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# Interview with David Conlon

**Toufik Mansour**

Photo by Elaine Doyle

David Conlon completed his Ph.D. at the University of Cambridge in 2009 under the supervision of Tim Gowers. After a Junior Research Fellowship at St John's College, Cambridge, he moved to the University of Oxford and was promoted to Professor in 2016. Since 2019, he has been a Professor at the California Institute of Technology. He has received several awards, including the European Prize in Combinatorics in 2011 and the LMS Whitehead Prize in 2019, and was among the sectional speakers on combinatorics at the International Congress of Mathematicians in 2014. He has broad research interests within combinatorics and related fields.

**Mansour**: Professor Conlon, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

**Conlon**: Combinatorics has become such a broad area that I think it means many different things to different people, be it extremal, probabilistic, enumerative, algebraic, or something else besides. For me, it's probably closest to discrete analysis, the application of analytic techniques to the study of finite discrete structures and in that has common ground with probability, analytic number theory, and theoretical computer science. But if one looks at the kinds of problems rather than the techniques, extremal combinatorics is perhaps the best description of my area of specialisation and is concerned with estimating the maximum or minimum size of discrete structures satisfying certain constraints.

**Mansour**: What do you think about the development of the relations between combinatorics and the rest of mathematics?

**Conlon**: I feel like all of mathematics moves forward together. Like we're all down in different parts of a coal mine, chipping away at our particular seams. And sometimes the seams run out and lead nowhere particular, but sometimes we find that we've dug all the way through to some unexpected place we couldn't have foreseen. There are now results in extremal combinatorics that draw on everything from group theory to algebraic geometry and topology to representation theory. And in some cases, many cases, we're still inside our seams waiting to see where they will lead.

**Mansour**: What have been some of the main goals of your research?

**Conlon**: The bulk of my research to date has been concerned with finding conditions for the appearance of substructures inside larger structures. But these kinds of problems come in many different guises and interact with a variety of different areas, so I've found myself pulled towards Ramsey theory, extremal graph theory, random structures, additive combinatorics, discrete geometry, theoretical computer science, and more besides.

**Mansour**: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

**Conlon**: I grew up in rural Ireland without


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much access to outside resources besides some small local libraries, so I didn't really come to mathematics in any serious way until I was 15, when I was invited to attend maths olympiad training. Before that, I used to spend my summers playing with my siblings and reading.

One of the first notable mathematical memories that I have was from a state exam I sat when I was 14. We had to study Euclidean geometry at school and sometimes the exams had interesting questions. One of these ultimately boiled down to the observation that the area of a triangle equals the inradius times the semiperimeter. It's a very simple fact of course, but I remember the thrill of realising how it worked very clearly. Though part of the reason I think I remember it so clearly is because it was my performance in that exam that resulted in my being invited to olympiad training in the first place.

Regarding early mentors, those who were involved in maths olympiad training in Ireland at the time, and particularly at Maynooth University, had more influence than they probably realise themselves.

**Mansour**: Were there specific problems that made you first interested in combinatorics?

**Conlon**: One of the first results they mentioned at olympiad training, as an illustration of the pigeonhole principle, was Ramsey's theorem. They showed us why $r(3) = 6$ and told us that $r(4) = 18$, but then mentioned that the value of $r(5)$ wasn't known. And I remember my mind being blown by this. How on earth was it possible for something so basic about seemingly small numbers to not be known? Well, at this point, having thought about these types of problems for years, I have a much better idea why!

**Mansour**: What was the reason you chose the University of Cambridge for your Ph.D. and your advisor Tim Gowers?

**Conlon**: As with so much in life, by accident. I was an undergraduate in Trinity College Dublin and, a year or two before I moved to Cambridge, a friend of mine had already moved across to do Part III, their master's course, and I decided to follow suit. When I arrived in Cambridge, I actually wasn't quite sure what I wanted to do. I had leanings towards physics as well as pure maths and was caught somewhere in the wide abyss between algebraic number theory and particle physics. But by far the best lectures I attended were those in combinatorics, given by Imre Leader and Tim Gowers, and that's what drew me in. It just seemed like so much fun! And I can now attest that it really is.

**Mansour**: What guides you in your research? A general theoretical question or a specific problem?

**Conlon**: I tend to start with a particular problem. I think it's good to have an anchor. But then I let myself wander in whatever direction it wants to take me. I've seen over time that it helps to be flexible in your approach to a problem and even with regard to what problem you're actually thinking about. Sometimes I'm not even quite sure myself what problem I'm thinking about. I'm just sort of drifting in the space around a problem and I think my subconscious is doing a lot of the heavy lifting. Like with a crossword. There I find it best to run rapidly through the clues, answering those I can, and then, on the second or third pass, more of the answers will pop out as my subconscious processes them. I think maths can work a little like that sometimes, though over longer timespans.

**Mansour**: When you are working on a problem, do you feel that something is true even before you have the proof?

**Conlon**: I try not to be too dogmatic about whether something is true or not. At one point, I convinced myself, for what seemed at the time to be quite reasonable reasons, that the Burr–Erdős conjecture, a famous question in graph Ramsey theory, was probably false. And then, soon after, it was proved by Choongbum Lee[1]. So now I always try to leave room for doubt. I tend to be particularly agnostic if there's no clear heuristic for why a conjecture should be true. I have that, for instance, with the Erdős–Hajnal conjecture[2], another very famous problem at the interface of Ramsey theory and structural graph theory. I think it could go either way.

**Mansour**: What three results do you consider the most influential in combinatorics during the last thirty years?

[1] C. Lee, *Ramsey numbers of degenerate graphs*, Ann. of Math. 185 (2017), 791–829.
[2] P. Erdős and A. Hajnal, *Ramsey-type theorems*, Discrete Appl. Math. 25:1-2 (1989), 37–52.

**Conlon**: I think, rather than individual results, I'd prefer to highlight trends in the field. And for me, some of the most important trends have been:

1. The push to extend results in graph theory to hypergraphs, a push which has thrown up a great number of interesting results and difficult questions and led to solutions to many long-standing problems. For example, Keevash's proof of the existence of designs[3] is one offshoot of the great advances that have been made in this direction. His approach uses a clever algebraic variant of the absorption technique, originally developed to resolve other basic problems such as that of finding an appropriate hypergraph extension of Dirac's theorem on Hamilton cycles. But this continues to be a very active and, if you'll forgive the pun, absorbing area of study.

2. The attempt to extend classic results in combinatorics to sparse settings. This is an ongoing effort, but has led to a sequence of revolutions in extremal combinatorics and related areas, from the proof of the Green–Tao theorem on progressions in primes[4], which has at its core a pseudorandom version of Szemerédi's theorem on arithmetic progressions in dense sets of integers, to the development of the hypergraph container method[5,6], which was originally, at least in part, motivated by the study of analogues of combinatorial theorems in random sets, but has since proved useful far beyond this setting.

3. A less well-defined trend is the increased interaction between extremal combinatorics and algebra. One prominent, reasonably well-understood, example is the work in additive combinatorics leading on from Gowers' proof of Szemerédi's theorem[7], in particular the resolution by Green, Tao, and Ziegler of the Gowers inverse conjecture[8]. But there are several places in discrete geometry and extremal graph theory where there are connections with algebraic geometry, finite geometry, and group theory which are not yet fully understood. For instance, what can we say about the structure of planar point sets with many collinear triples? Or $C_4$-free graphs of near-optimal size? It may be that questions like this are still a little too difficult for us to answer, but we seem to be brushing up against them repeatedly.

**Mansour**: What are the top three open questions on your list?

**Conlon**: 1. Until recently, I probably would have said that my number one question was the problem of giving an exponential improvement to the upper bound for diagonal Ramsey numbers. But now that that has been achieved by Campos, Griffiths, Morris, and Sahasrabudhe[9], my number one spot is held by the problem of estimating the diagonal Ramsey number of 3-uniform cliques. It's perhaps the problem that I've spent the most time thinking about. If we write $K_t^{(3)}$ for the complete 3-uniform hypergraph with $t$ vertices, the Ramsey number $r_3(t)$ is the smallest $n$ such that every two-colouring of the edges of $K_n^{(3)}$ contains a monochromatic copy of $K_t^{(3)}$. The problem then is to estimate $r_3(t)$. What's known is that there are positive constants $c$ and $c'$ such that

$$2^{ct^2} \leq r_3(t) \leq 2^{2^{c't}}.$$

So there's an exponential gap and it's a \$500 problem of Erdős to show that the upper bound is essentially correct. The funny thing is that such a lower bound is known if we allow four colours instead of two, so, instinctively, it seems that the situation shouldn't be so different for two colours. But we don't know. If anything, we are now more uncertain about it, because we, by which I mean myself, Jacob Fox and Vojta Rödl[10], found examples of hypergraphs, which we call hedgehogs, where the two-colour Ramsey number is polynomial in the number of vertices, but the four-colour Ramsey number is exponential.

2. I've also spent far too much time thinking about the Euclidean Ramsey problem. The problem is this: which finite sets $X \subset \mathbb{R}^d$ are Ramsey, in the sense that for every natural

[3] P. Keevash, *The existence of designs*, arXiv:1401.3665.

[4] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. 167 (2008), 481–547.

[5] J. Balogh, R. Morris, and W. Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. 28 (2015), 669–709.

[6] D. Saxton and A. Thomason, *Hypergraph containers*, Invent. Math. 201 (2015), 925–992.

[7] W. T. Gowers, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal. 11 (2001), 465–588.

[8] B. Green, T. Tao, and T. Ziegler, *An inverse theorem for the Gowers $U^{s+1}[N]$-norm*, Ann. of Math. 176 (2012), 1231–1372.

[9] M. Campos, S. Griffiths, R. Morris, and Julian Sahasrabudhe, *An exponential improvement for diagonal Ramsey*, arXiv:2303.09521.

[10] D. Conlon, J. Fox, and V. Rödl, *Hedgehogs are not colour blind*, J. Comb. 8 (2017), 475–485.

number $r$, if we $r$-colour the points of $\mathbb{R}^n$ for $n$ sufficiently large in terms of $X$ and $r$, then there is guaranteed to be a monochromatic isometric copy of $X$? The key point is the isometry condition. If we allow dilates, the problem becomes much easier and any $X$ works. This question was introduced in a series of papers by a Who's Who of extremal combinatorialists in the 1970s, Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus[11], but it remains wide open. Their main result was that any such $X$ has to be spherical, in that it has to be contained in a sphere of some dimension, and they also conjectured that the converse should hold, that every spherical set is Ramsey. This is incredibly attractive as a conjecture, but it's not at all clear that it's true. In fact, it's even open for cyclic quadrilaterals. A tantalising counterconjecture was made by Leader, Russell, and Walters[12], that a set $X$ is Ramsey if and only if it is contained in a transitive set. I'm honestly not sure which, if either, to believe.

3. The other thing that comes to mind is Sidorenko's conjecture[13], which says that if we fix a bipartite graph $H$, then the number of copies of $H$ in a graph with $n$ vertices and density $p$ is asymptotically at least the number of copies of $H$ in the random graph $G(n, p)$. There are more precise statements, but I won't get into them here. I've never quite believed this conjecture, but every time I think about it, I end up proving it for more cases. My first serious result on the conjecture was with Jacob Fox and Benny Sudakov[14] and says that if a bipartite graph has a vertex complete to the opposite side, then it satisfies the conjecture. I usually just say that the graph is Sidorenko if this happens. Then, over about a decade, this result was extended by myself, Jeong Han Kim, Choongbum Lee, and Joonkyung Lee[15] and also by Balász Szegedy[16] to show that one could glue graphs in various ways to produce more graphs that were Sidorenko. I thought that this might be everything until Joonkyung and I[17] managed to prove that every bipartite graph has a blowup for which the conjecture is true. What I mean by this is that if we fix a bipartite graph $H$ and one side $A$ of its bipartition, then we can take many copies of this graph, glue them along $A$ and the resulting graph is Sidorenko. Mainly because of this result, I'm now a little more agnostic about the conjecture. Maybe it is true after all.

**Mansour**: What kind of mathematics would you like to see in the next ten-to-twenty years as the continuation of your work?

**Conlon**: I'd really love to see the solutions to any of the three problems that I mentioned above! But, even without that, I'm quite excited about the future of extremal and probabilistic combinatorics. The field has attracted a huge number of incredibly strong, energetic young mathematicians in recent years and it seems that the possibilities are boundless. So maybe we will see solutions to these problems, but likely much more besides.

**Mansour**: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

**Conlon**: Well, yes, some things are more important and more central than others. One can't really deny that complex numbers are more central than surreal ones. However, I do think that there is interesting mathematics in almost every direction. In the way that the best 1% of almost any musical genre is genuinely good, I think that the best mathematics in any given area or on any given theme is bound to be interesting. One just has to have an open mind about it.

**Mansour**: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

**Conlon**: I do think the distinction is a meaningful one, though not always. Applied mathematicians inevitably develop techniques of rel-

[11] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, *Euclidean Ramsey theorems. I*, J. Combin. Theory Ser. A 14 (1973), 341–363.

[12] I. Leader, P. A. Russell, and M. Walters, *Transitive sets in Euclidean Ramsey theory*, J. Combin. Theory Ser. A 119 (2012), 382–396.

[13] A. F. Sidorenko, *A correlation inequality for bipartite graphs*, Graphs Combin. 9 (1993), 201–204.

[14] D. Conlon, J. Fox, and B. Sudakov, *An approximate version of Sidorenko's conjecture*, Geom. Funct. Anal. 20 (2010), 1354–1366.

[15] D. Conlon, J. H. Kim, C. Lee, and J. Lee, *Some advances on Sidorenko's conjecture*, J. Lond. Math. Soc. 98 (2018), 593–608.

[16] B. Szegedy, *An information theoretic approach to Sidorenko's conjecture*, arXiv:1406.6738.

[17] D. Conlon and J. Lee, *Sidorenko's conjecture for blow-ups*, Discrete Anal. (2021), Paper No. 2.

evance to pure mathematics and vice versa. I read once that all of Laplace's work was motivated by his study of celestial mechanics, but ultimately led to breakthroughs across almost all of pure and applied mathematics as it existed at the time. But I certainly see myself as a pure mathematician and am motivated by theoretical questions rather than practical applications. I'd be happy to have proved that a function is linear, even if the multiplying constant were astronomical.

**Mansour**: What advice would you give to young people thinking about pursuing a research career in mathematics?

**Conlon**: Like with any prolonged pursuit, you have to be quite passionate about it. Maths research can be quite frustrating at times, if things aren't going the way you might have hoped, so you have to have a lot of persistence. And I think having a deep love of your subject can help with that. It can even help others. During my own Ph.D., when I was feeling stuck or unmotivated, I'd go see my supervisor Tim Gowers, who was always full of enthusiasm for mathematics, and I'd feel instantly recharged by it.

**Mansour**: You have some experience with significant mathematics competitions such as International Mathematics Olympiads. What do you think about the importance of such competitions in inspiring young students for a research career?

**Conlon**: Being selected for the Irish Olympiad Team honestly changed the whole direction of my life. Before that, I hadn't really had much interest in mathematics. I enjoyed it and was good at it, but hadn't thought about it beyond what we did in school. To the point where I was a little surprised when I was eventually selected for the team. But I never really looked back after that. And I'm sure many others have had similar experiences. There is also, at least sometimes, correlation between success at olympiads and success in research. Combinatorics seems to have been particularly favoured by the attentions of many of the very best performers at these competitions, from Béla Bollobás and László Lovász in its early days down to Sergey Norin, Christian Reiher, Lisa Sauermann, and many more in recent years, including my own current student Jeck Lim, though I could easily name a dozen more.

**Mansour**: Would you tell us about your interests besides mathematics?

**Conlon**: I read very broadly, both fiction and non-fiction, though the amount I read tends to have a reciprocal relationship with how much research I'm engaged in at the time. I also walk a great deal, almost as a form of meditation. I can both focus better and let my mind wander more freely when walking and I think both are important to research. I keep meaning to walk from Caltech down to the coast, which is about 20 miles, just to get a real feel for the size of Los Angeles under my feet, but I still haven't gotten around to it!

**Mansour**: You have given talks at numerous conferences, workshops, and seminars. How important do you find such activities?

**Conlon**: I think the process of preparing for research talks can often help clarify your thinking about a problem, even after you think you've finished working on it. There's something about having to explain your work to others that really forces you to understand it yourself! I also quite enjoy the process of finding the right way to present something. But I think the most important aspect of these activities is the social one, the group and network building that happens at conferences and workshops that leads to collaborations and, often, close friendships. There's something about coming up through the ranks of mathematics together and seeing each other at conferences year after year that really cements a friendship. As another wise friend once put it, a fund of common experience is what friendships are built on.

**Mansour**: You spent most of your education and academic career at prestigious research institutes in the UK, but you moved to Caltech in 2019. How do you compare the two research cultures and institutions in the UK and USA? Which gives ambitious researchers more opportunities, freedom, and resources to pursue their interests?

**Conlon**: I've been very fortunate to spend my entire research career at exceptional institutions, Cambridge as a Ph.D. student and research fellow, Oxford as a lecturer, and now Caltech. They are each, in their own ways, special places. The physical environment in Cambridge and Oxford, the sheer beauty of the towns, while also being within easy reach of

London, make them wonderful places to live. When I moved from Dublin to Cambridge, I think I spent the first year just staring up at all the amazing buildings. On the other hand, the thing that I have found at Caltech is a huge amount of time and space for research, in part because the university is so small. And the weather in California doesn't hurt. Waking up to sunshine every day can really help one's motivation!

**Mansour**: *Ramsey numbers* appear in numerous of your recent research articles. However, there is one article that mentions *Ramsey games*[18]. What is a Ramsey game? How is it connected to Ramsey numbers? Would you please expand on this?

**Conlon**: The game in the paper you refer to is a variant of what's called the online Ramsey game, but let me talk instead about the original version of the game. Here we have two players, Builder and Painter, working on a board which starts out as the empty graph on the natural numbers. At each step, Builder adds an edge to this graph and Painter must immediately colour in one of two colours. Builder's goal is to force Painter to complete a monochromatic copy of some fixed graph $H$, while Painter wants to avoid doing so for as long as possible. How well can they do? Formally, the online Ramsey number $\tilde{r}(H)$ is defined as the smallest number of moves in which Builder can force a monochromatic copy of $H$. This is an interesting function, which has attracted considerable attention. For example, a result of mine[19] shows that for infinitely many $t$, the online Ramsey number $\tilde{r}(t) := \tilde{r}(K_t)$ is exponentially smaller than the trivial upper bound $\binom{r(t)}{2}$, trivial because Builder can just draw a complete graph on $r(t)$ vertices and then Painter is clearly guaranteed to draw a monochromatic $K_t$. On the other hand, a result by myself, Jacob Fox, Andrey Grinshpun, and Xiaoyu He[20] gives an exponential improvement to the lower bound $\tilde{r}(t) \geq \sqrt{2}^t$ coming from the analogous bound for ordinary Ramsey numbers.

**Mansour**: In one of your publications, *Rational exponents in extremal graph theory*[21], coauthored with Boris Bukh, you proved that for every rational number $r$ between 1 and 2, there is a family of graphs $\mathcal{H}_r$ such that $\mathrm{ex}(n, \mathcal{H}_r) = \Theta(n^r)$, where $\mathrm{ex}(n, \mathcal{H})$ is the largest $m$ for which there exists a graph with $n$ vertices and $m$ edges containing no graph from the family $\mathcal{H}$ as a subgraph. Would you explain your approach to the proofs?

**Conlon**: The starting point was a beautiful idea of Boris'[22], who showed how to use random varieties over finite fields to prove that for every $s$ there exists $t$ such that $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$. This technique turned out to be much more malleable than previous construction methods for extremal numbers. The first sign of this was a result of mine[23] where I adapted his technique to show that for every $s$ there exists $t$ such that $\mathrm{ex}(n, \theta_{s,t}) = \Theta(n^{1+1/s})$, where $\theta_{s,t}$ is the graph consisting of $t$ internally disjoint paths of length $s$ that share their two endpoints (the cycle $C_{2s}$ is just $\theta_{s,2}$). We then decided to work together to try and prove an old conjecture of Erdős and Simonovits saying that for every $r \in \mathbb{Q} \cap [1, 2]$ there exists a graph $H_r$ with $\mathrm{ex}(n, H_r) = \Theta(n^r)$. We almost succeeded, showing that this is true if one allows a finite family rather than just a single graph.

There has been a lot of subsequent work on trying to nail down the conjecture in its original form, in part because Erdős offered \$300 for a solution, but we still seem some way off. One funny thing about it is that the lower bound, which has always been the main issue for extremal numbers of bipartite graphs, follows directly from our methods, so the problem lies in proving the right upper bounds for the extremal numbers of certain graphs. To date, the best results[24,25] are that the conjecture holds close to 1, by which I mean for $r$ of the form $1 + a/b$ with $b > a^2$, and close to 2, for all $r$ of the form $2 - a/b$ with $b > a^2$.

[18] D. Conlon, S. Das, J. Lee, and T. Mészáros, *Ramsey games near the critical threshold*, Random Struct. Algorithms 57 (2020), 940–957.

[19] D. Conlon, *On-line Ramsey numbers*, SIAM J. Discrete Math. 23 (2009/10), 1954–1963.

[20] D. Conlon, J. Fox, A. Grinshpun, and X. He, *Online Ramsey numbers and the subgraph query problem*, Building bridges II – Mathematics of László Lovász, 159–194, Bolyai Soc. Math. Stud., 28, Springer, Berlin, 2019.

[21] B. Bukh and D. Conlon, *Rational exponents in extremal graph theory*, J. Eur. Math. Soc. 20 (2018), 1747–1757.

[22] B. Bukh, *Random algebraic construction of extremal graphs*, Bull. Lond. Math. Soc. 47 (2015), 939–945.

[23] D. Conlon, *Graphs with few paths of prescribed length between any two vertices*, Bull. Lond. Math. Soc. 51 (2019), 1015–1021.

[24] T. Jiang and Y. Qiu, *Many Turán exponents via subdivisions*, Combin. Probab. Comput. 32 (2023), 134–150.

[25] D. Conlon and O. Janzer, *Rational exponents near two*, Adv. Comb. (2022), Paper No. 9.

**Mansour**: You won the European Prize in Combinatorics in 2011 for your work in Ramsey theory and progress on Sidorenko's conjecture. Please tell us more about this work.

**Conlon**: In some part, I received the award for my improvement on the bounds for diagonal Ramsey numbers[26]. The diagonal Ramsey number $r(t)$ is the smallest $n$ such that every two-colouring of the edges of $K_n$ contains a monochromatic $K_t$. It was shown by Erdős and Szekeres[27] in 1935 that $r(t) \leq c4^t/\sqrt{t}$ and it took more than fifty years for this to be improved at all, first by Rödl[28] and then by Thomason[29] to $r(t) \leq c4^t/t$. During my Ph.D., I gave the superpolynomial improvement $r(t) \leq 4^t/t^{c\log t/\log\log t}$. I won't say too much about the proof, but in some ways it was a rendering of the methods of additive combinatorics in a graph-theoretical context. My method was recently optimised by Ashwin Sah[30], who showed that $r(t) \leq 4^t/t^{c\log t}$, which is the best one can possibly do using this method. The approach used in the recent breakthrough[9], giving an exponential improvement, is fundamentally (and necessarily) different.

I've already talked a little about my early work on Sidorenko's conjecture, which was also cited in the award, but let me add that this was, at the time, part of an exciting body of work applying the method of dependent random choice. This is a fairly simple, but remarkably powerful, lemma, originating in the work of Tim Gowers on Szemerédi's theorem, showing that any dense bipartite graph contains an induced subgraph where all or almost all $t$-tuples on one side of the bipartition have many neighbours on the other side. It was realised somewhat gradually how useful this lemma is, but in the late 2000s a number of people, including myself, Jacob Fox, and Benny Sudakov, made progress on a number of problems using the technique. Jacob and Benny ultimately wrote a very nice survey about it[31], which I'd highly recommend. But, getting back to the award, one of the applications we found was to Sidorenko's conjecture, where we showed that any bipartite graph with one vertex complete to the other side satisfies the conjecture[14]. This was then enough to prove an approximate version of the conjecture. I should add that dependent random choice has continued to be very important. For example, it's a central component in the recent breakthrough of Kelley and Meka[32] improving the bounds for Roth's theorem.

**Mansour**: You were one of the invited speakers at the International Congress of Mathematicians in 2014. Are there any recent breakthroughs related to the problems you discussed at the congress?

**Conlon**: My ICM survey was largely concerned with discussing analogues of combinatorial theorems relative to sparse random and pseudorandom graphs. This is an extremely active area of research, as it has been for over thirty years. For example, the celebrated Green–Tao theorem[4], that the primes contain arbitrarily long arithmetic progressions, is a result in this direction. At heart, what they actually prove is that Szemerédi's theorem[33], the statement that every dense subset of the integers contains arbitrarily long arithmetic progressions, continues to hold relative to a certain set of pseudoprimes. Then, because the primes are themselves dense in this set of pseudoprimes, the primes contain the required progressions. Rather than focusing just on pseudoprimes, Green and Tao actually prove a general statement saying that Szemerédi's theorem continues to hold relative to any sufficiently pseudorandom set of integers. With Jacob Fox and Yufei Zhao[34,35], we later found a simpler proof of this relative Szemerédi theorem which required weaker pseudorandomness assumptions. This simplification has been an important factor in several subsequent developments.

[26] D. Conlon, *A new upper bound for diagonal Ramsey numbers*, Ann. of Math. 170 (2009), 941–960.

[27] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compos. Math. 2 (1935), 463–470.

[28] R. L. Graham and V. Rödl, *Numbers in Ramsey theory*, Surveys in combinatorics 1987 (New Cross, 1987), 111–153, London Math. Soc. Lecture Note Ser., 123, Cambridge University Press, Cambridge, 1987.

[29] A. Thomason, *An upper bound for some Ramsey numbers*, J. Graph Theory 12 (1988), 509–517.

[30] A. Sah, *Diagonal Ramsey via effective quasirandomness*, Duke Math. J. 172 (2023), 545–567.

[31] J. Fox and B. Sudakov, *Dependent random choice*, Random Struct. Algorithms 38 (2011), 68–99.

[32] Z. Kelley and R. Meka, *Strong bounds for 3-progressions*, arXiv:2302.05537.

[33] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression,* Acta Arith. 27 (1975), 199–245.

[34] D. Conlon, J. Fox, and Y. Zhao, *A relative Szemerédi theorem*, Geom. Funct. Anal. 25 (2015), 733–762.

[35] D. Conlon, J. Fox, and Y. Zhao, *The Green–Tao theorem: an exposition*, EMS Surv. Math. Sci. 1 (2014), 249–282.

A more recent line of work in this direction is about proving analogues of combinatorial theorems relative to $C_4$-free graphs or, since we are talking about arithmetic problems, Sidon sets. For us, a Sidon set is a subset of $\{1, 2, \ldots, n\}$ with no non-trivial solution to the equation $x + y = z + w$. These are known to have size at most $O(\sqrt{n})$. In a recent paper with Jacob Fox, Benny Sudakov, and Yufei Zhao[36], we showed that the property of being Sidon is enough to sometimes allow us to prove sparse analogues of combinatorial theorems. For example, a variant of Szemerédi's theorem says that any set with no non-trivial solution to the equation $x + y + 2z = u + 3v$ has at most $o(n)$ elements. It turns out that an analogue of this result continues to hold relative to Sidon sets, saying that if a Sidon set has no non-trivial solution to the equation $x+y+2z = u+3v$, then it has $o(\sqrt{n})$ elements. This also disproves what's called the compactness conjecture for equations, because it gives two equations where it is easier to find solutions to one of the two equations than either of the two equations separately.

**Mansour**: Is graph theory a purely mathematical research interest for you or are you also interested in its applications to other fields? Can you give some examples of applications of graph theory that led to a solution to a challenging question in another research area?

**Conlon**: Graph theory is often unreasonably effective for solving problems in discrete geometry and additive combinatorics. The first example that comes to mind is the elementary bound of $O(n^{3/2})$ for the unit distance problem of estimating the maximum number of unit distances between $n$ points in the plane. This follows from just observing that the unit distance graph contains no copy of $K_{2,3}$ and then using a bound for the extremal number of this graph. A more modern example that comes to mind is the recent work on equiangular lines by Bukh[37], Balla–Dräxler–Keevash–Sudakov[38], and Jiang–Tidor–Yao–Zhang–Zhao[39], which culminated in very precise answers for the question of how many lines there can be passing through the origin in $\mathbb{R}^n$, where any two lines have angle $\alpha$ between them. Ultimately, this question can be rephrased in terms of graphs and then it boils down to understanding the maximum multiplicity of the second eigenvalue of their adjacency matrices.

**Mansour**: A joint paper by you, *An approximate version of Sidorenko's conjecture*[14], coauthored with Jacob Fox and Benny Sudakov, has some interesting connections to a broad range of topics, such as matrix theory, Markov chains, graph limits, and quasirandomness. Would you tell us more about this work?

**Conlon**: I've already brushed up against this paper a couple of times, but let me say a little about the connections. The first results on Sidorenko's conjecture go back to the 1960s and were matrix inequalities such as the Blakley–Roy inequality[40], which is essentially equivalent to Sidorenko's conjecture for paths. The connection to Markov chains also lies there. To bring in quasirandomness, let me mention a stronger conjecture, the forcing conjecture of Skokan and Thoma[41], which says that equality holds in Sidorenko's conjecture if and only if the graph in which we are counting is quasirandom. In the language of graph limits, a quasirandom graph corresponds to a graphon which is constant almost everywhere, so, in this context, the forcing conjecture just says that the homomorphism density for a fixed bipartite graph $H$ in a graphon $W$ of given density is minimised only by the constant graphon. This is such a clean statement that I would claim the natural language for Sidorenko's conjecture, or graph homomorphism inequalities in general, is that of graph limits.

**Mansour**: In one of your most cited research papers, *Combinatorial theorems in sparse random sets*[42], coauthored with W. T. Gowers, you give a very general method for proving

[36] D. Conlon, J. Fox, B. Sudakov, and Y. Zhao, *The regularity method for graphs with few 4-cycles*, J. Lond. Math. Soc. 104 (2021), 2376–2401.

[37] B. Bukh, *Bounds on equiangular lines and on related spherical codes*, SIAM J. Discrete Math. 30 (2016), 549–554.

[38] I. Balla, F. Dräxler, P. Keevash, and B. Sudakov, *Equiangular lines and spherical codes in Euclidean space*, Invent. Math. 211 (2018), 179–212.

[39] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, and Y. Zhao, *Equiangular lines with a fixed angle*, Ann. of Math. 194 (2021), 729–743.

[40] G. R. Blakley and P. Roy, *A Hölder type inequality for symmetric matrices with nonnegative entries*, Proc. Amer. Math. Soc. 16 (1965), 1244–1245.

[41] J. Skokan and L. Thoma, *Bipartite subgraphs and quasi-randomness*, Graphs Combin. 20 (2004), 255–262.

[42] D. Conlon and W. T. Gowers, *Combinatorial theorems in sparse random sets*, Ann. of Math. 184 (2016), 367–454.

*sparse random versions of combinatorial theorems*. Would you say a few lines about this work?

**Conlon**: This is part of a long line of work, going back to the 1980s, studying analogues of combinatorial theorems relative to random sets. For example, when does the binomial random graph $G(n,p)$ have the property that every two-colouring of its edges contains a monochromatic triangle? That is, when is the structure rich enough, when does it have enough triangles, to guarantee that there's always a monochromatic one? The answer is essentially when $p = C/\sqrt{n}$. This was originally proved by Frankl and Rödl[43] and their result was later generalised from triangles to any fixed graph $H$ in important work of Rödl and Ruciński[44], but open problems still remain.

Our work, and concurrent independent work by Mathias Schacht[45], which used very different techniques, gives a general method for transferring combinatorial theorems about fixed structures from the dense setting to the sparse setting. Given a sparse graph, our technique gives a dense model graph with similar properties. For example, if the sparse graph has $cpn^2$ edges and $c'p^3n^3$ triangles, then its dense model has roughly $cn^2$ edges and $c'n^3$ triangles. In particular, if the sparse graph has more than $(\frac{1}{4}+\epsilon)pn^2$ edges, the dense graph has more than $(\frac{1}{4}+\frac{\epsilon}{2})n^2$ edges and so, by Turán's theorem, must contain many triangles, at least $c'n^3$ for some $c'$. But then the sparse graph has at least $\frac{c'}{2}p^3n^3$ triangles, so we get an analogue of Turán's theorem inside the sparse graph. The same thing works if triangles are replaced by other fixed graphs, so we get a very general, and long conjectured, version of Turán's theorem within random graphs.

I should remark that the more recent work on hypergraph containers by Balogh, Morris, and Samotij[5] and, independently, Saxton and Thomason[6] also gives an approach to these problems. I say much more about all of this, and the strengths and weaknesses of the different approaches, in my ICM survey paper[46].

**Mansour**: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

**Conlon**: I wouldn't say that I make very strong use of enumerative techniques. Most of the problems that I think about have bounds that are very far apart, so much so that we are always applying analytic cutoff techniques to throw away parts of our structure that are badly behaved. For instance, there's that simple folklore lemma saying that any graph has a subgraph whose minimum degree is at least half the average degree of the original graph. Once you've applied that, you never need to worry about vertices of low degree.

On the other hand, enumerative problems figure constantly in extremal combinatorics. Sidorenko's conjecture is obviously an enumeration problem, but there are many fundamental enumeration questions which arise naturally in the area, especially when one is studying random structures. For instance, if you want to study random $d$-regular graphs, a first step is to understand, at least approximately, how many $d$-regular graphs there are. The same is true of studying, for example, random Steiner systems, on which great progress has been made recently.

**Mansour**: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "eureka moment"?

**Conlon**: I've occasionally had what might be described as eureka moments, a couple while staggering tired onto a bus after a long flight and another while laid low by a stomach ache. So maybe I just need to be at my lowest for any true epiphanies! But generally I don't believe in them. I think you have to do a lot of hard work first to understand a problem, to get at the heart of the issues involved, and then some background process starts ticking away at it. But perhaps to spell out one of these examples, it was about the Ramsey number of books[47]. A book is a collection of copies of $K_{k+1}$ glued

[43]P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without $K_4$*, Graphs Combin. 2 (1986), 135–144.
[44]V. Rödl and A. Ruciński, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. 8 (1995), 917–942.
[45]M. Schacht, *Extremal results for discrete random structures*, Ann. of Math. 184 (2016), 333–365.
[46]D. Conlon, *Combinatorial theorems relative to a random set*, Proceedings of the International Congress of Mathematicians – Seoul 2014, Vol. IV, 303–327.
[47]D. Conlon, *The Ramsey number of books*, Adv. Comb. (2019), Paper No. 3.

along a common $K_k$. These are surprisingly important objects in graph Ramsey theory. Indeed, every upper bound on diagonal Ramsey numbers, including the recent breakthrough[9], proceeds through first finding monochromatic books. I had been thinking about the Ramsey numbers of books on and off for a few months and had figured out how to prove an old conjecture of Erdős, Faudree, Rousseau, and Schelp[48] on their asymptotic behaviour up to a constant factor, but I was still trying to pin down the correct constant. And then, sitting at home one day in my flat in Oxford, sick enough that I couldn't attend the annual one-day meeting in combinatorics and certainly not feeling like I could do any serious thinking, the correct approach just popped into my head unbidden.

**Mansour**: Is there a specific problem you have been working on for many years? What progress have you made?

**Conlon**: I think I've already mentioned several such problems and there are many more, but let me mention one: what is the size-Ramsey number $\hat{r}(H)$ of a graph $H$ with $n$ vertices and maximum degree three? This is the minimum number of edges, rather than vertices, in a graph $G$ with the property that any two-colouring contains a monochromatic copy of $H$. I have no idea what the answer should be. It's fairly easy to show that $n \leq \hat{r}(H) \leq c'n^2$, but both bounds are quite hard to improve. Until recently, the state of the art was that $n \log^c n \leq \hat{r}(H) \leq n^{5/3+o(1)}$, with the lower bound a clever argument of Rödl and Szemerédi[49] and the upper bound an argument of Kohayakawa, Rödl, Schacht and Szemerédi[50] showing that the random graph $G(Cn, n^{-1/3+o(1)})$ has the property. Recently, with Rajko Nenadov and Milos Trujić[51], we improved the upper bound to $c'n^{8/5}$, which is the best a random graph can do, and Konstantin Tikhomirov[52] improved the lower bound to $ne^{c\sqrt{\log n}}$ using a clever variation on Rödl and Szemerédi's construction. Even more recently, Nemanja Draganić and Kalina Petrova[53] pushed our bound down to $n^{3/2+o(1)}$, using a random graph with certain dense subgraphs laid on top. Going further seems very difficult. Most researchers in the area, including myself, are inclined to believe that the answer should be at least $n^{1+\epsilon}$ for some $\epsilon > 0$, but it would be much more interesting if it were not.

**Mansour**: In a very recent short article[54], published in the Newsletter of the European Mathematical Society, Professor Melvyn B. Nathanson, while elaborating on the ethical aspects of the question "Who Owns the Theorem?", concluded that "Mathematical truths exist and mathematicians only discover them." On the other side, there are opinions that "mathematical truths are invented". As a third way, some people claim that it is both invented and discovered. What do you think about this old discussion? More precisely, do you believe that you *invent* or *discover* your theorems?

**Conlon**: My own viewpoint on this is probably closest to the third option. I think that we decide on which axioms to use, and they may or may not have some objective truth, but the only thing that really matters is their consistency. And then, once we decide on our axioms and assume their truth, the rest is discovery, chipping our way through all of the possible consequences of those axioms to find the interesting ones. The question of whether or not the axioms are themselves invented or discovered is a more difficult one, but, if I don't quite come down on the side of saying they're invented, I would still say that they're chosen. I see no particular reason why, if we were to start anew, we would begin with the same axioms. Should we really follow Euclid in thinking of a line as a 'breadthless length'? After all, it turned out to be seriously counterproductive to assume that there is exactly one line through every point which is parallel to a given line.

**Mansour**: Ireland has made notable contribu-

[48] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hungar. 9 (1978), 145–161.

[49] V. Rödl and E. Szemerédi, *On size Ramsey numbers of graphs with bounded degree*, Combinatorica 20 (2000), 257–262.

[50] Y. Kohayakawa, V. Rödl, M. Schacht, and E. Szemerédi, *Sparse partition universal graphs for graphs of bounded degree*, Adv. Math. 226 (2011), 5041–5065.

[51] D. Conlon, R. Nenadov, and M. Trujić, *The size-Ramsey number of cubic graphs*, Bull. Lond. Math. Soc. 54 (2022), 2135–2150.

[52] K. Tikhomirov, *On bounded degree graphs with large size-Ramsey numbers*, arXiv:2210.05818.

[53] N. Draganić and K. Petrova, *Size-Ramsey numbers of graphs with maximum degree three*, arXiv:2207.05048.

[54] M. B. Nathanson, *Who Owns the Theorem?* The best writing on Mathematics 2021, Princeton: Princeton University Press, 2022, 255–257.

tions to literature, music, and cinema. Can you recommend some of your favorite books, music, or movies to our readers?

**Conlon**: There was a very beautiful recent film made in Ireland, 'The Quiet Girl' or 'An Cailín Ciúin' as it is in Irish. And I honestly couldn't recommend it highly enough. It's mostly in Irish, with some English, but it really captures Ireland as it was in the 1970s and 80s. I believe it was the first Irish film to be nominated for Best International Film at the Oscars.

It's not hard to think of great Irish writers, but I've always been particularly partial to the works of Oscar Wilde. I had one of those huge Complete Works when I was a kid, 1000 pages long, and I read the whole thing cover to cover. Though rather different in tone to his delightful comedic plays, his long poem 'The Ballad of Reading Gaol' is the thing I return to most often.

With music, I've recently found myself listening to a lot of Sinéad O'Connor, who died this past summer. Her biggest hit was her beautiful cover of 'Nothing Compares 2 U', but she made a lot of wonderful, passionate music. The world is a poorer place without her.

**Mansour**: Professor Conlon, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

**Conlon**: And thank you, I very much enjoyed it!