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Enumerative Combinatorics and Applications

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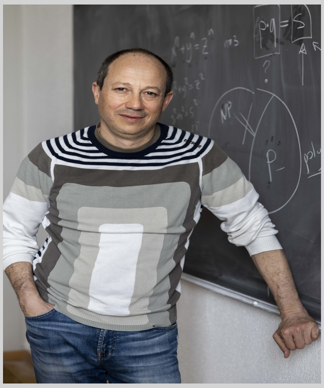
Interview with Benny Sudakov

Toufik Mansour



Benny Sudakov received his Ph.D. from Tel Aviv University in 1999 under the supervision of Noga Alon. Over the course of his career, he has held positions at Princeton University, the Institute for Advanced Studies, and the University of California, Los Angeles. Since 2013, Sudakov has been a professor of mathematics at ETH Zurich. He has received several prestigious awards, including a Sloan Fellowship, NSF CAREER Award, and the Humboldt Research Award. Sudakov is a Member of the Academia Europaea and a Fellow of the American Mathematical Society. He was also an invited speaker at the 2010 International Congress of Mathematicians. With more than 300 scientific publications to his name, Sudakov serves on the editorial board of 14 research journals. His primary scientific interests lie in combinatorics and its applications to other areas of mathematics and computer science.

Mansour: Professor Sudakov, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Sudakov: Combinatorics is a fundamental mathematical discipline that focuses on the study of discrete objects and their properties. Examples of such objects are graphs, hypergraphs, finite sets, subsets of integers or other algebraic structures, and permutations.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Sudakov: Combinatorics has experienced remarkable growth in the past eighty years, evolving into a vibrant and mature area with its own distinctive set of problems, approaches, and methodologies. This rapid expansion can be attributed, in large part, to the development of powerful techniques, based on ideas from many mathematical fields such as probability, algebra, harmonic analysis, and topology. These tools play an important organizing role in combinatorics, similar to the one that deep theorems of great generality play in more classical areas of mathematics. Moreover, the relationship between combinatorics and other areas of mathematics is mutually beneficial, with combinatorics finding significant applications in various mathematical disciplines. These include number theory, probability, geometry, functional analysis, logic, and theoretical computer science.

Mansour: What have been some of the main goals of your research?

Sudakov: While my typical focus lies in resolving fairly specific combinatorial problems, I tend to choose the ones where one can use probabilistic and algebraic methods. Moreover, I try to choose questions whose solution might potentially lead to the development of new techniques that are relevant to a broad array of other combinatorial problems.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Sudakov: My interest in mathematics was influenced by two individuals. First and fore-

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most, my father played a pivotal role. From a very early age (as far back as I can remember), he engaged me in discussions about mathematics and presented me with various captivating mathematical puzzles. Later on, he introduced me to the journal "Kvant" (Russian for "quantum"), a popular science magazine covering physics and mathematics for school students in the USSR. My interest in mathematics was further nurtured by my middle school math teacher. Recognizing that I was finding the standard curriculum somewhat boring, she provided me with intriguing and demanding problems to solve instead.

Mansour: Were there specific problems that made you first interested in combinatorics?

Sudakov: I believe that the first significant combinatorial problem that captured my interest was the well known Erdős-Faber-Lovász conjecture[1], which is a problem on graph coloring. I saw it in some popular article during my initial year as an undergraduate. During that time, I briefly thought about it and even managed to prove some very special cases. However, due to the limited presence of combinatorialists in Tbilisi, where I pursued my undergraduate studies, I eventually transitioned to exploring differential topology and geometry.

Mansour: What was the reason you chose Tel Aviv University for your Ph.D. and your advisor Noga Alon?

Sudakov: My reason for choosing Tel Aviv University was very simple: I lived in Tel Aviv, and it was nearby. During university, I began conversing with Gregory Gutin, who was Noga's student at the time. He recommended that I take Noga's course on the probabilistic method. I enjoyed the subject greatly and consequently asked Noga if I could pursue my master's thesis under his guidance.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Sudakov: In my research, I typically begin with a specific question. This makes it a bit easier to initiate thinking. Having said that, I always attempt to select a problem whose solution I believe can shed some light on other problems in the area. Indeed, there have been several instances where solving one problem has enabled me to develop techniques that had applicability beyond the original question.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Sudakov: Many times, this is indeed the case and it motivates me to work harder in an attempt to prove something that I believe is true. However, I should also acknowledge that I have been wrong several times. Therefore, I have learned over time not to be overly confident, and if I cannot prove something for an extended period, I also search for counterexamples.

Mansour: What three results do you consider the most influential in graph theory during the last thirty years?

Sudakov: There are undoubtedly more than three such results that one could mention. However, to be specific, here are three results that I find extremely influential:

The Szemerédi regularity lemma[2], which I will discuss a bit later in this interview.

Graph minor theory and the Robertson-Seymour theorem[3], which states that every family of graphs closed under taking minors can be characterized by a finite set of excluded minors.

My third choice will be the Lovász local lemma[4], which is a powerful probabilistic tool for solving problems with dependencies between events.

Mansour: What are the top three open questions in your list?

Sudakov: The first question concerns Ramsey numbers of 3-uniform hypergraphs. The Ramsey number $r^{(3)}(n)$ is the smallest positive integer N such that any 2-coloring of the edges of the complete 3-uniform N-vertex hypergraph contains a monochromatic set of size n, mean-

[1] D. Y. Kang, T. Kelly, D. Kühn, A. Methuku, and D. Osthus, *A proof of the Erdős-Faber-Lovász conjecture*, Ann. of Math. 198(2) (2023), 537–618.

[2] J. Komlós and M. Simonovits, Szemerédi's, Regularity lemma and its applications in graph theory, Bolyai Society Mathematical Studies 2, Combinatorics, Paul Erdős is Eighty (Volume 2) (D. Miklós, V. T. Sós, T. Szönyi eds.), Keszthely (Hungary) (1993), Budapest (1996), 295-352.

[3] N. Robertson and P. D. Seymour, *Graph Minors. XX. Wagner's conjecture*, J. Combin. Theory, Ser. B 92(2) (2004), 325–357.

[4] N. Alon and J. Spencer, *The probabilistic method*, Chapter 5, Wiley, 4th Ed., 2015.

[5] D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory*, in: Surveys in Combinatorics 2015, Cambridge University Press, 2015, 49-118.

ing a set in which all triples have the same color. This problem has a long history[5], dating back many decades, with the central question being whether these numbers grow exponentially or doubly-exponentially. The best lower bound we have is of the order of magnitude 2^{cn^2}, while the best upper bound is $2^{2^{cn}}$, where c denotes an absolute constant.

Another problem[6] I am interested in is related to Turán numbers of bipartite graphs. Formally, the Turán number of a fixed graph H, denoted by $ex(n, H)$, is the maximum possible number of edges in an n-vertex graph that does not contain H as a subgraph. When H is bipartite, there are relatively few results that determine the order of magnitude of these numbers. Furthermore, we lack a good understanding of which graph parameters of H we should examine to estimate $ex(n, H)$. A well-known conjecture by Erdős from 1967 suggests that the local density of a graph should be one of these parameters. A graph H is *s-degenerate* if every one of its subgraphs contains a vertex of degree at most s. Specifically, for an s-degenerate bipartite graph H, this conjecture says that $ex(n, H) = O(n^{2-1/s})$.

My third favorite problem is attributed to Brown, Erdős and Sós. We denote by $f(n, v, e)$ the maximum number of edges a 3-uniform hypergraph with n vertices can have without containing a set of v vertices that span at least e edges. In 1973, Brown, Erdős and Sós[7] initiated the study of this function and conjectured that $f(n, e+3, e) = o(n^2)$. A celebrated result by Ruzsa and Szemerédi[8], which has surprising applications in number theory, settles the first non-trivial instance of this conjecture, namely the case $v = 6, e = 3$. However, even the next case, when $(v = 7, e = 4)$, remains completely open.

Mansour: What kind of mathematics would you like to see in the next ten to twenty years as the continuation of your work?

Sudakov: In my research, I always strive to emphasize methods rather than the specific problems I am solving. Therefore, I am hopeful that over the next ten to twenty years, we will witness further development of general techniques with broad applicability.

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Sudakov: I believe that at any given moment, there are both more and less active areas in mathematics. I think that fields experiencing exciting developments become notably more active and draw the attention of many young researchers. As a result, one hears more about the results emerging from these areas, which might create an impression that these areas are more important. However, with the passage of time, the focus shifts, and different areas take the lead in activity.

For instance, consider the state of additive combinatorics about 40 years ago. At that time, I think only a few people were working in this field. However, the situation changed rapidly, and today, it is a highly central area with extensive connections to numerous other domains of mathematics.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Sudakov: I believe it depends on the specific subject within applied mathematics. Certain areas of applied mathematics closely resemble pure mathematics. In these domains, scientists also engage in theorem proving and establish rigorous results. However, there are equally significant areas of applied mathematics where the focus lies on developing heuristics and practical algorithms. These approaches might not always provide definitive solutions, but they perform well in real-world instances and enable efficient simulation of natural phenomena. In my research, I primarily focus on proving theorems, although some problems that capture my interest can be motivated by applications.

Mansour: You have supervised several in their Ph.D. thesis. What do you think about the importance of working with Ph.D. students and passing knowledge to them? Do you follow

[6] B. Sudakov, *Recent developments in extremal combinatorics: Ramsey and Turan type problems*, Proc. International Congress of Mathematicians, Hyderabad, India, 2010, Volume 4, 2579–2606.

[7] W. G. Brown, P. Erdős, and V. Sós, *Some extremal problems on r-graphs*, New directions in the theory of graphs, 53–63, 1973.

[8] I. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, in Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18, Volume II, 939–945.

your students after they complete their thesis?

Sudakov: I find it extremely important to have Ph.D. students and to collaborate with them. Firstly, this collaboration plays a crucial role in passing the existing knowledge within our field to the next generation of scientists. Additionally, Ph.D. students often introduce new and innovative viewpoints to the research projects. Their unique ideas and enthusiasm frequently result in fresh methods and problem-solving approaches, ultimately driving progress in our area. Many of my Ph.D. students have become my frequent collaborators, and our collaborations often extend well beyond the completion of their theses.

Mansour: What advice would you give to young researchers thinking about pursuing a research career in mathematics?

Sudakov: I believe that the most crucial aspect is to pursue a research career only if one genuinely derives pleasure from contemplating problems, meaning that the enjoyment should come also from the process itself, not only from the eventual outcome. Mathematics demands a considerable amount of patience. It is akin to a marathon rather than a sprint, and it is important not to lose hope if progress seems slow at the outset, especially when observing others who appear to possess greater knowledge or accomplish tasks more quickly. Just as in a marathon, it does not matter what is happening after the first five kilometers. What's important is the sustained effort and dedication over the entire course.

Mansour: Would you tell us about your interests besides mathematics?

Sudakov: I enjoy going hiking; this is an especially convenient hobby to have when one lives in Switzerland. Additionally, I have an interest in opera and classical music, and I take pleasure in reading good books.

Mansour: You gave talks at numerous conferences, workshops, and seminars. What do you think about the importance of such activities for researchers?

Sudakov: I believe that both giving and listening to talks at conferences are very important. It often occurs that while the idea behind the solution to a problem is conceptually simple, this simplicity can be lost when one writes the formal proof. Conference talks provide a good platform to communicate and learn such ideas directly from the author of the paper.

Mansour: *Ramsey numbers* for various families of graphs are discussed in your articles. Would you tell us about some significant recent results in this direction? What are your favorite open problems in Ramsey Theory?

Sudakov: Several remarkable achievements in Ramsey theory have been made even this year alone. To recap, the Ramsey number $r(t,n)$ is the smallest positive integer N such that any red/blue edge-coloring of the complete N-vertex graph contains either a red set of size t or a blue set of size n. A fundamental result by Erdős and Szekeres[9] states that $r(n,n) \leq 4^n$ in the diagonal case, and $r(t,n) \leq n^{t-1}$ in the off-diagonal case.

A major open problem that I find very intriguing is to determine the limit of $c := R(n,n)^{1/n}$. Until very recently, the best-known bounds were $\sqrt{2} \leq c \leq 4$. However, Campos, Griffiths, Morris, and Sahasrabudhe[10] made a dramatic improvement by establishing an upper bound of $c \leq 3.999$.

As for the off-diagonal case, the best general lower bound at the moment is $r(t,n) \geq \tilde{\Omega}(n^{(t+1)/2})$, where the $\tilde{\Omega}$-notation accounts for a polylogarithmic factor. This bound is tight when $t = 3$, and until this year, it was the only case where tight bounds were known. In a major breakthrough, Mattheus and Verstraete[11] recently proved that $r(4,n) = \tilde{\Omega}(n^3)$, matching the Erdős and Szekeres upper bound. This provides strong evidence for the possibility that $r(t,n) = \tilde{\Theta}(n^{t-1})$ for every t, which is another one of my favorite questions in graph Ramsey numbers.

Mansour: In a joint paper with Asaf Ferber and Michael Krivelevich, *Counting and packing Hamilton cycles in dense graphs and oriented graphs*[12], you presented a general method for counting and packing Hamilton cycles in dense graphs and oriented graphs based on permanent estimates. Please explain the main ideas behind this method and the results.

[9] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compos. Math. 2 (1935), 463–470.

[10] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe, *An exponential improvement for diagonal Ramsey*, arXiv:2303.09521.

[11] S. Mattheus and J. Verstraete, *The asymptotics of $r(4,t)$*, arXiv:2306.04007.

[12] A. Ferber, M. Krivelevich, and B. Sudakov, *Counting and packing Hamilton cycles in dense graphs and oriented graphs*, J. Combin. Theory, Ser. B 122 (2017), 196–220.

Sudakov: Suppose we have a d-regular graph G on n vertices, with $d \geq n/2$. A celebrated theorem by Dirac then states that G has a Hamilton cycle. It appears that one can actually demonstrate much more: such a graph G contains $d/2$ edge-disjoint Hamilton cycles and exponentially many distinct Hamilton cycles. In their most robust forms, these results were proven by Csaba, Kühn, Lo, Osthus, and Treglown[13], as well as by Cuckler and Kahn[14]. Our paper, co-authored with Ferber and Krivelevich, introduces an alternative approach for establishing similar results.

Given a d-regular graph, we consider $Per(A_G)$, where A_G is the adjacency matrix of G. For an $n \times n$ matrix B, its permanent $Per(B)$ is defined as $Per(B) = \sum_\sigma \prod_{i=1}^n b_{i\sigma(i)}$, where the sum is taken over all permutations of $[n]$. From the definition of the permanent, one can observe that for the matrix A_G, the permanent counts the number of 2-factors in G, which are collections of vertex-disjoint cycles covering all the vertices. Since G is regular, we can use well-known estimates for the permanent, such as the Van der Waerden and Minc conjectures (now theorems, due to Egorychev[15]–Falikman[16] and Brégman[17], respectively), to obtain very precise estimates of $Per(A_G)$. These estimates readily imply that G has, for example, exponentially many 2-factors. By leveraging these estimates more rigorously, we can demonstrate that G possesses exponentially many 2-factors, each of which contains very few cycles, say fewer than $n^{2/3}$ cycles.

Given that the degree of the graph, d, is greater than or equal to $n/2$, we can employ standard rotation techniques, similar to those used in proving Dirac's theorem[18], to convert each such 2-factor into a genuine Hamilton cycle. While these cycles may not necessarily be distinct, we can utilize the fact that only a few edges are altered in the process to demonstrate that the number of distinct Hamiltonian cycles we obtain is exponentially large. Using these techniques, we can also find approximately $d/2$ edge-disjoint 2-factors with only a few cycles in each. This can be achieved even if we initially set aside a small number of random edges from G. We then proceed to transform each of these 2-factors into a Hamilton cycle, utilizing a few freshly reserved random edges that were set aside at the beginning of the process.

Mansour: In a joint paper *Graph Products, Fourier Analysis, and Spectral Techniques*[19], coauthored with Noga Alon, Irit Dinur, and Ehud Friedgut, by using Fourier analysis on abelian groups and spectral techniques, you studied powers of regular graphs defined by the weak graph product and provided a characterization of maximum-size independent sets for a wide family of base graphs. Would you tell us about it?

Sudakov: Consider a road junction equipped with k switches that control the traffic lights. Each switch has three states, and the states of the switches determine the color of the traffic lights: red, yellow, or green. You've been informed that whenever you adjust the position of *all* the switches simultaneously, the color of the traffic lights changes. Prove that, in fact, the traffic lights are controlled by only one of these switches.

This problem represents a specific case related to our research, where we focused on characterizing optimal colorings and maximal independent sets in powers of regular graphs. In this context, the "weak k-th power" of a given graph G, denoted as G^k, is a graph whose vertex set consists of all possible vectors $(u_1, \ldots, u_k)$, where $u_i \in G$. Two vertices, $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$, are connected in G^k if and only if $u_i v_i$ forms an edge in G for all $1 \leq i \leq k$. The configuration space of the switches described above can be modeled by the k-fold product of a triangle K_3.

In our paper, we study the powers of regular graphs, including complete graphs, line graphs of regular graphs with perfect matchings, and Kneser graphs, among others. We

[13] B. Csaba, D. Kühn, A. Lo, D. Osthus, and A. Treglown, *Proof of the* 1*-factorization and Hamilton decomposition conjectures*, Memoirs AMS 244 (2016), Monograph 1154.

[14] B. Cuckler and J. Kahn, *Hamiltonian cycles in Dirac graphs*, Combinatorica 29(3) (2009), 299–326

[15] G. Egorychev, *The solution of the Van der Waerden problem for permanents*, Dokl. Akad. Nauk SSSR 258 (1981), 1041–1044.

[16] D. Falikman, *A proof of the Van der Waerden problem for permanents of a doubly stochastic matrix*, Mat. Zametki 29 (1981), 931–938.

[17] L. M. Brégman, *Some properties of non-negative matrices and their permanents*, Sov. Mat. Dokl. 14 (1973), 945–949

[18] R. Diestel, *Graph theory*, Chapter 10, Springer 5th Ed., 2017.

[19] N. Alon, I. Dinur, E. Friedgut, B. Sudakov, *Graph products, Fourier analysis and spectral techniques*, Geom. Funct. Anal. 14 (2004), 913–940.

provide a comprehensive characterization of maximum-size independent sets and, in many cases, determine optimal colorings for these products. Our findings reveal that the independent sets induced by the base graph are the only maximum-size independent sets. Furthermore we give a qualitative stability statement: any independent set of size close to the maximum is close to some independent set of maximum size.

Our approach relies on Fourier analysis applied to Abelian groups and makes use of spectral techniques. To facilitate this, we develop fundamental lemmas concerning the Fourier transform of functions on the set $\{0, 1, \ldots, r-1\}^n$, extending and generalizing useful results from the $\{0, 1\}^n$ case.

Moreover, our paper suggests that many tools from harmonic analysis hold for general tensor products of some fixed orthogonal set of vectors, and not only for characters of abelian groups. Although this fact is not difficult and its proof essentially amounts to a change of basis, it seems very powerful and may well lead to additional interesting consequences.

Mansour: One of your research interests includes *Application of Combinatorics to Theoretical Computer Science*. Would you tell us about some specific problems that originated in theoretical computer science but became popular among combinatorialists?

Sudakov: I may not have a specific problem in my mind that originated in computer science and transitioned to combinatorics, but there are many problems that are central to both fields. In some of these cases, interesting methods developed by computer scientists later find valuable applications in proving combinatorial results.

One such problem is the Max-Cut problem, which involves finding a partition of a graph's vertices into two complementary sets, S and T, such that the number of edges between S and T is maximized. Finding the exact solution to the Max-Cut problem is known to be NP-hard, which means that we do not anticipate the existence of a polynomial-time algorithm that can guarantee finding the optimal solution for all instances of the problem. Consequently, in computer science, substantial efforts have been devoted to developing efficient approximation algorithms for Max-Cut. The most well-known algorithm for this purpose is attributed to Goemans and Williamson[20], and it is based on semidefinite programming.

In combinatorics, there is a long history of research aimed at providing the best lower bounds for the Max-Cut of various families of graphs. One extensively studied family is the set of H-free graphs for some fixed, small graph H. Many such extremal results for the Max-Cut problem rely on intricate probabilistic arguments. Recently, in collaboration with Carlson, Kolla, Li, Mani, and Trevisan[21], we proposed a different approach to establishing lower bounds on the Max-Cut of sparse H-free graphs using approximation through semidefinite programming (SDP).

This approach is intuitive and computationally straightforward. The main inspiration came from the celebrated approximation algorithm by Goemans and Williamson, mentioned above. In this approach, given a graph G with m edges, we initially construct an explicit solution for the standard Max-Cut SDP relaxation of G, which has a value of at least $(1/2+W)m$ for some positive surplus W. Subsequently, we apply Goemans–Williamson randomized rounding, based on the sign of the scalar product with a random unit vector, to extract a cut in G whose surplus is within a constant factor of W.

Mansour: In a survey paper *Dependent random choice*[22] coauthored with Jacob Fox, you bring attention to what you call *a simple and yet surprisingly powerful probabilistic technique that shows how to find in a dense graph a large subset of vertices in which all (or almost all) small subsets have many common neighbors*. Would you tell us more about this work?

Sudakov: The basic idea of the dependent random choice technique is very simple. In order to find a large subset U of graph G in which every set of, say, d vertices has many common neighbors, one can take U to be the set of all common neighbors of an appropriately chosen

[20] M. X. Goemans and D. P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM 42(6) (1995), 1115–1145

[21] C. Carlson, A. Kolla, R. Li, N. Mani, B. Sudakov, and L. Trevisan, *Lower Bounds for Max-Cut in \$H\$-Free Graphs via Semidefinite Programming*, SIAM J. Discrete Math. 35:3 (2021), 1557–1568.

[22] J. Fox and B. Sudakov, *Dependent random choice*, Random Struct. Algor. 38:1-2 (2010), 68–99.

random subset of vertices R. Intuitively, it is clear that if some set of d vertices has only a few common neighbors, then it is unlikely that all the members of R will be chosen among these neighbors. Hence, we do not expect U to contain any such subset of d vertices. Moreover, if G has many edges then one can expect U to be large.

The main idea of this methodology is that in the course of a probabilistic proof, it is often more effective not to make the choices uniformly at random, but to try and make them depend on each other in a way tailored to the specific argument needed. While this sounds somewhat ambiguous, this simple reasoning and its various extensions have already found many applications in extremal graph theory, additive combinatorics, Ramsey theory and combinatorial geometry. Jacob and I detail many of these applications in our survey.

Mansour: Are there any results in graph theory that you consider very surprising or unintuitive?

Sudakov: The result that I consider extremely surprising and not a priori intuitive is the Szemerédi regularity lemma. It roughly states that the vertices of any sufficiently large graph can be partitioned into a bounded number of parts, so that the edges between most pairs of different parts exhibit behavior that is nearly random. This result has revolutionized the field of graph theory and has many applications in other areas of mathematics. I find it truly amazing that such a useful result, applicable to arbitrary graphs, can be proven.

Mansour: An interesting article *A New Path to Equal-Angle Lines*[23], published in *Quanta Magazine*, a popular science magazine, explains some of your research works. Please tell us more about the topics mentioned in that article.

Sudakov: A basic result in geometry states that the maximum number of equidistant points in d-dimensional Euclidean space is $d+1$, a configuration realized by a simplex. A closely related question is to find the maximal number of lines through the origin such that any pair of lines defines the same angle. We call such lines equiangular. This natural question appears to be very difficult. It is easy to show that in the plane we can have at most 3 lines. But for large dimensions we do not know exact or even asymptotic estimates. We only know that the maximum number of equiangular lines is quadratic in the dimension d. Intriguingly, in all constructions with quadratically many lines, the cosine of the common angle between the lines tends to zero when d tends to infinity. On the other hand, all known constructions of equiangular lines with a fixed common angle have significantly smaller sizes. Therefore, about 50 years ago, Lemmens and Seidel[24] asked to determine the maximum number of equiangular lines with a fixed common angle which does not depend on the dimension.

Interestingly, this geometric problem can be attacked using spectral graph theory and Ramsey theory. Indeed, together with Balla, Dräxler and Keevash[25], we answered this question and showed that the maximum number of equiangular lines with a fixed angle is $2d-2$. Moreover, this can only be achieved for the angle with cosine $1/3$. More recently, following our work, Jiang, Tidor, Yao, Zhang, and Zhao[26] wrote a beautiful paper, where they determine how the maximum number of lines varies based on each given fixed angle.

Mansour: In a very recent paper *Counting H-free orientations of graphs*[27], co-authored with Matija Bucić and Oliver Janzer, you provided an answer to one of Erdős' problems (1975), asking to determine or estimate the maximum possible number of H-free orientations of an n-vertex graph for a given oriented graph H. Would you elaborate more on this result?

Sudakov: Given a fixed directed graph H and an undirected graph G, an orientation of G is called H-free if it does not contain H as a directed subgraph. Let $D(n,H)$ denote the maximum possible number of H-free orientations that n-vertex graph G might have. Erdős' question was to estimate $D(n,H)$. For an undirected graph F, let $ex(n,F)$ be the max-

[23] See, `https://www.quantamagazine.org/a-new-path-to-equal-angle-lines-20170411/`.

[24] P. W. H. Lemmens and J. J. Seidel, *Equiangular lines*, J. Alg. 24:3 (1973), 494–512.

[25] I. Balla, F. Dräxler, P. Keevash, and B. Sudanov, *Equiangular lines and spherical codes in Euclidean space*, Invent. math. 211 (2018), 179–212.

[26] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, and Y. Zhao, *Equiangular lines with a fixed angle*, Ann. of Math. 194 (2021), 729–743.

[27] M. Bucić, O. Janzer, and B. Sudakov, *Counting H-free orientations of graphs*, Math. Proc. Cambridge Philosophical Soc. 174(1) (2023), 79–95.

imum number of edges in an n-vertex F-free graph. Writing F for the underlying undirected graph of H, we have a trivial lower bound $D(n,H) \geq 2^{ex(n,F)}$ since if G is an F-free graph, then any orientation of G is H-free. When H is a tournament or H is a directed odd cycle, this simple lower bound gives the correct answer, which was shown by Alon and Yuster[28] and by myself and my coauthors respectively. Using a short and beautiful argument involving a version of the classical Sauer–Shelah lemma on the VC dimension of set systems, Kozma and Moran[29] proved that the number of orientations of a *fixed* graph G without H is always at most the number of F-free subgraphs of G, where as usual F is the underlying graph of H. Hence, one can obtain upper bounds for $D(n,H)$ from known results on the number of n-vertex F-free graphs, which is an extensively studied subject on its own.

The above results provide us with a good understanding of $D(n,H)$ whenever the underlying graph of H contains a cycle. This leads to the natural question of what happens in the remaining case, namely when H is an orientation of a forest F. The previous pargarph also suggests that $D(n,H)$ should always be $2^{\Theta(ex(n,F))}$. However, perhaps surprisingly, it turns out that this is not the case. Together with Matija and Oliver we managed to resolve all remaining cases by showing that for every oriented forest H with at least two edges, either $D(n,H) = 2^{\Theta(n)}$ or $D(n,H) = 2^{\Theta(n \log n)}$. We also provide a precise characterisation for when each case occurs.

We call an oriented graph H *1-almost antidirected* if there exists a bipartition $V(H) = A \cup B$ of the vertex set such that there are no edges inside $H[A]$ and $H[B]$, every $u \in A$ has at most one incoming edge and every $v \in B$ has at most one outgoing edge. For example, the path with the usual orientation is 1-almost antidirected, but there are other orientations of the path that are not. We showed that for an oriented forest H with at least two edges, $D(n,H) = 2^{\Theta(n)}$ if H is 1-almost antidirected and $2^{\Theta(n \log n)}$ otherwise.

Mansour: If you were asked to list three or four famous results on finite sets, what would be on your list and why?

Sudakov: I would begin with Sperner's theorem[30], which characterizes the largest possible collection of finite sets, none of which contains any other set within the collection. This is likely the oldest result in extremal set theory. It has several unexpected applications, for example, to the theory of random discrete matrices. There even exists an entire book titled "Sperner Theory" dedicated to this theorem and its numerous variations and extensions.

The next result I want to mention is the celebrated Erdős–Ko–Rado theorem[30], which bounds the number of k-element subsets of an n-element ground set, such that any two sets share at least one element. This theorem has been extended to various types of mathematical objects other than sets, including linear subspaces, permutations, and strings. There exist several proofs of this theorem, employing important tools such as probabilistic methods and spectral techniques.

Finally, I would also like to mention Lovász's theorem[31], which determines the chromatic number of Kneser graphs. These graphs have vertices represented by the k-element subsets of an n-element set, and two vertices are connected if and only if the corresponding sets are disjoint. This theorem stands out as one of the most elegant and surprising applications of topological methods in combinatorics. It laid the foundation for the field of topological combinatorics.

Mansour: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

Sudakov: I work on enumerative problems from time to time. The previously mentioned result concerning the counting of H-free orientations serves as an example. However, in my research, I don't extensively rely on classical enumerative techniques like bijective proofs or generating functions. In the problems I have studied, the typical approach involves proving that the majority of objects I intend to count possess a very specific structure. Once this structure is established, the counting process becomes relatively straightforward. For

[28] N. Alon and R. Yuster, *The number of orientations having no fixed tournament*, Combinatorica 26:1 (2006), 1—16.
[29] L. Kozma and S. Moran, *Shattering, graph orientations, and connectivity*, Electron. J. Combin. 20:3 (2013), P.44.
[30] P. Frankl and N. Tokushige, *Extremal problems for finite sets*, AMS, 2010.
[31] J. Matoušek, *Using the Borsuk-Ulam theorem*, Chapter 3, Springer, 2003.

example, one can prove that the majority of triangle-free graphs are bipartite. Providing a good estimate on the number of bipartite graphs is already not a particularly challenging problem.

Mansour: In the BennyFest, a combinatorial meeting celebrating your 50th birthday, during your thank you speech, you said: “There is a very wise saying that for a successful life, you need four things: a loving family, good friends, wise teachers, and talented students.” How does each of them play an essential role in our lives?

Sudakov: Let me begin with the importance of having wise teachers. Among the many things I learned from my teachers, two stand out in terms of their significance. Firstly, they helped me develop a refined taste in mathematics. I learned how to recognize elegant and beautiful proofs and statements, as well as how to select interesting problems worthy of study. Secondly, I gained an understanding of the caliber of research I should aspire to. It's evident that not every result one produces will be groundbreaking, but having a clear understanding of the level of achievement to strive for is crucial.

Having students has been incredibly important to me, as I derive great satisfaction from collaborating with them. Since early on in my career, students have served as my primary collaborators, and I've been fortunate over the years to work with exceptionally talented individuals. I believe that a significant portion of the research I take pride in has been accomplished alongside my Ph.D. students and postdocs.

Finally, I was fortunate to have a loving family and good friends who provided a foundation of support, care, and emotional wellbeing that significantly contributed to my success and overall happiness.

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a “eureka moment”?

Sudakov: I will tell below a story that I believe answers this and the next questions together.

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Sudakov: One of my favorite problems, which I worked on intermittently for about ten years, was a conjecture by Erdős about Ramsey numbers of graphs with a fixed size. Recall that the Ramsey number of a graph G is the smallest positive integer n such that, in any 2-edge coloring of the complete graph on n vertices, contain a monochromatic copy of G. This conjecture states that if G has m edges and no isolated vertices, then its Ramsey number is at most $2^{O(\sqrt{m})}$. In other words, up to a constant factor in the exponent, the complete graph with m edges has the maximum Ramsey number among all m-edge graphs.

Together with Noga Alon and Michael Krivelevich, we proved this result with an extra logarithmic factor in the exponent. However, the general case eluded me for quite some time. I remember being at a conference with David Conlon where we discussed this problem again. On the plane back home from the conference, I started thinking about this problem once more. Suddenly, I had indeed a “eureka moment”. I realized that one could attempt to use an old result of Erdős-Szemerédi to tackle this problem. By the time the flight was over, I had managed to improve the logarithmic factor from $\log m$ to $\log \log m$ and was quite confident that one could push this even further. Indeed, very soon after, using the same approach, I was able to prove the entire conjecture. Later, Jacob Fox suggested to me a very elegant approach for writing the proof of the result.

Mansour: In a very recent short article[32] published in the Newsletter of the European Mathematical Society, Professor Melvyn B. Nathanson, while elaborating on the ethical aspects of the question “Who Owns the Theorem?” concluded that “Mathematical truths exist, and mathematicians only discover them.” On the other side, there are opinions that “mathematical truths are invented”. As a third way, some people claim that it is both invented and discovered. What do you think about this old discussion? More precisely, do you believe that you *invent* or *discover* your theorems?

[32] M. B. Nathanson, *Who Owns the Theorem?* The best writing on Mathematics 2021, Princeton: Princeton University Press, 2022, 255–257.

Sudakov: I believe that depending on the result, either perspective can be valid. For instance, when one successfully proves an old conjecture that has been discussed for a while, it is accurate to say that a mathematical truth has been established. Conversely, there are situations when someone uncovers a surprising connection between previously unrelated concepts or establishes a fact that nobody had suspected before (as in the case of the Regularity Lemma I previously mentioned). In such instances, I would suggest that the mathematical truth has been invented.

Mansour: Professor Sudakov, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Sudakov: Thank you for this interesting discussion. I appreciate the opportunity to share some of my thoughts.