \mu)$ corresponding to u and u' are of opposite sign. Hence, if v^* is the sum of the columns of S , the entries of $A^*(\lambda, \mu)$ associated to u will cancel.

It remains to check the contribution of sources and sinks to v^* . There are two cases.

Case 1: (G' is of type (C.2.1)) Every vertex of G' that is not a source of $G(\lambda, \mu)$ is incident to exactly one horizontal arc, so when adding up the columns of S , the only ± 1 values that do not cancel out are the sources of G' . However, these are all sources of $G(\lambda, \mu)$, so by Proposition 2.4 they contribute at most μ_i^* 1's to row i . Hence $v_i^* \leq \mu_i^*$. This says $A^*(\lambda, \mu)$ is $*$ -reducible provided $S \subsetneq [\lambda_1] := \{1, 2, \dots, \lambda_1\}$. Indeed this is the case since if $S = [\lambda_1]$ then G' contains a vertex in each column. Then since all vertices in a column are connected, we would see that $G' = G(\lambda, \mu)$ which is a contradiction.

Case 2: (G' is of type (C.2.2)) The only vertices in G' that are not incident to a horizontal arc in G' are the sink and the sources. All other vertices of G' are incident to a unique horizontal arc. By the definition of (C.2.2), one of the sources is a 1 that is in the same row as the unique sink, which is a -1 in $A^*(\lambda, \mu)$. As a result, the only values that do not cancel out when adding the columns in S are the other sources of G' . Since these vertices are also sources of G , by appeal to Proposition 2.4 (as in Case 1), we see $A^*(\lambda, \mu)$ is $*$ -reducible, provided $S \subsetneq [\lambda_1]$. To see this proper containment holds, notice that the first column of $A^*(\lambda, \mu)$ contains a single 1 and hence must correspond to a sink vertex q . Since G' has a unique sink (which is a -1), q is not in G' and thus $1 \notin S$. \square

In fact, the converse is true; see Theorem 3.2.

Example 2.8. Our conservative subtree uses $S = \{2, 3, 4, 8\}$. The reader can check that with this S , $A^*(\lambda, \mu)$ is $*$ -reducible, with

$$v = (0, 1, 0, 0, 0, 0, 2)^T \leq \mu^* = (0, 3, 0, 0, 0, 0, 4)^T.$$

Conclusion of the proof of Theorem 2.1: The first claim about the planar forest structure of $G(\lambda, \mu)$ is Proposition 2.5. The second claim follows from Proposition 2.6, Proposition 2.3, and Proposition 2.2. \square

3. Proof of the Width Bound (Theorem 1.3)

Our main application of Theorem 2.1 is to prove Theorem 1.3. First, let us state some facts we will need:

Lemma 3.1. If a column of $G(\lambda, \mu)$ of (matrix index) c contains exactly one vertical arc, and in addition, the arc's northmost endpoint is in row n , there is no horizontal arc $y \leftarrow x$ (necessarily leftward) such that $\text{row}(x) = \text{row}(y) < n$ with $\text{col}(x) > c$ and $\text{col}(y) \leq c$.

Proof. Since there is only a single vertical arc, this means that at step $\lambda_1 - c + 1$ of Ryser's algorithm, a column C in $\mu^{(\lambda_1 - c)}$ is shortened to length n (in $\mu^{(\lambda_1 - c + 1)}$), and no other column was deleted. Therefore, we must fall into Lemma 2.3(2). Thus C is the rightmost and shortest column in $\mu^{(\lambda_1 - c)}$. C remains the shortest in $\mu^{(\lambda_1 - c + 1)}$. However, the existence of the arc $y \leftarrow x$ implies that at step $\lambda_1 - c + 1$ of the algorithm, there is a shorter column in $\mu^{(\lambda_1 - c)}$ of length $\text{row}(x) = \text{row}(y) < n$. This contradicts the previous sentence. \square

Lemma 3.2. *Suppose $G(\lambda, \mu)$ has a source s . There is no vertical arc of the form:*

- (a) $\begin{matrix} y \\ \uparrow \\ x \end{matrix}$ with $\text{row}(x) > \text{row}(s)$ and $\text{row}(y) \leq \text{row}(s)$; or
- (b) $\begin{matrix} x \\ \downarrow \\ y \end{matrix}$ with $\text{row}(x) < \text{row}(s)$ and $\text{row}(y) \geq \text{row}(s)$,

such that $\text{col}(x) = \text{col}(y) > \text{col}(s)$.

Proof. By Proposition 2.4, since s is a source, it corresponds to a 1. We claim that at step $t_1 = \lambda_1 - \text{col}(x) + 1$ of Ryser's algorithm, there is a column C of length $\text{row}(s)$ that is unaltered by the step. Indeed, what we know is that at step $t_2 = \lambda_1 - \text{col}(s) + 1$ there is a column of length $\text{row}(s)$. If no such column exists at step t_1 , then between step t_1 and step t_2 some longer column must have been shortened, at step t' , to length $\text{row}(s)$ which is then shortened or deleted at step t_2 . This implies $A^*(\lambda, \mu)_{\text{row}(s), \lambda_1 - t' + 1} = -1$ and $A^*(\lambda, \mu)_{\text{row}(s), \lambda_1 - t + 1} = 0$ for $t' < t < t_2$. However, we know $A^*(\lambda, \mu)_{\text{row}(s), \lambda_1 - t_2 + 1} = 1$ which implies s is not a source, a contradiction.

Suppose a vertical arc of type (a) occurs. Then examining the possibilities of Lemma 2.4, we conclude that some column of length $\text{row}(x) > \text{row}(s)$ in $\mu^{(t_1-1)}$ was shortened and is now (in $\mu^{(t_1)}$) shorter/equal in length than C . This contradicts the last sentence of Lemma 2.3. Hence an arc of type (a) cannot occur.

Now suppose a vertical arc of type (b) occurs. By Lemma 2.4 there must be another vertical arc $\begin{matrix} y \\ \uparrow \\ x' \end{matrix}$ in $\text{col}(x) = \text{col}(y)$. Thus at step t_1 we must be in case (3) of Lemma 2.3. However, in $\mu^{(t_1)}$, the shortened column S has length $\text{row}(y) > \text{row}(s)$ and the column D deleted in $\mu^{(t_1-1)}$ is of length $\text{row}(x) < \text{row}(s)$. Therefore D cannot be of length equal to the one to the immediate right of S (since C is in-between), as demanded by Lemma 2.3(3), a contradiction. Hence no arc of type (b) can occur either. \square

First suppose that $\lambda_1 > r$. If $G(\lambda, \mu)$ is not connected, let G' be any connected component. Then G' is a conservative subtree by Lemma 2.8 and we are done by Theorem 2.1.

Therefore we may assume $G(\lambda, \mu)$ is connected. Therefore, Lemma 2.7 says every column other than the leftmost one has a leftward horizontal arc coming out of it. That is, there are at least $\lambda_1 - 1$ horizontal arcs. No horizontal arc appears in the bottom row (since $A^*(\lambda, \mu)$ has no -1 's in that row). Hence the arcs appear in at most $r - 1$ rows. Since we assume $\lambda_1 - 1 > r - 1$, by pigeonhole, some row contains two horizontal arcs $u \rightarrow u'$ and $v \rightarrow v'$. We may suppose $v \rightarrow v'$ is to the left of $u \rightarrow u'$.

Lemma 3.3. *If there exist vertices $u', v \in G(\lambda, \mu)$ such that v is a -1 , u' is a 1 , $\text{row}(v) = \text{row}(u')$ and $\text{col}(v) < \text{col}(u')$, then $G(\lambda, \mu)$ has a conservative subtree.*

Proof. If $G(\lambda, \mu)$ is not connected, then we are done by Lemma 2.8. Therefore, we can assume that $G(\lambda, \mu)$ is connected.

Claim 3.1. *There is a directed path from u' to v in $G(\lambda, \mu)$.*

Proof of Claim 3.1: Suppose not. Since $G(\lambda, \mu)$ is connected, there is an *undirected* path (with distinct vertices) P^+ from u' to v . We assert that P^+ looks like the one in Figure 1 (or its vertical reflection, for which our argument is the same, only flipped, except that we apply the definition of $G(\lambda, \mu)$ whenever we refer to Lemma 3.1 in that case). More precisely, let P be the initial part of P^+ from u' to 1 (in Figure 1, this is represented by the dashed blue lines). We claim P is a *directed path* from u' to 1. These assertions about P^+ and P are straightforward by inspecting the definition of $G(\lambda, \mu)$ and Lemmas 2.2, 2.5 and 2.6. Let R be the region (strictly) bounded from above by P , (weakly) bounded from the left and right by the columns of v and u' , respectively, and unbounded from below.

Consider the induced subgraph \overline{G} of $G(\lambda, \mu)$ consisting of all vertices that have a directed path to v . Let s be the unique vertex of \overline{G} in R that is northmost then rightmost among all choices (such an s exists since v itself is in R).

Observe that s corresponds to a 1 in $A^*(\lambda, \mu)$: otherwise if it corresponds to a -1 , it must have a vertical arc e coming into it (Lemma 2.2). If e comes from above and is entirely inside R , then we contradict the maximality of s . If the other endpoint x is on P then there is a directed path $u' \dashrightarrow x \dashrightarrow s \dashrightarrow v$, which contradicts the assumption of this argument. Finally, if neither is true, we contradict Theorem 2.1 (planarity of $G(\lambda, \mu)$). If e only comes into s from below, then by Lemma 2.2 this is the unique vertical arc in its column, and we contradict Lemma 3.1.

By a similar argument to the previous paragraph, s is a source, since otherwise there is a horizontal arc coming into s from the right, and we would either violate the maximality of s , planarity of $G(\lambda, \mu)$, or the initial assumption that there is no path from u' to v .

We assert $\text{row}(s) < \text{row}(v)$: Since v corresponds to a -1 , by Proposition 2.4, it is not a source and hence has a vertical arc e entering it (it cannot be a horizontal arc because of the definition of $G(\lambda, \mu)$). By Lemma 3.1

By pigeonhole, since $r + 1 > s$ there must be two of these integers with the same remainder modulo s . Moreover,

$$b_1 \not\equiv b_2 \pmod{s}$$

since $0 < b_2 < b_1 < s$. So the difference of our two congruent integers from (7) is of the form $\sum_{k \in I} b_k$. Then (λ, μ) has a decomposition where λ^\bullet is given columns of lengths $b_k, k \in I$, and μ^\bullet has $\frac{1}{s} \sum_{k \in I} b_k \in \mathbb{Z}$ columns of length s each. Since $a_1 = b_1 \leq s$ it follows that

$$(\mu^\bullet)' \geq_{\text{Dom}} (\lambda^\bullet)' \text{ and } (\mu^\circ)' \geq_{\text{Dom}} (\lambda^\circ)',$$

where $(\lambda^\circ, \mu^\circ) = (\lambda, \mu) - (\lambda^\bullet, \mu^\bullet)$. □

The proof of Theorem 1.3 is now complete. □

Using the results of this section and the last we derive an additional result:

Theorem 3.2. *Ryser's canonical matrix $A(\lambda, \mu)$ is reducible if and only if $G(\lambda, \mu)$ has a conservative subtree.*

Proof. First, let us establish the converse to Proposition 2.6. That is we claim that if $A^*(\lambda, \mu)$ is $*$ -reducible then $G(\lambda, \mu)$ has a conservative subtree. If $A^*(\lambda, \mu)$ is $*$ -reducible, there is some subset $\emptyset \subset S \subset [\lambda_1]$ of the columns of $A^*(\lambda, \mu)$ whose sum v^* satisfies $0 \leq v_i^* \leq \mu_i^*$ for all i . We may assume that $1 \notin S$, as otherwise we can consider $[\lambda_1] \setminus S$.

If $G(\lambda, \mu)$ is not connected, then we are done, as any connected component of $G(\lambda, \mu)$ is a conservative subtree, by Lemma 2.8. Therefore, we can assume that $G(\lambda, \mu)$ is connected. Lemma 2.7 says that there must be a horizontal edge starting in every column besides column 1, and since every horizontal edge starts at a -1 , every column except column 1 must have a -1 in it. Let $c = \min(S)$. Since $c > 1$, there is some vertex in column c , denoted w , that is a -1 . Since $v_{\text{row}(w)}^* \geq 0$, there must be some vertex u' that is a 1 such that $\text{row}(w) = \text{row}(u')$ and $\text{col}(u') \in S$, so in particular $\text{col}(u') > \text{col}(w)$. As a result, Lemma 3.3 implies that there is a conservative subtree, completing the argument.

The result follows by combining the above argument with Proposition 2.6 and Proposition 2.3. □

Theorem 3.2 combined with Example 2.6 shows that (λ, μ) being reducible does not imply $G(\lambda, \mu)$ has a conservative subtree.

4. Classification of the rays of the Kostka cone

We break up the proof into two propositions. The first verifies that our candidates are indeed extremal rays. The second shows that all extremal rays are of this form.

Proposition 4.1. *Let a, b, ℓ satisfy $a \geq b > 0, \ell \geq 0$, and $\ell + a \leq r$. Then*

$$(\lambda, \mu) = \left(\underbrace{a, \dots, a}_{b+\ell}, 0, \dots, 0; \underbrace{a, \dots, a}_\ell, \underbrace{b, \dots, b}_a, 0, \dots, 0 \right)$$

gives an extremal ray of Kostka_r .

Proof. The only way to write (λ, μ) as a sum of two real partitions is

$$\sum_{i=1}^2 \left(\underbrace{a_i, \dots, a_i}_{b+\ell}, 0, \dots, 0; \underbrace{c_i, \dots, c_i}_\ell, \underbrace{b_i, \dots, b_i}_a, 0, \dots, 0 \right)$$

where $a = a_1 + a_2 = c_1 + c_2$ and $b = b_1 + b_2$ and $c_1 \geq b_1, c_2 \geq b_2$. Suppose that both summands belong to the cone. Then

$$\ell \cdot a_1 \geq \ell \cdot c_1$$

by (1) for $t = \ell$; likewise

$$\ell \cdot a_2 \geq \ell \cdot c_2.$$

These two inequalities add to make $\ell a = \ell a$, so they both hold with equality. Thus $a_1 = c_1$ and $a_2 = c_2$ (or $\ell = 0$, in which case the c_i don't exist). Now we also know

$$\begin{aligned} a_1(\ell + b) &= \ell \cdot c_1 + a \cdot b_1, \\ a_2(\ell + b) &= \ell \cdot c_2 + a \cdot b_2, \end{aligned}$$

which rearrange to $a_1/b_1 = a/b$ and $a_2/b_2 = a/b$ (independent of ℓ). Thus our two summands are parallel. □

Lemma 4.1. *Suppose (λ, μ) lies on an extremal ray of Kostka_r and that all inequalities (1) hold strictly for $1 \leq t < r$. Then (λ, μ) is parallel to (a^b, b^a) for suitable integers $r \geq a \geq b > 0$.*

Proof. Define b to be the length of λ and a that of μ . Consider the families of real partitions

$$\lambda_{\pm\epsilon} := \lambda \pm \frac{1}{b}\epsilon(1^b)$$

and

$$\mu_{\pm\epsilon} := \mu \pm \frac{1}{a}\epsilon(1^a).$$

Then $|\lambda_{+\epsilon}| = |\mu_{+\epsilon}|$ (same for $-\epsilon$), and $\lambda_{\pm\epsilon}, \mu_{\pm\epsilon}$ are all real partitions for $\epsilon > 0$ small enough. Furthermore, for ϵ sufficiently small, all inequalities (1) hold for the pairs $(\lambda_{\pm\epsilon}, \mu_{\pm\epsilon})$. Since (λ, μ) is extremal, it must be true that $\lambda_{+\epsilon}$ and $\lambda_{-\epsilon}$ are parallel and, furthermore, parallel to λ . This implies that, for any $j \leq b$,

$$(\lambda_j + \epsilon/b)/\lambda_j = (\lambda_1 + \epsilon/b)/\lambda_1,$$

hence $\lambda_1 = \lambda_j$. Similarly, $\mu_1 = \mu_j$ for any $j \leq a$. Now scale (λ, μ) so that $\lambda_1 = a$. Since

$$a\mu_1 = |\mu| = |\lambda| = ba,$$

we must have $b = \mu_1$, and the statement is proven. □

Definition 4.1. *Suppose $\nu = (\nu_1 \geq \dots \geq \nu_s)$ and $\xi = (\xi_1 \geq \dots \geq \xi_t)$ are two partitions. If $\nu_s \geq \xi_1$, then*

$$(\nu_1 \geq \dots \geq \nu_s \geq \xi_1 \geq \dots \geq \xi_t)$$

is a partition of $s + t$ parts, the concatenation of ν and ξ , which we denote by $\nu \# \xi$.

Combined with Proposition 4.1 and Lemma 4.1, the next statement completes the proof of Theorem 1.4.

Proposition 4.2. *Suppose (λ, μ) is an extremal ray of Kostka_r where at least one inequality (1) with $t < r$ holds with equality. Then (λ, μ) is parallel to*

$$\left(\underbrace{a, \dots, a}_{b+\ell}, 0, \dots, 0; \underbrace{a, \dots, a}_{\ell}, \underbrace{b, \dots, b}_a, 0, \dots, 0 \right)$$

for suitable $a \geq b > 0$ and $\ell \geq 0$.

Proof. We proceed by induction on $r \geq 1$. The base case $r = 1$ is trivial. Now suppose $r > 1$. Take (λ, μ) to lie on an extremal ray of Kostka_r . For simplicity, assume that $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$ (since the inequalities defining $\text{Kostka}_r \subseteq \mathbb{R}^{2r}$ have rational coefficients, this is possible; cf. [13, §16.2]). Given that at least one inequality (1) holds with equality, let J consist of the integers $j \in \{1, \dots, r-1\}$ such that

$$\sum_{i=1}^j \lambda_i = \sum_{i=1}^j \mu_i.$$

There are two possibilities:

- (A) For all $j \in J$, $\lambda_{j+1} = 0$.
- (B) For some $j \in J$, $\lambda_{j+1} > 0$.

In case (A), take $r' = \min J$. Note that $\lambda_{r'+1} \geq \mu_{r'+1}$ by (1) for $t = r' + 1$, so $\lambda_k = \mu_k = 0$ for all $r' + 1 \leq k \leq r$. Then (λ, μ) comes from $\text{Kostka}_{r'}$ by appending trailing 0s. Furthermore, the inequalities (1) hold strictly for all $1 \leq j < r'$, so $(\lambda, \mu) = (a^b, b^a)$, after scaling, by Lemma 4.1.

Now assume we are in case (B). Pick any $j \in J$ with $\lambda_{j+1} > 0$. Therefore $\lambda_j > 0$ as well. Set $\lambda_{\leq j}$ to be the partition consisting just of $\lambda_1, \dots, \lambda_j$. Let $\lambda_{> j}$ be the rest of λ . That is,

$$\lambda = \lambda_{\leq j} \# \lambda_{> j}.$$

Likewise, $\mu = \mu_{\leq j} \# \mu_{> j}$. Define rational partitions

$$\lambda^{(1)} := (1^j) \# \left(\frac{1}{\lambda_j} \right) \lambda_{> j}$$

$$\lambda^{(2)} := (\lambda_{\leq j} - (1^j)) \# \left(\frac{\lambda_j - 1}{\lambda_j} \right) \lambda_{> j};$$

note that $\lambda = \lambda^{(1)} + \lambda^{(2)}$. The sequence $\lambda^{(1)}$ is a rational partition since

$$1 \geq \frac{1}{\lambda_j} \lambda_{j+1}.$$

Likewise $\lambda^{(2)}$ is nonincreasing since

$$\lambda_j - 1 \geq (\lambda_j - 1) \cdot \frac{\lambda_{j+1}}{\lambda_j}.$$

Furthermore, $\lambda^{(2)}$ has no negative entries since $\lambda_j - 1 \geq 0$ (λ_j is an integer greater than 0), so $\lambda^{(2)}$ is a rational partition.

Observe that

$$\mu_j \geq \lambda_j \tag{8}$$

since

$$\sum_{i=1}^j \lambda_i = \sum_{i=1}^j \mu_i \quad \text{and} \quad \sum_{i=1}^{j-1} \lambda_i \geq \sum_{i=1}^{j-1} \mu_i. \tag{9}$$

In particular, $\mu_j > 0$ since we already saw above that $\lambda_j > 0$. Define

$$\begin{aligned} \mu^{(1)} &:= (1^j) \# \left(\frac{1}{\lambda_j} \right) \mu_{> j} \\ \mu^{(2)} &:= (\mu_{\leq j} - (1^j)) \# \left(\frac{\lambda_j - 1}{\lambda_j} \right) \mu_{> j}; \end{aligned}$$

once again $\mu^{(1)}, \mu^{(2)}$ are rational partitions such that $\mu = \mu^{(1)} + \mu^{(2)}$. To see this for $\mu^{(1)}$, note that

$$\lambda_j \geq \lambda_{j+1} \geq \mu_{j+1}, \tag{10}$$

where the rightmost inequality holds by the equality in (9). For $\mu^{(2)}$, we have

$$\mu_j - 1 \geq \lambda_j - 1 = \frac{\lambda_j - 1}{\lambda_j} \lambda_j \geq \frac{\lambda_j - 1}{\lambda_j} \mu_{j+1},$$

where the last inequality is by (10).

Claim 4.1. *We have $\lambda^{(1)} \geq_{\text{Dom}} \mu^{(1)}$ and $\lambda^{(2)} \geq_{\text{Dom}} \mu^{(2)}$.*

Proof. We will show that $(\lambda^{(1)}, \mu^{(1)})$ satisfy inequalities (1). For any $t \leq j$, the inequality (1) for t holds (trivially) with equality for $(\lambda^{(1)}, \mu^{(1)})$. Therefore the inequalities (1) for $j < t < r$ are equivalent to the inequalities

$$\frac{1}{\lambda_j} \sum_{i=j+1}^t \lambda_i \geq \frac{1}{\lambda_j} \sum_{i=j+1}^t \mu_i, \tag{11}$$

which hold by the assumption

$$\sum_{i=1}^j \lambda_i = \sum_{i=1}^j \mu_i$$

and (1). An analogous argument shows $(\lambda^{(2)}, \mu^{(2)})$ satisfy (1). □

Therefore we have written

$$(\lambda, \mu) = (\lambda^{(1)}, \mu^{(1)}) + (\lambda^{(2)}, \mu^{(2)})$$

as the sum of elements of Kostka_r , so these summands must be parallel. This has several implications: apparently $\lambda_1 = \lambda_2 = \dots = \lambda_j$ and $\mu_1 = \mu_2 = \dots = \mu_j$. Furthermore,

$$\lambda_1 \geq \mu_1 \geq \mu_j \geq \lambda_j,$$

where the last inequality is (8). Thus,

$$\lambda_a = \lambda_b = \mu_a = \mu_b \text{ for all } a, b \in [1, j]. \tag{12}$$

Claim 4.2. $\lambda_{j+1} = \lambda_j$.

Proof. Assuming $\lambda_{j+1} < \lambda_j$, we also have

$$\mu_{j+1} \leq \lambda_{j+1} < \lambda_j = \mu_j,$$

where the leftmost inequality follows from (1) and the fact that $j \in J$, and the equality holds by (12). Hence

$$(\lambda, \mu) \pm \epsilon(1^j, 1^j) \in \text{Kostka}_r$$

for all $0 < \epsilon \leq 1$. Since $\lambda_{j+1} > 0$, we know $(1^j, 1^j)$ is not parallel to (λ, μ) . This implies (λ, μ) is not extremal in Kostka_r , a contradiction. Hence $\lambda_{j+1} = \lambda_j$. \square

Recall the truncated partitions $\lambda_{>j} = (\lambda_{j+1} \geq \dots \geq \lambda_r)$ and $\mu_{>j} = (\mu_{j+1} \geq \dots \geq \mu_r)$. Since $\sum_{i=1}^j \lambda_i = \sum_{i=1}^j \mu_i$, we know that $|\lambda_{>j}| = |\mu_{>j}|$. Moreover, $\lambda_{>j} \geq_{\text{Dom}} \mu_{>j}$ by (11). So $(\lambda_{>j}, \mu_{>j}) \in \text{Kostka}_{r-j}$.

Claim 4.3. $(\lambda_{>j}, \mu_{>j})$ lies on an extremal ray of Kostka_{r-j} .

Proof. If not, we can decompose $\lambda_{>j} = \bar{\lambda}^{(1)} + \bar{\lambda}^{(2)}$ and $\mu_{>j} = \bar{\mu}^{(1)} + \bar{\mu}^{(2)}$ where $(\bar{\lambda}^{(1)}, \bar{\mu}^{(1)})$ and $(\bar{\lambda}^{(2)}, \bar{\mu}^{(2)})$ are nonparallel elements of Kostka_{r-j} . Define the following concatenations of real partitions:

$$\lambda^{(1)} := \left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \lambda_{\leq j} \# \bar{\lambda}^{(1)}$$

$$\lambda^{(2)} := \left(\frac{\bar{\lambda}_1^{(2)}}{\lambda_1} \right) \lambda_{\leq j} \# \bar{\lambda}^{(2)}$$

and

$$\mu^{(1)} := \left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \mu_{\leq j} \# \bar{\mu}^{(1)}$$

$$\mu^{(2)} := \left(\frac{\bar{\lambda}_1^{(2)}}{\lambda_1} \right) \mu_{\leq j} \# \bar{\mu}^{(2)}$$

Subclaim 4.4. Each of $\lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}$ is a real partition with at most r nonzero parts.

Proof. In each concatenation, the two pieces are separately real partitions. We need only verify that at each “#” the last element of the first piece is weakly bigger than the first element of the second piece.

- $(\lambda^{(1)})$: Using (12), $\left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \lambda_j = \left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \lambda_1 = \bar{\lambda}_1^{(1)}$, so the concatenation is good and $\lambda^{(1)}$ is a real partition.
- $(\lambda^{(2)})$: Use the same proof as for $\lambda^{(1)}$, switching upper indices everywhere.
- $(\mu^{(1)})$: Once again by (12), we have $\mu_j = \lambda_1$. Since $\bar{\lambda}^{(1)} \geq_{\text{Dom}} \bar{\mu}^{(1)}$, $\bar{\lambda}_1^{(1)} \geq \bar{\mu}_1^{(1)}$. Putting these together, we get $\left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \mu_j = \left(\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} \right) \lambda_1 = \bar{\lambda}_1^{(1)} \geq \bar{\mu}_1^{(1)}$, as needed.
- $(\mu^{(2)})$: Use the same proof as for $\mu^{(1)}$. \square

Since the first j entries of $\lambda^{(1)}$ and $\mu^{(1)}$ are the same, $\bar{\lambda}^{(1)} \geq_{\text{Dom}} \bar{\mu}^{(1)} \implies \lambda^{(1)} \geq_{\text{Dom}} \mu^{(1)}$. Likewise $\lambda^{(2)} \geq_{\text{Dom}} \mu^{(2)}$. Observe that

$$\frac{\bar{\lambda}_1^{(1)}}{\lambda_1} + \frac{\bar{\lambda}_1^{(2)}}{\lambda_1} = \frac{(\lambda_{>j})_1}{\lambda_1} = \frac{\lambda_{j+1}}{\lambda_1} = 1,$$

where the last equality comes from Claim 4.2 and (12). This implies that $\lambda = \lambda^{(1)} + \lambda^{(2)}$ and $\mu = \mu^{(1)} + \mu^{(2)}$. Since $(\bar{\lambda}^{(1)}, \bar{\mu}^{(1)})$ and $(\bar{\lambda}^{(2)}, \bar{\mu}^{(2)})$ are nonparallel, $(\lambda^{(1)}, \mu^{(1)})$ and $(\lambda^{(2)}, \mu^{(2)})$ are also nonparallel. Therefore we have obtained a nonparallel decomposition of (λ, μ) inside Kostka_r , which contradicts our assumption that (λ, μ) is an extremal ray. \square

By the induction hypothesis, there exist integers $a \geq b > 0$ and $k \geq 0$ such that $(\lambda_{>j}, \mu_{>j}) \in \text{Kostka}_{r-j}$ is parallel to $(a^{k+b}, a^k b^a)$. Hence there exists a positive real number q such that

$$(\lambda_{>j}, \mu_{>j}) = q(a^{k+b}, a^k b^a).$$

This makes $\lambda_{j+1} = qa$. By Claim 4.2 and (12), $\lambda = q(a^{j+k+b})$ and $\mu = q(a^{j+k} b^a)$, as desired (let $\ell = j + k$). This completes the proof of Proposition 4.2. \square

5. Conjectures

5.1 Generalized Catalan sequences

Let $\vec{x} = (x_1, \dots, x_t)$ be a sequence of nonzero integers. We say \vec{x} is *generalized Catalan* if

$$\sum_{i=1}^t x_i = 0 \tag{13}$$

and

$$\sum_{i=1}^q x_i \geq 0 \text{ for all } 1 \leq q \leq t. \tag{14}$$

Define \vec{x} to be *reducible* if there is a Catalan sublist $\vec{x}^\circ = (x_{i_1}, x_{i_2}, \dots, x_{i_a})$ such that the complementary sublist \vec{x}^\bullet is also Catalan.

Example 5.1. $\vec{x} = (\underline{3}, 2, 1, -2, 1, -2, \underline{-1}, \underline{-1}, 2, -1, 2, 1, -2, -1, -1, \underline{-1})$ is a Catalan list. The underlined elements and non-underlined elements separately define two Catalan sublists $\vec{x}^\circ = (3, -1, -1, -1)$ and $\vec{x}^\bullet = (2, 1, -2, 1, -2, 2, -1, 2, 1, -2, -1, -1)$ that decompose \vec{x} .

A maximal consecutive subsequence of \vec{x} consisting of integers of the same sign is called a *run*. By condition (13) there are an even number $2y$ of runs. Let $a_k > 0$ be the maximum (in absolute value) of any x_i in run k . Define

$$\text{cost}(\vec{x}) = \sum_{k=1}^{2y} a_k \text{ and } \text{width}(\vec{x}) = y.$$

Example 5.2. Continuing Example 5.1, $y = 4$, $\text{cost}(\vec{x}) = 3 + 2 + 1 + 2 + 2 + 1 + 2 + 2 = 15$ and $\text{width}(\vec{x}) = 16$.

Conjecture 5.1. If $\text{cost}(\vec{x}) < \text{width}(\vec{x})$ then \vec{x} is reducible.[†]

Definition 5.1. Let $(\lambda, \mu) \in \text{Kostka}_r^{\mathbb{Z}}$. A decomposition $(\lambda, \mu) = (\lambda^\bullet, \mu^\bullet) + (\lambda^\circ, \mu^\circ)$ is commonly reducible if one can choose common columns of λ and μ to give λ^\bullet and μ^\bullet .

This asserts a strengthening of Theorem 1.3:

Conjecture 5.2. If $\lambda_1 > r$ then (λ, μ) is commonly reducible.

Example 5.3. Let $r = 4$ and consider the decomposition

$$(\lambda, \mu) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = (\lambda^\bullet, \mu^\bullet) + (\lambda^\circ, \mu^\circ).$$

This demonstrates the common reducibility of (λ, μ) . Here we interpret $(\lambda^\bullet, \mu^\bullet)$ as obtained from (λ, μ) by using the (possibly empty) columns 1, 5 and $(\lambda^\circ, \mu^\circ)$ as obtained from (λ, μ) by using the complementary columns. Notice that

$$(\lambda, \mu) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

is another decomposition that does not exhibit common reducibility.

The hypothesis of Conjecture 5.2 cannot be dispensed with. For instance,

$$(\lambda, \mu) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

is not in the Hilbert basis of Kostka_4 , but is not commonly reducible.

Proposition 5.1. Conjecture 5.1 \implies Conjecture 5.2.

[†]After the original version of this preprint was posted to the arXiv, J. Kim [10, Theorem 1.1] proved this conjecture.

Proof. Assume $\lambda \geq_{\text{Dom}} \mu$. Define a sequence \vec{x} of length λ_1 by $x_j := \mu'_j - \lambda'_j$. Since $|\lambda| = |\mu|$, (13) holds. If $x_j = 0$ for any j , then (λ, μ) is trivially commonly reducible, so we may assume otherwise. Now notice that $\lambda' \leq_{\text{Dom}} \mu'$ (equivalently, $\mu \leq_{\text{Dom}} \lambda$) is clearly equivalent to (14). Furthermore, from the definitions one checks $\text{cost}(\vec{x}) \leq \ell(\mu)$. Hence, under the hypothesis of Conjecture 5.2,

$$\text{cost}(\vec{x}) \leq \ell(\mu) < r < \lambda_1 = \text{width}(\vec{x}).$$

Therefore Conjecture 5.1's conclusion is that \vec{x} is decomposable into generalized Catalan sequences \vec{x}^\bullet and \vec{x}° , which correspond to, say, columns C and $[\lambda_1] - C$. Then define λ^\bullet and λ° to be columns C and $[\lambda_1] - C$ of λ , respectively. Similarly define μ^\bullet and μ° . Since \vec{x}^\bullet is Catalan, by the equivalence stated earlier in this proof, $\lambda^\bullet \geq_{\text{Dom}} \mu^\bullet$. Similarly $\lambda^\circ \geq_{\text{Dom}} \mu^\circ$. Thus $(\lambda, \mu) = (\lambda^\bullet, \mu^\bullet) + (\lambda^\circ, \mu^\circ)$ witnesses the common reducibility. \square

Using an argument similar to the one for [15, Theorem 6.1] we have a proof of Conjecture 5.2 in the case $y = 1$.

Let us also remark that one can study the decision problem of whether a generalized Catalan sequence \vec{x} is reducible. A modification of the argument for Theorem 1.1 shows that this problem is also NP-complete.

5.2 The Littlewood-Richardson cone

For partitions

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \mu = (\mu_1, \mu_2, \dots, \mu_r), \nu = (\nu_1, \nu_2, \dots, \nu_r),$$

let $c_{\lambda, \mu}^\nu$ be the *Littlewood-Richardson coefficient*. Combinatorially, $c_{\lambda, \mu}^\nu$ counts the number of semistandard tableaux T of skew shape ν/λ of content μ such that the right to left, top to bottom, row reading word is a ballot sequence [14]. Define

$$\text{LR}_r^{\mathbb{Z}} = \{(\lambda, \mu, \nu) : c_{\lambda, \mu}^\nu > 0\}.$$

Like $\text{Kostka}_r^{\mathbb{Z}}$, $\text{LR}_r^{\mathbb{Z}}$ is a finitely generated semigroup; more precisely, $\text{LR}_r^{\mathbb{Z}}$ are the lattice points of a pointed polyhedral cone LR_r defined by the celebrated *Horn inequalities*; we refer to the survey [6] and the references therein. A. Zelevinsky [16] raised the question of studying the Hilbert basis of $\text{LR}_r^{\mathbb{Z}}$. Analogous to Theorem 1.1, we conjecture that the decision problem of deciding if (λ, μ, ν) is a Hilbert basis element is also NP-complete.

In a previous preprint version of this work, we conjectured:

Conjecture 5.3. *If (λ, μ, ν) is in the Hilbert basis of $\text{LR}_r^{\mathbb{Z}}$ then $\nu_1 \leq r$.*

However, we now know this to be false, due to an explicit family of counterexamples that were constructed by J. Kiers and G. Orelowitz; see Appendix B.[‡]

A. NP-completeness and the Hilbert basis

The *Subset Sum problem* (**SubsetSum**) takes as input positive integers a_1, \dots, a_d and b . The output is “yes” if there exists $S \subseteq \{1, 2, \dots, d\}$ such that

$$\sum_{i \in S} a_i = b. \tag{15}$$

This problem is well-known to be NP-complete.

Clearly $\text{KostkaHilbert} \in \text{NP}$: given a proposed decomposition (3) it takes polynomial time to check it. Thus, to prove that KostkaHilbert is NP-complete we give a polynomial-time reduction of **SubsetSum** to it.

Given the input a_1, \dots, a_d, b to **SubsetSum**, let

$$A = \sum_{i=1}^d a_i.$$

After sorting (which takes $O(d \log d)$ -time), we may assume $a_1 \geq a_2 \geq \dots \geq a_d$. We may also assume $b \leq A$, since otherwise **SubsetSum** is trivial; this condition can be checked in $O(d)$ -time. Now define $\bar{\lambda}$ to have columns $A + 1, a_1, a_2, \dots, a_d$. Set $\bar{\mu}$ to have columns $A + 1 + (A - b)$ and b .

Lemma A.1. $\bar{\lambda} \geq_{\text{Dom}} \bar{\mu}$.

Proof. By construction, $|\bar{\lambda}| = (A + 1) + A = A + 1 + (A - b) + b = |\bar{\mu}|$. Since \leq_{Dom} is an anti-automorphism with respect to conjugation [14, Section 7.2], it is equivalent to show that $\bar{\mu}' \geq_{\text{Dom}} \bar{\lambda}'$. Since $\bar{\mu}'$ only has two rows, this inequality is immediate since $\bar{\mu}'_1 = A + 1 + (A - b) \geq A + 1$. \square

[‡]An anonymous referee has suggested a different conjecture, namely, if (λ, μ, ν) is in the Hilbert basis of $\text{LR}_r^{\mathbb{Z}}$ then either $\lambda_1 \leq r$ or $\mu_1 \leq r$. A computer verification shows that both Conjecture 5.3 and this alternative conjecture are true for $r \leq 9$.

In view of Lemma A.1, $(\bar{\lambda}, \bar{\mu})$ is not a Hilbert basis element if it has a nontrivial decomposition (3). This next observation is clear:

Lemma A.2. (λ, μ) is not a Hilbert basis element if and only if both of the following hold:

- (I) the columns of λ^\bullet (resp. μ^\bullet) and λ° (resp. μ°) have multiset union equal to the columns of λ (resp. μ); and
- (II) $\lambda^\bullet \geq_{\text{Dom}} \mu^\bullet$ and $\lambda^\circ \geq_{\text{Dom}} \mu^\circ$.

To complete the proof of Theorem 1.1, it remains to show:

Proposition A.1. $(\bar{\lambda}, \bar{\mu})$ is not a Hilbert basis if and only if `SubsetSum` returns `Yes`.

Proof. (\Rightarrow) Suppose we have a nontrivial decomposition (3) of $(\bar{\lambda}, \bar{\mu})$. By Lemma A.2(I), since μ has precisely two columns (of length $A + 1 + (A - b)$ and b), we may assume $\bar{\mu}^\bullet = (1^b)$. For the same reason, the columns of $\bar{\lambda}^\bullet$ must be a subset S' of the columns of $\bar{\lambda}$. However, since the first column of $\bar{\lambda}$ is of length $A + 1 > b$, that column cannot be in S' . Hence we may take S to be the columns of λ used in S' (after reindexing). By construction, these columns are of length a_1, \dots, a_d and since

$$\sum_{i \in S} a_i = |\bar{\lambda}^\bullet| = |\bar{\mu}^\bullet| = |(1^b)| = b,$$

`SubsetSum` returns `Yes`, as desired.

(\Leftarrow) Suppose $S \subseteq \{1, 2, \dots, d\}$ satisfies (15). Let $\bar{\lambda}^\bullet$ be the partition formed by taking the columns of λ indexed by S ; $\bar{\lambda}^\circ$ is the partition consisting of the remaining columns of λ together with the first column (of length $A + 1 + (A - b)$) of $\bar{\lambda}$. Now, let $\bar{\mu}^\bullet = (1^b)$ and $\bar{\mu}^\circ = 1^{A+1+(A-b)}$. Clearly, (3) holds. Moreover, since $|\bar{\lambda}^\bullet| = b = |\bar{\mu}^\bullet|$ and $\bar{\mu}^\bullet$ is a column partition, $\bar{\mu}^\bullet \leq_{\text{Dom}} \bar{\lambda}^\bullet$ is immediate. Similarly, $\bar{\mu}^\circ \leq_{\text{Dom}} \bar{\lambda}^\circ$ is true. Hence $(\bar{\lambda}, \bar{\mu})$ is reducible and therefore not a Hilbert basis element. \square

Theorem 1.1 can be compared to two related facts. By (1) one has a polynomial-time algorithm (using the same input encoding) for deciding if $(\lambda, \mu) \in \text{Kostka}_{r, \mu}^{\mathbb{Z}}$. In contrast, *counting* $K_{\lambda, \mu}$ is $\#P$ -complete (using an encoding of partitions by rows) [11].

B. Counterexample to Conjecture 5.3, relationship to P. Belkale's conformal block question (by J. Kiers and G. Orelowitz)

The following result gives a counterexample to Conjecture 5.3. In fact, it also shows that one cannot replace the bound of r by any linear function of r .

Theorem B.1. For each $r \in \mathbb{Z}_{>0}$ there exists (λ, μ, ν) in the Hilbert basis of $\text{LR}_r^{\mathbb{Z}}$ such that $\nu_1 > r^2/9$.

The proof for Theorem B.1 uses an algorithm of P. Belkale [1] as recalled in [9, Algorithm 7] that constructs "type I" extremal rays for LR_r . We recall the algorithm below.

The inputs to the algorithm are four sets and an integer (I, J, K, L, a) such that:

- $I, J, K \subset [r]$ with $0 < |I| = |J| = |K| < r$ and $c_{\tau(I), \tau(J)}^{\tau(K)} = 1$,
- L is equal to one of I, J , or K ,
- $a \in L$ such that
 - if $L = K$, $a > 1$ and $a - 1 \notin L$,
 - otherwise, $a < r$ and $a + 1 \notin L$.

and the output is a type I extremal ray of LR_r .

The algorithm proceeds as follows:

- B1. Transform (I, J, K) into (I', J', K') by replacing the a in L with $a + 1$ (or with $a - 1$ if $L = K$), and leaving the other two sets unchanged.
- B2. Construct $\lambda \in \text{Par}$ by setting $\lambda_{b-1} - \lambda_b = 0$ if $b \notin I'$ or $b - 1 \in I'$, and otherwise $\lambda_{b-1} - \lambda_b = c_{\tau(I''), \tau(J')}^{\tau(K')}$ where $I'' = I' \cup \{b - 1\} \setminus \{b\}$, and $\lambda_r = 0$.

B3. Construct $\mu \in \text{Par}$ in the same way as λ , using J' instead of I' .

B4. Construct $\nu \in \text{Par}$, by setting $\nu_b - \nu_{b+1} = 0$ if $b \notin K'$ or $b+1 \in K'$, and otherwise $\nu_b - \nu_{b+1} = c_{\tau(I'), \tau(J')}^{\tau(K'')}$, where $K'' = K' \cup \{b+1\} \setminus \{b\}$, and $\nu_{r+1} = 0$.

The algorithm returns the triple (λ, μ, ν) .

Proof of Theorem B.1. For each $r \equiv 2 \pmod{3}$, we will use Belkale's algorithm to show that there exist type I extremal rays (λ, μ, ν) in LR_r such that $\nu_1 \in \Omega(r^2)$.

Fix $k \geq 2$, and let $r = 3k - 1$. Let

$$I = \{1, 2, 3, \dots, k-2, k-1, 2k\}, J = \{1, 4, 7, \dots, 3k-2\}, K = \{2, 5, 8, \dots, 3k-1\}, L = I,$$

and $a = k - 1$. The only condition for (I, J, K, L, a) to be an input to the algorithm that is not immediate is if $c_{\tau(I), \tau(J)}^{\tau(K)} = 1$. We have that

$$\tau(I) = (k), \tau(J) = (2k-2, 2k-4, \dots, 2), \text{ and } \tau(K) = (2k-1, 2k-3, \dots, 3, 1).$$

Clearly, $\tau(K)/\tau(J)$ is a horizontal strip of size k so by Pieri's rule, $c_{\tau(I), \tau(J)}^{\tau(K)} = 1$. Therefore, (I, J, K, L, a) gives a valid input for the algorithm

Step B1: With the given inputs, $I' = \{1, 2, 3, \dots, k-2, k, 2k\}$, $J' = J$, and $K' = K$. Note that $\tau(I') = (k, 1)$.

Step B2: There are only two elements of I' that can be reduced by 1: the k and the $2k$. If we subtract 1 from the k , then $I'' = I$, so $\lambda_{k-1} - \lambda_k = c_{\tau(I''), \tau(J')}^{\tau(K')} = c_{\tau(I), \tau(J)}^{\tau(K)} = 1$.

On the other hand, if we subtract 1 from the $2k$, then $I'' = \{1, 2, 3, \dots, k-2, k, 2k-1\}$, so $\tau(I'') = (k-1, 1)$.

According to the Littlewood-Richardson rule this means that $c_{\tau(I''), \tau(J')}^{\tau(K')}$ is the number of ballot tableaux of shape $(2k-1, 2k-3, \dots, 3, 1)/(2k-2, 2k-4, \dots, 4, 2)$ filled with $k-1$ 1's and one 2. As long as the 2 is placed in any row other than the first, the filling will be ballot, and since $(2k-1, 2k-3, \dots, 3, 1)/(2k-2, 2k-4, \dots, 4, 2)$ has k rows, $\lambda_{2k-1} - \lambda_{2k} = c_{\tau(I''), \tau(J')}^{\tau(K')} = k - 1$.

For all other $i \in [r]$, either $i \in I'$ or both $i, i+1 \notin I'$, so $\lambda_i - \lambda_{i+1} = 0$ for all $i \neq k-1, 2k-1$. Therefore, $\lambda = ((k)^{k-1}, (k-1)^k)$.

Step B3: For each $i \in [k] \setminus \{1\}$, $3i-2 \in J'$, but $3i-3 \notin J'$, so we can subtract 1 from $3i-2$ to let $J'' = \{1, 4, \dots, 3i-5, 3i-3, 3i+1, 3i+4, \dots, 3k-2\}$. According to the Littlewood-Richardson rule this means that $c_{\tau(I'), \tau(J'')}^{\tau(K')}$ is the number of ballot tableaux filled with k 1's and one 2 and of shape $(2k-1, 2k-3, \dots, 3, 1)/(2k-2, 2k-4, \dots, 2i, 2i-3, 2i-4, 2i-6, \dots, 2)$. If the two is in the rightmost box of any row other than the first, the filling will be ballot, and otherwise it will not be, and since $(2k-1, 2k-3, \dots, 3, 1)/(2k-2, 2k-4, \dots, 4, 2)$ has k rows, $\mu_{3i-3} - \mu_{3i-2} = c_{\tau(I'), \tau(J'')}^{\tau(K')} = k - 1$.

For all other $i \in [r]$, either $i \in J'$ or both $i, i+1 \notin J'$, so $\mu_i - \mu_{i+1} = 0$ for all $i \neq 3j-2$ for any j . Therefore, $\mu = (((k-1)(k-1))^3, ((k-2)(k-1))^3, \dots, (k-1)^3)$.

Step B4: For each $i \in [k]$, $3i-1 \in K'$, but $3i \notin K'$, so we can add 1 to $3i-1$ to let $K'' = \{2, 5, \dots, 3i-4, 3i, 3i+2, 3i+5, \dots, 3k-1\}$. According to the Littlewood-Richardson rule this means that $c_{\tau(I'), \tau(J')}^{\tau(K'')}$ is the number of ballot tableaux filled with k 1's and one 2 and of shape $(2k-1, 2k-3, \dots, 2i+1, 2i, 2i-3, 2i-5, \dots, 3, 1)/(2k-2, 2k-4, \dots, 2)$. If the two is in the rightmost box of any row other than the first, the filling will be ballot, and otherwise it will not be, and since $(2k-1, 2k-3, \dots, 2i+1, 2i, 2i-3, 2i-5, \dots, 3, 1)/(2k-2, 2k-4, \dots, 2)$ has k rows, $\nu_{3i-1} - \nu_{3i} = c_{\tau(I'), \tau(J')}^{\tau(K'')} = k - 1$.

For all other $i \in [r]$, either $i \in K'$ or both $i, i+1 \notin K'$, so $\nu_i - \nu_{i+1} = 0$ for all $i \neq 3j-1$ for any j . Therefore, $\nu = (((k(k-1))^2, ((k-1)(k-1))^3, ((k-2)(k-1))^3, \dots, (k-1)^3)$.

Since $r = 3k - 1$, $k = \frac{r+1}{3}$, and so we have constructed a type I extremal ray of LR_r such that $\nu_1 \in \frac{r+1}{3}(\frac{r+1}{3} - 1) \in \Omega(r^2)$. Since $\lambda_1 = k$ and $\lambda_k = k - 1$ are relatively prime, (λ, μ, ν) is not a multiple of any other element of LR_r . Therefore, (λ, μ, ν) is in the Hilbert basis for LR_r , completing the proof. \square

In fact, Theorem B.1 has an application to a more general setting. Analogous to the Littlewood-Richardson coefficients $c_{\lambda, \mu}^{\nu}$ are the spaces of conformal blocks $\mathbb{V}(\lambda, \mu, \nu, \ell)$ and their dimensions $c_{\lambda, \mu}^{\nu}(\ell)$; here λ, μ, ν are still partitions of length at most r , and ℓ is a natural number. These always satisfy

$$\begin{aligned} c_{\lambda, \mu}^{\nu}(\ell) &\leq c_{\lambda, \mu}^{\nu}; \\ \lim_{\ell \rightarrow \infty} c_{\lambda, \mu}^{\nu}(\ell) &= c_{\lambda, \mu}^{\nu}. \end{aligned} \tag{16}$$

Akin to the cone LR_r , there is a pointed rational cone CB_r generated by all $(\lambda, \mu, \nu, \ell)$ such that $c_{\lambda, \mu}^{\nu}(\ell) \neq 0$. By (16), there is a natural surjective map $\text{CB}_r \twoheadrightarrow \text{LR}_r$ by forgetting the ℓ -coordinate.

The extremal rays of CB_r have been studied by Belkale in [2], and the rays formulas of [1] and [2] are compatible in that every type I extremal ray of LR_r has a lift to a type I extremal ray of CB_r .

A trivial property of the coefficients $c_{\lambda, \mu}^{\nu}(\ell)$ is that they are 0 unless $\max\{\lambda_1 - \lambda_r, \mu_1 - \mu_r, \nu_1 - \nu_r\} \leq \ell$. So in the proof of Theorem B.1, the type I ray (λ, μ, ν) of LR_{3k-1} lifts to a type I ray $(\lambda, \mu, \nu, \ell)$ of CB_{3k-1} for which $\ell \geq (k-1)^2$ (in fact, $\ell = (k-1)^2 + 1$). This violates the original speculation that $\ell < r$ for the type I extremal rays put forth in the preprint [2, Remark 1.6.5].

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