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On Weighted and Bounded Multidimensional Catalan Numbers

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Abstract: Motivated by the literature on weighted Catalan numbers and their arithmetic properties, we introduce novel weighted analogs of multidimensional Catalan numbers (A060854 in the OEIS). We prove that they are eventually periodic modulo a positive integer m, assuming certain conditions, and derive formulas for several examples of these numbers. We accomplish this by studying k-dimensional Balanced ballot paths, which generalize Dyck paths to the k-dimensional space. Building on this framework, we introduce two integer sequences for any $k \geq 2$: the first counts k-dimensional Balanced ballot paths by exact height, generalizing the enumeration of the Dyck paths by their exact heights; the second counts k-dimensional Balanced ballot paths with α peaks, generalizing Narayana numbers.

Keywords: Multidimensional Catalan numbers; Narayana numbers; Periodicity; Weighted Catalan numbers
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1. Introduction

1.1 Catalan Numbers

The sequence of Catalan numbers $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is one of the most studied sequences in enumerative combinatorics. The Catalan numbers (A000108 in the OEIS [12]) count various objects, such as triangulations of a convex polygon with $n+2$ edges and rooted binary trees with $2n$ nodes, among others [13]. Most notably, they enumerate Dyck paths of $2n$ steps, which are sequences of 2-dimensional steps $(\vec{s}_1, \vec{s}_2, \dots, \vec{s}_{2n})$, each consisting of n steps of form $(1,1)$, which we call *up-steps*, and n steps of form $(1,-1)$, which we call *down-steps*, such that any intermediate point $\vec{v}_i = \sum_{j=1}^{i} \vec{s}_j$ is on or above the x-axis (see Figure 1 for an example). One famous refinement of the Catalan numbers is the Narayana numbers (A001263 in the OEIS [12]). The Narayana number $N(n,\alpha)$ is the number of Dyck paths of $2n$ steps with exactly α peaks, i.e., α instances of an up-step followed immediately by a down-step.

Figure 1: Left: Intermediate points of an 8-step Dyck path. Right: Dyck path on the left, but with contributions to the weights next to each step in the direction $(1,1)$. The path has weight $wt_{\vec{b}}(P) = b_0^2 b_1^2$ with respect to the infinite sequence of integers $\vec{b} = (b_0, b_1, \dots)$.

1.2 Weighted Catalan Numbers

We now introduce weighted Catalan numbers, which we generalize in the paper. They first appeared in the work of Goulden and Jackson [5]. For a fixed infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$ and a Dyck path P of $2n$ steps, we define the *weight* $wt_{\vec{b}}(P)$ of P as the product $b_{u_1} b_{u_2} \cdots b_{u_n}$, where u_i is the *height*, i.e., y-coordinate, of the starting point of the ith up-step of P. The corresponding *nth weighted Catalan number for* $\vec{b}$ is defined as $C_n^{\vec{b}} = \sum_P wt_{\vec{b}}(P)$, where the sum is over all Dyck paths of $2n$ steps. In particular for $\vec{b} = (1, 1, \ldots)$, we have $C_n^{\vec{b}} = C_n$. Examples of Dyck paths with their weights are displayed in Figures 1 and 2.

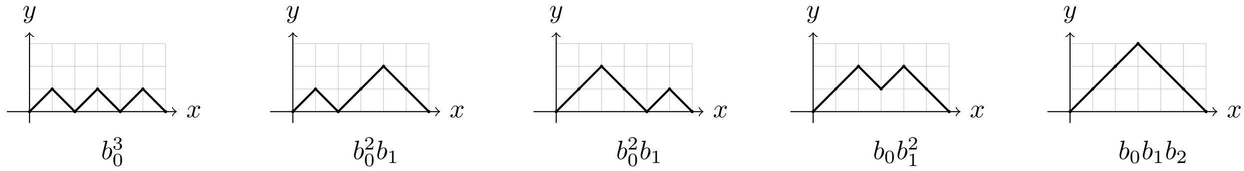

Figure 2: Five Dyck paths of 6 steps with their weights with respect to $\vec{b} = (b_0, b_1, \ldots)$. The third weighted Catalan number with respect to $\vec{b}$ is $C_3^{\vec{b}} = b_0^3 + 2b_0^2 b_1 + b_0 b_1^2 + b_0 b_1 b_2$.

For a particular $\vec{b}$, the weighted Catalan numbers have combinatorial interpretations.

Example 1.1. *When the weight vector is $\vec{b} = (1, q, q^2, \ldots)$, the corresponding weighted Catalan number $C_n^{\vec{b}}$ is the nth q-Catalan number, whose formulation is attributed to Carlitz and Riordan [3] (see also equation 1.19 of Haglund's reference [6]). The q-Catalan number is defined as*

$$C_n(q) = \sum_P q^{Area(P)}$$

where the sum is over all Dyck paths $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{2n})$ of $2n$ steps and $Area(P)$ is the number of whole unit squares in the lattice $\mathbb{Z}^2_{\geq 0}$ that lie under the path obtained by connecting consecutive intermediate points $\sum_{j=1}^{i} \vec{s}_j$ and $\sum_{j=1}^{i+1} \vec{s}_j$ with line segments for each $i \in \{0, 1, \ldots, 2n-1\}$.

In another example, weighted Catalan numbers count topological objects.

Example 1.2. *Postnikov [8] proved that when $\vec{b} = (1^2, 3^2, 5^2, \ldots)$, the weighted Catalan number $C_n^{\vec{b}}$ counts combinatorial types of Morse links of order n. Postnikov conjectured that for any integer $r \geq 3$, the sequence $C_0^{\vec{b}} \pmod{3^r}, C_1^{\vec{b}} \pmod{3^r}, C_2^{\vec{b}} \pmod{3^r}, \ldots$ has a period of $2 \cdot 3^{r-3}$. In other words, he proposed that $2 \cdot 3^{r-3}$ is the smallest positive integer τ such that $C_{n+\tau}^{\vec{b}} - C_n^{\vec{b}}$ is a multiple of 3^r. This conjecture was partially proven by Gao and Gu [4] in 2021.*

The arithmetic properties of weighted Catalan numbers have also been extensively studied. In 2006, Postnikov and Sagan [9] derived a condition under which the 2-adic valuation of the weighted Catalan numbers is equal to that of the corresponding unweighted ones. In 2007, Konvalinka [7] proved an analogous result for a generalization of Catalan numbers. He also conjectured sufficient conditions for the 2-adic valuation of $C_n^{\vec{b}}$ to be equal to that of C_n, under the assumption that there is a polynomial $f(x)$ with integer coefficients such that $\vec{b} = (f(0), f(1), \ldots)$. In 2010, An [1] proved Konvalinka's conjecture and studied other divisibility properties of weighted Catalan numbers using matrices. Later, in 2012, Shader [11] considered the periodicity modulo p^r for prime p of specific weighted Catalan numbers. In 2021, Gao and Gu [4, Theorem 4.2] proved that if m is a positive integer and $\vec{b} = (b_0, b_1, \ldots)$ is an infinite sequence of integers, then $C_0^{\vec{b}} \pmod{m}, C_1^{\vec{b}} \pmod{m}, C_2^{\vec{b}} \pmod{m}, \ldots$ is eventually periodic if and only if $m \mid b_0 b_1 \ldots b_\ell$ for some positive integer ℓ.

1.3 Our Contributions and Paper Outline

Our goal in this paper is to extend the study of weighted Catalan numbers to the multidimensional setting, in which multidimensional Catalan numbers (A060854 in the OEIS [12]) are defined geometrically; to the best of our knowledge, this has not been done before. We introduce the k-dimensional weighted Catalan numbers for $k \geq 2$. We prove their eventual periodicity modulo an integer, assuming certain conditions, and derive formulas for several classes of them. We also introduce novel analogs in the k-dimensional setting of the enumeration of Dyck paths by exact height (A080936 in the OEIS [12]) and of the Narayana numbers. We organize the paper as follows.

In Section 2, we begin by formally defining Balanced ballot paths and height for those paths. Using this notation, we define k-dimensional weighted Catalan numbers and k-dimensional u-bounded weighted Catalan numbers for all $k \geq 2$ and $u \geq 0$. Additionally, we reprove a substatement of Theorem 4.2 from [4] using a matrix-based approach, which serves to inspire the techniques used in subsequent sections.

In Section 3, we prove that for any positive integer m and any infinite sequence of integers $\vec{b}$, whose terms and m satisfy certain divisibility conditions, the sequence of the k-dimensional weighted Catalan numbers for $\vec{b}$ is eventually periodic modulo m.

In Section 4, we compute closed-form and recursive formulas for several cases of the multidimensional bounded and weighted Catalan numbers.

In Section 5, we introduce the k-dimensional Balanced-Ballot-Path-Height triangle, our generalization to k-dimensions of the enumeration of Dyck paths by height. We illustrate these by computing the first several values for $k = 3$ and $k = 4$. We also prove the eventual periodicity of the sequence of entries of the triangle modulo an integer.

In Section 6, we construct analogs of the Narayana numbers for the k-dimensional Balanced ballot paths. We prove relations between our analog of the Narayana numbers, the multidimensional Catalan numbers, and the Balanced-Ballot-Path-Height triangle. We illustrate the triangles by computing the first several values for $k = 3$ and $k = 4$. We conclude with open problems concerning the triangle for $k = 3$.

All Python scripts used to compute examples for Balanced-Ballot-Path-Height triangles and our analog of the Narayana triangle are publicly available in the GitHub repository `https://github.com/Ryota-IMath/Inagaki_Pramatarova_multidim_height_Catalan`.

2. Preliminaries and Definitions

2.1 Our Setup

We begin by discussing variants of the Dyck path and their extensions to higher dimensions. In this paper, unless stated otherwise, we work in the k-dimensional setting $\mathbb{Z}^k$, where $k \geq 2$ is an integer.

First, we formally state some basic terminology. A **point** in the k-dimensional lattice $\mathbb{Z}^k$ is a k-tuple $\vec{x} = (x_1, x_2, \ldots, x_k)$. We define a **path** as a sequence of vectors, called **steps**. For a path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ we define the ith **intermediate point** of P to be $\vec{v}_i = \sum_{j=1}^{i} \vec{s}_j$, with the 0th intermediate point being $\vec{v}_0 = \vec{0}$ by convention. Typically, in the paths we study, steps are in the positive coordinate directions, along the standard basis vectors $\vec{e_i} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, in which the ith coordinate is 1, and the others are 0.

We now state our generalization of Dyck paths to k-dimensional space.

Definition 2.1. *A k-**dimensional Balanced ballot path** of kn steps is a path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{kn})$ whose steps are in $\mathbb{Z}^k$ and satisfy the following conditions:*

- *(Balanced Property) For each $i \in \{1, 2, \ldots, k\}$, exactly n steps of P are equal to $\vec{e}_i$.*
- *(Ballot Property) Each intermediate point $\vec{x} = (x_1, \ldots, x_k) = \vec{0} + \sum_{j=1}^{i} \vec{s}_j$ in the path satisfies $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0$.*

*An **up-step** is any step equal to $\vec{e_1} = (1, 0, \ldots, 0)$. A **down-step** is any step equal to $\vec{e}_i$ for some $i \neq 1$. Given a Balanced ballot path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{kn})$, we define the **starting point of the ith step of** P as the sum of all steps in the Balanced ballot path before the ith term, i.e., $\vec{0} + \sum_{j=1}^{i-1} \vec{s}_j$.*

In particular, the 2-dimensional Balanced ballot paths are equivalent to Dyck paths. This equivalence can be seen by scaling, rotating, and flipping 2-dimensional Balanced ballot paths.

Remark 2.1. *The set of k-dimensional Balanced ballot paths of kn steps is equinumerous to the set of sequences of length kn that consist of n instances of i for each $i \in \{1, 2, \ldots, k\}$. The latter are called **ballot sequences** in [14], which is why we call the second item in Definition 2.1 the **ballot property**.*

Why steps equal to $\vec{e}_1$ are called up-steps will be explained later in this section.

We similarly define k-dimensional sub-ballot paths from $\vec{a}$ to $\vec{c}$.

Definition 2.2. *Given points $\vec{a}, \vec{c} \in \mathbb{Z}_{\geq 0}^k$, we define a k-**dimensional sub-ballot path from** $\vec{a}$ **to** $\vec{c}$ to be a path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ whose steps are in $\mathbb{Z}^k$ and satisfy the following conditions:*

- *Each step satisfies $\vec{s}_i \in \{\vec{e}_1, \ldots, \vec{e}_k\}$.*
- *Starting from $\vec{a}$ and taking those steps leads to $\vec{c}$, i.e., $\vec{a} + \vec{s}_1 + \vec{s}_2 + \cdots + \vec{s}_\ell = \vec{c}$.*
- *For all $i \in \{0, 1, 2, \ldots, \ell\}$, the point $\vec{x} = (x_1, \ldots, x_k) = \vec{a} + \sum_{j=1}^{i} \vec{s}_j$ satisfies $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0$.*

As in Definition 2.1, any step in the direction of $\vec{e_1} = (1, 0, \ldots, 0)$ *is an* ***up-step****. Likewise, a* ***down-step*** *is any step equal to* $\vec{e}_i$ *for some* $i \neq 1$*. For sub-ballot path* $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ *from* $\vec{a}$ *to* $\vec{c}$*, the* ***starting point of the*** i***th step*** *is defined to be* $\vec{a}$ *plus the sum of all steps in the sequence before* i*th term, i.e.,* $\vec{a} + \sum_{j=1}^{i-1} \vec{s}_j$*.*

In particular, observe that any k-dimensional Balanced ballot path with kn steps is a k-dimensional sub-ballot path from $\vec{0}$ to $(n, n, \ldots, n)$.

Using k-dimensional Balanced ballot paths, we define the k-dimensional Catalan number:

Definition 2.3 (A060854 in the OEIS [12])**.** *For* n *and* k*, the* n***th*** k***-dimensional Catalan number****, denoted by* $C_{k,n}$*, is the number of* k*-dimensional Balanced ballot paths of* kn *steps.*

This definition yields the formula

$$C_{n,k} = C_{k,n} = \frac{(0! \cdot 1! \cdots (n-1)!) \cdot (kn)!}{k!(k+1)! \cdots (k+n-1)!}.$$

Remark 2.2. *The* n*th* k*-dimensional Catalan number is the number of Standard Young Tableaux of shape* $k \times n$*, as derived by the hook length formula [14].*

We now extend the notion of weighted and bounded Catalan numbers to k-dimensional Catalan numbers. To that end, we define the height function as follows.

Definition 2.4. *For* $k \geq 2$*, we define the* ***height*** *function* $h_k : \mathbb{Z}^k \to \mathbb{R}$ *by*

$$h_k(x) = x_1 - x_2 + x_1 - x_3 + \ldots + x_1 - x_k = (k-1)x_1 - \sum_{i=2}^{k} x_i.$$

Given a k*-dimensional Balanced ballot path* $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{kn})$ *of* kn *steps, we define the* ***height of the Balanced ballot path*** P *to be* $\max\{h_k(\sum_{j=1}^{i} \vec{s}_j) : i \in \{0, 1, \ldots, kn\}\}$*. Similarly, when given a* k*-dimensional sub-ballot path* $P' = (\vec{s}'_1, \vec{s}'_2, \ldots, \vec{s}'_\ell)$ *from* $\vec{a}$ *to* $\vec{c}$*, we define the* ***height of the sub-ballot path*** P' *to be* $\max\{h_k(\vec{a} + \sum_{j=1}^{i} \vec{s}'_j) : i \in \{0, 1, \ldots, \ell\}\}$*.*

By convention, any Balanced ballot path with 0 *steps has a height of* 0*.*

The above definition of height is a natural analog of the height of intermediate points of Dyck paths; the latter is the difference between the number of up-steps and down-steps, i.e., $x_1 - x_2$. The height function is the Manhattan distance [10] from the point $(x_1, x_2, \ldots, x_k)$ to the point $(x_1, x_1, \ldots, x_1)$. The definition of height roughly measures how far $(x_1, x_2, \ldots, x_k)$ deviates from the line $\text{Span}\{(1, 1, \ldots, 1)\}$, which is the natural analog of the line $y = x$ from the 2-dimensional case.

Example 2.1. *A* 3*-dimensional Balanced ballot path is shown in Figure 3, where the black arrows correspond to* $\vec{e}_1$*, the dark gray arrows to* $\vec{e}_2$*, and the light gray arrows to* $\vec{e}_3$*. We label each intermediate point* $\vec{v}_i$ *with its corresponding height* $h_3(\vec{v}_i)$*. The height of this Balanced ballot path is* 3*, as it is the maximum height attained by any of its intermediate points.*

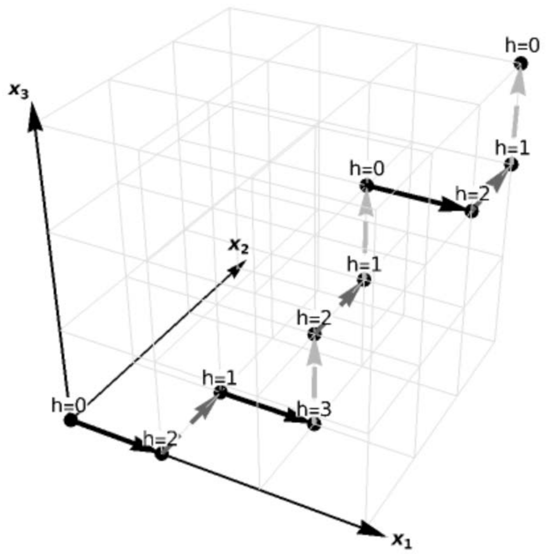

Figure 3: A 3-dimensional Balanced ballot path from $(0, 0, 0)$ to $(3, 3, 3)$ with the heights of each intermediate point along the path indicated. We use the formula $h_3(x) = x_1 - x_2 + x_1 - x_3$ to calculate the heights.

Definition 2.5. *For integers $k \geq 2$ and $u \geq 0$, we define the n**th** k**-dimensional** u**-bounded Catalan number**, denoted by $C_{k,u,n}$, as the number of k-dimensional Balanced ballot paths P of kn steps, satisfying the following condition: the height of the Balanced ballot path P, as in Definition 2.4, is less than or equal to u.*

The k-dimensional Balanced ballot paths of height at most u can be visualized as follows: These Balanced ballot paths are equivalent to the sub-ballot paths from $\vec{0}$ to $(n, \ldots, n)$ such that each node is between the hyperplanes

$$x_1 = \frac{x_2 + \cdots + x_k}{k-1} \text{ and } x_1 = \frac{x_2 + \cdots + x_k + u}{k-1}.$$

Here, steps in the $\vec{e}_1$ direction make positive contributions to the height of the resulting intermediate point. For this reason, we call steps equal to $\vec{e}_1$ up-steps. On the other hand, for $i \in \{2, 3, \ldots, k\}$, steps in the $\vec{e}_i$ direction make negative contributions to the height of the resulting intermediate point. This is why we call these steps down-steps.

Building again on our notions of height and Balanced ballot paths, we define the k-dimensional weighted Catalan numbers below. Note that, when calculating the weight of the Balanced ballot path, only the up-steps contribute multiplicatively to the path's total weight.

Definition 2.6. *Given a fixed infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$ and a k-dimensional Balanced ballot path P of kn steps, we define the **weight of** P **with respect to** $\vec{b}$, denoted by $wt_{\vec{b}}(P)$, as the product $b_{u_1} b_{u_2} \cdots b_{u_n}$, where u_i is the height (as in Definition 2.4) of the starting point of the ith up-step of P. The corresponding n**th** k**-dimensional weighted Catalan number** is*

$$C^{\vec{b}}_{k,n} = \sum_P wt_{\vec{b}}(P),$$

where the sum is over all k-dimensional Balanced ballot path P of kn steps.

In particular, we have $C^{(1,1,\ldots)}_{k,n} = C_{k,n}$ and $C^{\vec{b}'}_{k,n} = C_{k,u,n}$ for $\vec{b}' = (1, 1, \ldots, 1, 0, 0, \ldots)$, where the first $u - k + 2$ entries are equal to 1 and the rest are equal to 0.

We also consider the following auxiliary definition.

Definition 2.7. *Given an infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$, two points $\vec{a}$ and $\vec{c}$ in $\mathbb{Z}^k_{\geq 0}$, and a k-dimensional sub-ballot path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ from $\vec{a}$ to $\vec{c}$, we define the **weight of** P **with respect to** $\vec{b}$, denoted by $wt_{\vec{b}}(P)$, as the product $b_{u_1} b_{u_2} \cdots b_{u_{\ell'}}$, where u_i is the height (as in Definition 2.4) of the starting point of the ith up-step of P. The product is taken over all up-steps (i.e., steps of type $\vec{e}_1$), where ℓ' is the number of up-steps, indexed in their order of appearance. The corresponding n**th** k**-dimensional weighted sub-Catalan number** is*

$$C^{\vec{b}}_{k,\vec{a},\vec{c}} = \sum_P wt_{\vec{b}}(P),$$

where the sum is over all k-dimensional sub-ballot paths P from $\vec{a}$ to $\vec{c}$.

In particular, we have $C^{\vec{b}}_{k,\vec{0},(n,n,\ldots,n)} = C^{\vec{b}}_{k,n}$ because k-dimensional ballot paths are equivalent to k-dimensional sub-ballot paths from $\vec{0}$ to $(n, n, \ldots, n)$.

Finally, we define the k-dimensional u-bounded weighted Catalan numbers, a special case of k-dimensional weighted Catalan numbers.

Definition 2.8. *Let $k \geq 2$ and $u \geq 0$ be integers. For a fixed infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$, the corresponding n**th** k**-dimensional** u**-bounded weighted Catalan number** $C^{\vec{b}}_{k,u,n}$ is defined by $C^{\vec{b}}_{k,u,n} = \sum_P wt_{\vec{b}}(P)$ where the sum is over all k-dimensional Balanced ballot paths P of kn steps such that the height of P is at most u.*

In particular, we obtain $C^{\vec{b}}_{k,u,n} = C^{(b_0, b_1, \ldots, b_{u-k+1}, 0, 0, \ldots)}_{k,n}$ and $C^{(1,1,\ldots)}_{k,u,n} = C_{k,u,n}$, and $C_{k,u,n} = C^{\vec{b}'}_{k,u,n}$ for the infinite sequence of integers $\vec{b}' = (1, 1, \ldots, 1, 0, 0, \ldots)$, where the first $u - k + 1$ entries are equal to 1 and the remaining entries are equal to 0.

2.2 Matrix Recurrences for Weighted Catalan Numbers

To provide a foundation for examining periodicity, we use matrix-based recursive formulations for the weighted Catalan numbers to partially reprove the following substatement of Theorem 4.2 of [4].

Proposition 2.1. *Let m be any positive integer. If m divides $b_0 b_1 \ldots b_u$ for some non-negative integer u, then the sequence $C^{\vec{b}}_0 \pmod{m}, C^{\vec{b}}_1 \pmod{m}, C^{\vec{b}}_2 \pmod{m}, \ldots$ is eventually periodic.*

To that end, we derive a tridiagonal-matrix-based recurrence for the 2-dimensional weighted Catalan numbers analogous to An [1] and Shader [11]. We introduce some notation.

Definition 2.9. *For points $\vec{a}, \vec{c} \in \mathbb{Z}^2$, we define a* ***sub-Dyck path*** *from $\vec{a}$ to $\vec{c}$ as a sequence $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ of 2-dimensional steps of the form $(1,1)$ and $(1,-1)$ such that $\vec{a} + \sum_{j=0}^{\ell} \vec{s}_j = \vec{c}$ and, for any $i \in \{0,1,2,\ldots,\ell\}$, the point $\vec{a} + \sum_{j=1}^{i} \vec{s}_j$ is above or on the x-axis. We define the* ***starting point of the ith step of*** *P to be $\vec{a} + \sum_{j=1}^{i-1} \vec{s}_j$.*

Definition 2.10. *Given an infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$, two points $\vec{a}, \vec{c}$ in $\mathbb{Z}^2$, and a sub-Dyck path $P = (\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_\ell)$ from $\vec{a}$ to $\vec{c}$, the* ***weight of*** *P* ***with respect to*** *$\vec{b}$, denoted by $wt_{\vec{b}}(P)$, is the product $b_{u_1} b_{u_2} \cdots b_{u_{\ell'}}$, where u_i is the height of the ith up-step of P and the product is over all up-steps.*

For nonnegative integers i and n, we define the corresponding ***weighted sub-Catalan number*** *as $S^{\vec{b}}_{n,i} = \sum_P wt_{\vec{b}}(P)$ where the sum is over all sub-Dyck paths from $(0,i)$ to $(2n,0)$.*

In particular, we have $S^{\vec{b}}_{n,0} = C^{\vec{b}}_n$ since Dyck paths of length $2n$ are equivalent to sub-Dyck paths from $(0,0)$ to $(2n,0)$.

Lemma 2.1. *The weighted sub-Catalan numbers satisfy the following recurrence:*

$$\begin{pmatrix} S^{\vec{b}}_{n,0} \\ S^{\vec{b}}_{n,2} \\ S^{\vec{b}}_{n,4} \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 & b_0 b_1 & 0 & \cdots & \cdots & \cdots \\ 1 & b_1 + b_2 & b_2 b_3 & 0 & \cdots & \cdots \\ 0 & 1 & b_3 + b_4 & b_4 b_5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \begin{pmatrix} S^{\vec{b}}_{n-1,0} \\ S^{\vec{b}}_{n-1,2} \\ S^{\vec{b}}_{n-1,4} \\ \vdots \end{pmatrix}.$$

Proof. We have $S^{\vec{b}}_{n,0} = b_0 S^{\vec{b}}_{n-1,0} + b_0 b_1 S^{\vec{b}}_{n-1,2}$ due to the possible two-step subsequences we can append together to form a Dyck path of $2n$ steps. For our first two steps of the Dyck path, we may take the two-step sub-Dyck path $((1,1),(1,1))$, which has weight $b_0 b_1$, from $(0,0)$ to $(2,2)$ or the two-step sub-Dyck path $((1,1),(1,-1))$, which has weight b_0, from $(0,0)$ to $(2,0)$.

If we take the former, the set of possible sub-Dyck paths that can be appended to it to construct a Dyck path of $2n$ steps is exactly the set of all sub-Dyck paths $(\vec{s}_1', \vec{s}_2', \ldots)$ from $(2,2)$ to $(2n,0)$. This set is the same as the set of all sub-Dyck paths from $(0,2)$ to $(2n-2,0)$. This collection of sub-Dyck paths contributes the summand $b_0 b_1 S^{\vec{b}}_{n-1,2}$.

If we take the latter, the set of possible sub-Dyck paths that can be appended to complete the Dyck path of $2n$ steps is exactly the set of all sub-Dyck paths from $(2,0)$ to $(2n,0)$. This set is the same as the set of all sub-Dyck paths from $(0,0)$ to $(2n-2,0)$. This set of sub-Dyck paths yields the summand $b_0 S^{\vec{b}}_{n-1,0}$.

For $i \geq 1$, we have

$$S^{\vec{b}}_{n,2i} = S^{\vec{b}}_{n-1,2i-2} + (b_{2i} + b_{2i-1}) S^{\vec{b}}_{n-1,2i} + b_{2i} b_{2i+1} S^{\vec{b}}_{n-1,2i+2},$$

due to the possible two-step sub-Dyck paths we can take from $(0,2i)$ and their corresponding weights. The four possible sub-Dyck paths are $((1,1),(1,1))$, $((1,1),(1,-1))$, $((1,-1),(1,1))$, and $((1,-1),(1,-1))$.

We first consider the sub-Dyck path $((1,1),(1,1))$ from $(0,2i)$ to $(2,2i+2)$, which has weight $b_{2i} b_{2i+1}$. The possible suffixes that can follow to create a sub-Dyck path from $(0,2i)$ to $(2n,0)$ are exactly the set of all sub-Dyck paths from $(0,2i+2)$ to $(2n-2,0)$. Hence, this case contributes the summand $b_{2i} b_{2i+1} S^{\vec{b}}_{n-1,2i+2}$.

Next, consider the two-step sub-Dyck paths $((1,1),(1,-1))$ and $((1,-1),(1,1))$. The weight of the sub-Dyck path $((1,1),(1,-1))$ from $(0,2i)$ to $(2,2i)$ is b_{2i}, and the product of the weights of the sub-Dyck path $((1,-1),(1,1))$ from $(0,2i)$ to $(2,2i)$ is b_{2i-1}. Then, observe that the set of possible suffixes that can follow to complete a sub-Dyck path from $(0,2i)$ to $(2n,0)$ is exactly the set of all sub-Dyck paths from $(0,2i)$ to $(2n-2,0)$. This set of sub-Dyck paths yields the summand $(b_{2i-1} + b_{2i}) S^{\vec{b}}_{n-1,2i}$.

Finally, we consider the two-step sub-Dyck path $((1,-1),(1,-1))$. This sub-Dyck path from $(0,2i)$ to $(2,2i-2)$ has weight 1. Then observe that the set of possible suffixes that can be appended to complete a sub-Dyck path from $(0,2i)$ to $(2n,0)$ is the set of all sub-Dyck paths from $(0,2i-2)$ to $(2n-2,0)$. This contributes the summand $S^{\vec{b}}_{n-1,2i-2}$. □

Using the matrix from the proof of Lemma 2.1, which we call the *transition matrix*, we prove Proposition 2.1.

Proof of Proposition 2.1. We prove this periodicity property by using an argument similar to that employed by An to prove Theorem 20 of [1].

Recall that $S^{\vec{b}}_{n,i}$ is the sum of weights of all weighted Dyck paths from $(0,i)$ to $(2n,0)$. Observe that the weight of each Dyck path for $i > \left\lfloor \frac{u}{2} \right\rfloor$ is divisible by $b_0 b_1 \ldots b_u$. Together with Lemma 2.1, this fact implies that the transition matrix modulo m is of finite size $(\ell+1)\times(\ell+1)$, where $\ell = \left\lfloor \frac{u}{2} \right\rfloor$. If u is even, then the last element of the matrix is $a_{2\ell} = b_{u-1}$, and if u is odd, then it is $a_{2\ell} = b_{u-2} + b_{u-1}$. Consequently, Lemma 2.1 yields the following relation.

$$\begin{pmatrix} S^{\vec{b}}_{n,0} \\ S^{\vec{b}}_{n,2} \\ S^{\vec{b}}_{n,4} \\ \vdots \\ S^{\vec{b}}_{n,2\ell} \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_1 & 0 & \dots & \dots & \dots \\ 1 & b_1+b_2 & b_2b_3 & 0 & \dots & \dots \\ 0 & 1 & b_3+b_4 & b_4b_5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 1 & a_{2\ell} \end{pmatrix} \begin{pmatrix} S^{\vec{b}}_{n-1,0} \\ S^{\vec{b}}_{n-1,2} \\ S^{\vec{b}}_{n-1,4} \\ \vdots \\ S^{\vec{b}}_{n-1,2\ell} \end{pmatrix}.$$

There exist positive integers t and t' such that $(S^{\vec{b}}_{t,0}, \dots, S^{\vec{b}}_{t,2\ell}) \equiv (S^{\vec{b}}_{t+t',0}, \dots, S^{\vec{b}}_{t+t',2\ell}) \pmod{m}$ because, by the pigeonhole principle, there are at most $m^{\ell+1}$ possible values of $(S^{\vec{b}}_{n,0}, \dots, S^{\vec{b}}_{n,2\ell}) \pmod{m}$. The matrix is of finite size, and thus the sequence will eventually be periodic, with $S^{\vec{b}}_{n+jt',0} \equiv S^{\vec{b}}_{n,0} \pmod{m}$ for any integer $j \geq 1$ and sufficiently large n. □

3. Periodicity of multidimensional Catalan numbers

In this section, we derive general results on the periodicity of the multidimensional weighted Catalan numbers from Definition 2.6. We obtain a result analogous to Proposition 2.1 on the periodicity of the k-dimensional, bounded weighted Catalan numbers.

Theorem 3.1. *Let m be a positive integer and $\vec{b} = (b_0, b_1, \dots)$ be a fixed infinite sequence of integers. The sequence of corresponding k-dimensional u-bounded weighted Catalan numbers $C^{\vec{b}}_{k,u,0} \pmod{m}, C^{\vec{b}}_{k,u,1} \pmod{m}, C^{\vec{b}}_{k,u,2} \pmod{m}, \dots$ is eventually periodic.*

Proof. We extend the arguments used in Lemma 2.1 and Proposition 2.1 to the k-dimensional case. First, we consider the transition matrix, constructed using the same method as used to derive the matrix in Lemma 2.1, except that we consider the previous k steps rather than two. Note that the transition matrix for $C^{\vec{b}}_{k,u,n}$ is finite because we consider Balanced ballot paths of height at most u. Then, consider $(A_n, A'_n, \dots A^{(\ell)}_n)$, which is a vector whose coordinates are sums of the weights $wt_{\vec{b}}(P)$ of sub-ballot paths P from a finite set of starting points. One of the coordinates, A_n, is equal to $C^{\vec{b}}_{k,u,n}$, which is the sum of weights of all of the sub-ballot paths starting from $\vec{0}$ to $(n, n, \dots, n)$ with height at most u. There are at most $m^{\ell+1}$ possible values of $(A_n, A'_n, \dots A^{(\ell)}_n) \pmod{m}$; hence, by the pigeonhole principle, there exist positive integers t and t' such that $(A_t, A'_t, \dots A^{(\ell)}_t) \equiv (A_{t+t'}, A'_{t+t'}, \dots A^{(\ell)}_{t+t'}) \pmod{m}$. Thus, $A_{n+st'} \equiv A_n \pmod{m}$ for any positive integer s and sufficiently large n. □

Using Theorem 3.1, we obtain the following result on the periodicity of the k-dimensional weighted Catalan numbers.

Theorem 3.2. *Let m be any positive integer, and let $\vec{b} = (b_0, b_1, \dots)$ be any infinite sequence of integers. If there exists a positive integer u such that $b_u, b_{u+1}, \dots b_{u+k-2}$ are all divisible by m, then the sequence $C^{\vec{b}}_{k,0} \pmod{m}, C^{\vec{b}}_{k,1} \pmod{m}, C^{\vec{b}}_{k,2} \pmod{m}, \dots$ is eventually periodic.*

Proof. By our definition of height (Definition 2.4), the weight of the Balanced ballot path changes only at each up-step (see Definition 2.6). We examine what occurs after taking one up-step. For a k-dimensional Balanced ballot path P to reach a height greater than $u + k - 2$, there must be an up-step whose starting point has height between u and $u+k-2$ inclusive. This is because taking an up-step increases the height of the resulting intermediate point by $k - 1$. If the Balanced ballot path P has an up-step starting at a point within that range of heights, then its weight $wt_{\vec{b}}(P)$ is divisible by m. Thus, for any k-dimensional Balanced ballot path P with height greater than $u + k - 2$, the weight $wt_{\vec{b}}(P) \equiv 0 \pmod{m}$. Then it is enough to consider only the paths such that $h_k(x) \leq u + k - 2$ for each intermediate point x, i.e., the Balanced ballot paths with height at most $u + k - 2$; this is because all other contributions vanish modulo m. By Theorem 3.1, the sequence of the weighted Catalan numbers corresponding to those paths modulo m, which is $C^{\vec{b}}_{k,(u+k-2),0} \pmod{m}, C^{\vec{b}}_{k,(u+k-2),1} \pmod{m}, C^{\vec{b}}_{k,(u+k-2),2} \pmod{m}, \dots$, is eventually periodic. □

Similar statements hold when m divides the product of several entries of $\vec{b} = (b_0, b_1, \dots)$. Below is another specific, sufficient condition for periodicity of the sequence of k-dimensional weighted Catalan numbers modulo m.

Theorem 3.3. *Let m be any positive integer and let $\vec{b} = (b_0, b_1, \dots)$ be any infinite sequence of integers. If there exists a positive integer u such that $m \mid b_{u-j} b_{u+k-1-j'}$ for all $j \in \{0, 1, 2 \dots, k-2\}$ and $j' \in \{j, \dots k-2\}$, the sequence $C^{\vec{b}}_{k,0} \pmod{m}, C^{\vec{b}}_{k,1} \pmod{m}, C^{\vec{b}}_{k,2} \pmod{m}, \dots$ is eventually periodic.*

Proof. For any Balanced ballot path P that has height greater than or equal to $u+k$, the last two up-steps before we have the first intermediate point above height $u+k$ are always of the following form: a step in the $\vec{e}_1$ direction from height $u-j$ for some $j \in \{0, 1, 2, \dots, k-2\}$ to height $u-j+k-1$ and then an up-step from height ℓ for some $\ell \in \{u+1, u+2, \dots, u+k-1-j\}$ to height $\ell+k-1$. We note that $\ell+k-1 \geq u+k$; taking an up-step whose starting point has height ℓ yields an intermediate point of height at least $u+k$.

Therefore, we find that any summand in $C^{\vec{b}}_{k,n} \equiv \sum_P wt_{\vec{b}}(P) \pmod{m}$ from any Balanced ballot path that exceeds height $u+k-1$ is $0 \pmod{m}$. Hence, $C^{\vec{b}}_{k,n} \equiv C^{\vec{b}}_{k,u+k-1,n} \pmod{m}$. We know from Theorem 3.1 that $C^{\vec{b}}_{k,u+k-1,n} \pmod{m}$ is eventually periodic. This completes the proof. □

4. Examples of k-dimensional weighted Catalan numbers

In this section, we obtain formulas for specific sequences of the k-dimensional u-bounded and weighted Catalan numbers $C^{\vec{b}}_{k,u,n}$. We start with a general k, but later in this section we focus primarily on $k=3$, for ease of computation.

We begin with the following general observations on k-dimensional Balanced ballot paths.

Proposition 4.1. *For $k \geq 2$, every nonzero-length k-dimensional Balanced ballot path must have height at least $k-1$.*

Proof. Every nonzero-length k-dimensional Balanced ballot path must begin with an up-step, i.e., a step of form $\vec{e}_1$. By our definition of height (Definition 2.4), the intermediate point, $\vec{v}_1 = \vec{e}_1$, resulting from the first step has height $h_k(\vec{v}_1) = k-1$. Hence, every k-dimensional Balanced ballot path must have a height of at least $k-1$. □

Proposition 4.2. *For any $k \geq 2$, we have $C_{k,k-1,n} = 1$.*

Proof. For every $n \geq 1$, there is only one k-dimensional Balanced ballot path from $\vec{0}$ to $(n, n, \dots, n)$ that does not exceed height $k-1$, namely the Balanced ballot path formed by repeating n times the sequence of steps $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k)$. We call this specific path P_0.

We can describe all intermediate points of P_0 by noting that, for all $d \in \{0, 1, 2, \dots, n-1\}$ and $r \in \{0, 1, \dots, k\}$, the $(dk+r)$th intermediate point can be written as

$$\vec{v}_{dk+r} = \sum_{i=1}^{r} (d+1)\vec{e}_i + \sum_{i'=r+1}^{k} d\vec{e}_{i'}.$$

From this formula, we find that, for $d \in \{0, 1, 2, \dots, n-1\}$ and $r=1$, the $(dk+r)$th intermediate point has height $k-1$ and for all other values of $r \in \{0, 1, \dots, k\}$, its height is strictly less than $k-1$.

We also show that any Balanced ballot path P from $\vec{0}$ to $(n, n, \dots, n)$ with $P \neq P_0$ has height greater than $k-1$. Suppose for the sake of contradiction that there existed a Balanced ballot path P with height $k-1$ that is not equal to P_0. We show that for $d \in \{1, 2, \dots k-1\}$, the subsequence of steps between the dth and $(d+1)$th up-step in P must be $(\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k)$.

We proceed by induction on d. We observe that between the first and second up-steps (i.e., instances of $\vec{e}_1$) in path $P = (\vec{s}_1, \vec{s}_2, \dots, \vec{s}_{kn})$, there must be at least $k-1$ down-steps; otherwise, the height of the path at the $(k+1)$th intermediate point $\vec{v}_{k+1}$ is $h_k(\vec{v}_{k+1}) > k-1$. From the ballot property of Balanced ballot paths (second item of Definition 2.1) and because the first step of P is the first up-step, the sequence of steps after the first and second up-steps must be $(\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k)$. For the inductive step, we make a similar argument and show that the subsequence of steps between the dth and $(d+1)$th up-step in P must be $(\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k)$. By our inductive hypothesis, we observe that the dth up-step is the $((d-1)k+1)$th step. Then, the $((d-1)k+1)$th intermediate point of P is $d\vec{e}_1 + \sum_{i=2}^{k} (d-1)\vec{e}_i$ and has height $k-1$. Thus, as in the base case, there must be $k-1$ down-steps between the dth and $(d+1)$th up-step in P. Furthermore, those steps in order are $\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k$. Now, because we know that for any $d \in \{0, 1, 2, \dots, n-1\}$, the subsequence of steps between the dth and $(d+1)$th up-steps of P is $(\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k)$, the first $(n-1)k+1$ steps of P consist of the sequence of steps $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k)$ being repeated $n-1$ consecutive times followed by a step equal to $\vec{e}_1$. Finally, we find that the last $k-1$ steps of P in order are $\vec{e}_2, \vec{e}_3, \dots, \vec{e}_k$, because of the ballot property. Therefore, we obtain that P consists of the sequence $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k)$ being repeated n consecutive times. Hence, $P = P_0$ and we have reached a contradiction. □

In sum, by Proposition 4.1, all k-dimensional Balanced ballot paths of nonzero steps have height at least k, and, by Proposition 4.2, there is only one k-dimensional Balanced ballot path of height at most $k-1$. Thus, we dedicate the rest of the section to deriving formulas for the k-dimensional, u-bounded weighted Catalan numbers for certain values of k and $u \geq k-1$. We do this using matrix-based methods similar to the one used to obtain Lemma 2.1.

We now find a closed-form expression for the k-dimensional, k-bounded, weighted Catalan numbers.

Theorem 4.1. *For any infinite sequence of integers $\vec{b} = (b_0, b_1, \dots)$ and integer $n \geq 1$, we have*

$$C^{\vec{b}}_{k,k,n} = b_0(b_0 + (k-1)b_1)^{n-1}.$$

Proof. Since $C^{\vec{b}}_{k,k,1} = b_0$, it suffices to prove that the k-dimensional k-bounded and weighted Catalan numbers satisfy the recurrence

$$C^{\vec{b}}_{k,k,n} = (b_0 + (k-1)b_1)C^{\vec{b}}_{k,k,n-1}.$$

Thus, we dedicate the rest of the proof to proving the recurrence.

We first observe that, in order to prove the recurrence, it is sufficient to prove $C^{(b_0,b_1,0,0,\dots)}_{k,n} = (b_0 + (k-1)b_1)C^{(b_0,b_1,0,0,\dots)}_{k,n-1}$. This is because $C^{\vec{b}}_{k,k,n} = C^{(b_0,b_1,0,0,\dots)}_{k,n}$ and $C^{(b_0,b_1,0,0,\dots)}_{k,n-1} = C^{\vec{b}}_{k,k,n-1}$.

For brevity, let $A_n = C^{\vec{b}}_{k,k,n}$. Let $B_{n-1} = C^{\vec{b}}_{k,(2,1,1,\dots,1,0),(n,n,\dots,n)}$, i.e., the sum of the weights of all k-dimensional sub-ballot paths from $(2,1,1,\dots,1,0)$ to $(n,n,\dots,n)$ with height at most k.

We first prove that $A_n = b_0A_{n-1} + b_0b_1B_{n-1}$. We begin by observing that there is only one k-step sub-ballot path from $(a,\dots,a)$ to $(a+1,\dots,a+1)$ with height at most k. This sub-ballot path is $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k)$. This sub-ballot path has a weight of b_0. Then, observe the set of k-dimensional sub-ballot paths from $(a+1, a+1, \dots, a+1)$ to $(n,n,\dots,n)$ is the same as the set of k-dimensional Balanced ballot paths of length $k(n-a-1)$. This yields the summand b_0A_{n-1}. Then, there is only one k-step sub-ballot path from $(a,\dots,a)$ to $(a+2, a+1, \dots, a+1, a)$ with height at most k. These sub-ballot paths are all the length-k sequences of steps that can begin the k-dimensional Balanced ballot paths of kn steps. This has a weight contribution of b_0b_1. This yields the summand $b_0b_1B_{n-1}$.

Now, we prove that $B_n = (k-1)A_{n-1} + (k-1)b_1B_{n-1}$. We begin by observing that there are $k-1$ ways to go from $(a, a-1, \dots, a-1, a-2)$ to $(a,\dots,a)$ with weight contribution of 1. Then, observe that the set of k-dimensional sub-ballot paths from $(a,\dots,a)$ to $(n,\dots,n)$ having height of at most k is the same as the set of k-dimensional sub-ballot paths from $(0,\dots,0)$ to $(n-a,\dots,n-a)$ with height at most k. This yields the summand $(k-1)A_{n-1}$. Then, we observe that there are $k-1$ sub-ballot paths of k steps from $(a, a-1, \dots, a-1, a-2)$ to $(a+1, a, \dots, a, a-1)$ of height at most k. In each of these paths, every instance of $\vec{e}_i$ must be before $\vec{e}_{i+1}$ for each $i \in \{2, 3, \dots k-2\}$, and the last step is $\vec{e}_1$. Also, in each of these paths, there are $k-1$ possibilities for when the $\vec{e}_k$ step will occur. Thus, each of these sub-ballot paths has a weight of b_1, and they yield us the summand $(k-1)b_1B_{n-1}$.

Using these relations, we obtain the recurrence:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_1 \\ k-1 & (k-1)b_1 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}.$$

From $A_n = b_0A_{n-1} + b_0b_1B_{n-1}$ it follows $B_{n-1} = \dfrac{A_n - b_0A_{n-1}}{b_0b_1}$. Substituting into the second row we get $B_n = (k-1)A_{n-1} + (k-1)\dfrac{A_n - b_0A_{n-1}}{b_0} = \dfrac{k-1}{b_0}A_n$. Hence, $A_n = (b_0 + (k-1)b_1)A_{n-1}$. □

If we set $\vec{b} = (1, 1, 0, 0, \dots)$, Theorem 4.1 immediately implies the following.

Corollary 4.1. *For any integer $n \geq 1$, we have $C_{k,k,n} = k^{n-1}$.*

We now focus on finding formulas for examples of 3-dimensional weighted, u-bounded Catalan numbers. For the rest of the section, let $\vec{b} = (b_0, b_1, \dots)$ and $\vec{b}' = (b_0, b_1, \dots, b_{u-2}, 0, 0, \dots)$ be infinite sequences of integers, and let u be a nonnegative integer. We define $A_n = C^{\vec{b}'}_{3,n} = C^{\vec{b}'}_{3,u,n}$, by $B_{n-1} = C^{\vec{b}'}_{3,(2,1,0),(n,n,n)}$, by $C'_{n-2} := C^{\vec{b}'}_{3,(3,3,0),(n,n,n)}$, $D_{n-1} := C^{\vec{b}'}_{3,(3,0,0),(n,n,n)}$, and $E_{n-2} = C^{\vec{b}'}_{3,(4,2,0),(n,n,n)}$, and $F_{n-3} = C^{\vec{b}'}_{3,(5,4,0),(n,n,n)}$. Note that the initial conditions are $(A_0, B_0, C'_0, D_0, E_0) = (1, 0, 0, 0, 0)$. Also, note that, by definition, $A_n = C^{\vec{b}}_{3,u,n}$. We derive recursive relations for $A_n = C^{\vec{b}}_{3,u,n}$ for the cases $u = 4$, $u = 5$, and $u = 6$.

We first find recursive relations describing $A_n = C^{\vec{b}}_{3,u,n}$ for the case where $u = 3$.

Proposition 4.3. *For any infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$, the 3-dimensional 4-bounded weighted Catalan numbers satisfy the recurrence*

$$\begin{pmatrix} A_n \\ B_n \\ C'_n \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_1 + b_0b_2 & 0 \\ 2 & 2b_1 + 2b_2 & b_2 \\ 1 & b_1 + b_2 & b_2 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C'_{n-1} \end{pmatrix}.$$

Proof. As in the proof of Theorem 4.1, we begin by observing the possible 3-step sub-ballot paths that can be concatenated to construct 3-dimensional Balanced ballot paths. The possible forms of intermediate points after every 3-step, 3-dimensional sub-ballot path not exceeding height 4 are displayed in Figure 4.

We first show that $A_n = b_0A_{n-1} + (b_0b_1 + b_0b_2)B_{n-1}$. By the definition of a Balanced ballot path, there is one 3-step 3-dimensional sub-ballot path from a point of form (a, a, a) to $(a + 1, a + 1, a + 1)$, namely the path $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ with a weight contribution of b_0. Then, observe that the set of sub-ballot paths from $(a+1, a+1, a+1)$ to (n, n, n) is the same as the set of 3-dimensional Balanced ballot paths of $3(n - a - 1)$ steps. This contributes the summand b_0A_{n-1}. Next, observe that there are two sequences of steps that one can take from (a, a, a) to $(a + 2, a + 1, a)$ with height at most 5, namely sub-ballot paths $(\vec{e_1}, \vec{e_2}, \vec{e_1})$ and $(\vec{e_1}, \vec{e_1}, \vec{e_2})$, which have weights b_0b_1 and b_0b_2 respectively. These are all of the length-3 sub-ballot paths with height at most 4 that can begin the 3-dimensional Balanced ballot paths of length $3n$.

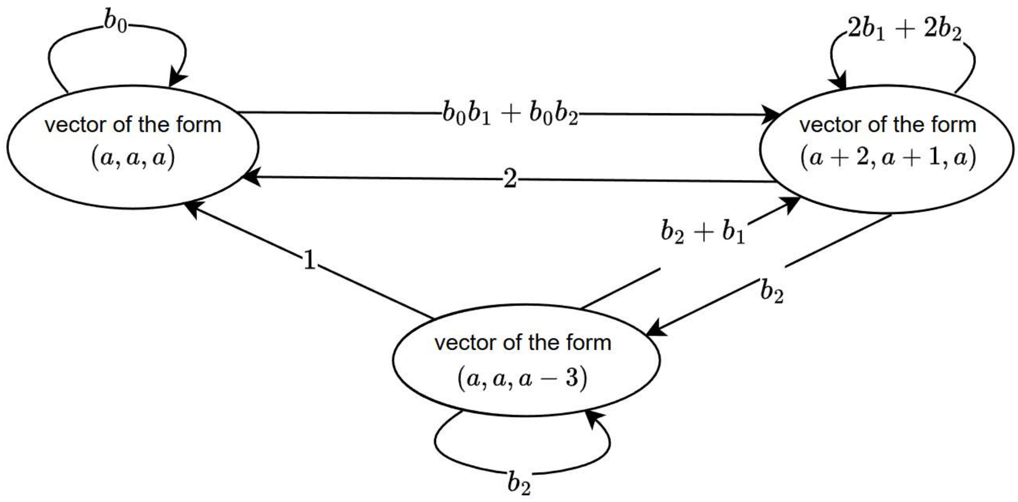

Figure 4: Possible forms of intermediate points for Balanced ballot paths whose weights are summed in $C^{\vec{b}}_{3,5,n}$ with $\vec{b} = (b_0, b_1, \ldots)$. We label each arrow with the sum of the weights of 3-step sub-ballot paths with height at most 4 that transition between the forms.

Likewise, we prove that $B_n = 2A_{n-1} + (2b_1 + 2b_2)B_{n-1} + b_2C'_{n-1}$. Firstly, the sum of the weight contributions of all 3-step sub-ballot paths from $(a, a - 1, a - 2)$ to (a, a, a) is 2. The set of sub-ballot paths from (a, a, a) to (n, n, n) with height at most 4 is the same as the set of sub-ballot paths from $(0, 0, 0)$ to $(n - a, n - a, n - a)$ with height at most 4. This yields the summand $2A_{n-1}$. Then, we observe that the sum of weight contributions of all 3-step sub-ballot paths from $(a, a - 1, a - 2)$ to $(a + 1, a, a - 1)$ with height at most 4 equals $2b_1 + 2b_2$. The set of sub-ballot paths from $(a + 1, a, a - 1)$ to (n, n, n) with height at most 4 is the set of sub-ballot paths from $(2, 1, 0)$ to $(n - a + 1, n - a + 1, n - a + 1)$ with height at most 4. Thus, we have the summand $(2b_1 + 2b_2)B_{n-1}$. Lastly, we observe that the sum of weight contributions of all 3-step sub-ballot paths from $(a, a - 1, a - 2)$ to $(a + 1, a + 1, a - 2)$ with height at most 4 is b_2. The set of sub-ballot paths from $(a + 1, a + 1, a - 2)$ to (n, n, n) with height at most 4 is equal to the set of sub-ballot paths from $(3, 3, 0)$ to $(n - a + 2, n - a + 2, n - a + 2)$ with height at most 4. Hence, we have the summand $(b_2)C'_{n-1}$.

Finally, we show that $C'_n = A_{n-1} + (b_1 + b_2)B_{n-1} + b_2C'_{n-1}$. To prove this relation, we begin by observing that the sum of the weight contributions of all sub-ballot paths from $(a, a, a - 3)$ to (a, a, a) with height at most 4 is 1. The set of sub-ballot paths from (a, a, a) to (n, n, n) with height at most 4 is the same as the set of sub-ballot paths from $(0, 0, 0)$ to $(n - a, n - a, n - a)$ with height at most 4. Thus, we obtain the summand A_{n-1}. Also, we find that the sum of the weights of all sub-ballot paths from $(a, a, a - 3)$ to $(a + 1, a, a - 1)$ with height at most 4 is $b_2 + b_1$. The set of sub-ballot paths from $(a + 1, a, a - 1)$ to (n, n, n) with height at most 4 is the set of sub-ballot paths from $(2, 1, 0)$ to $(n - a + 1, n - a + 1, n - a + 1)$ with height at most 4. Thus, we have the summand $(b_2 + b_1)B_{n-1}$. Lastly, we find that the sum of the weight contributions of all sub-ballot paths from $(a, a, a - 3)$ to $(a + 1, a + 1, a - 2)$ is b_2. The set of sub-ballot paths from $(a + 1, a + 1, a - 2)$ to (n, n, n)

with height at most 4 is the same as the set of sub-ballot paths from $(3,3,0)$ to $(n-a+2, n-a+2, n-a+2)$ with height at most 4. Thus, we have the summand $b_2 C'_{n-1}$. □

In the case $\vec{b} = (1,1,0,0\ldots)$, Theorem 4.3 gives $C^{\vec{b}}_{3,4,n} = (\frac{1}{6}) \cdot ((3+\sqrt{6})^{n+1} + (3-\sqrt{6})^{n+1})$. The sequence with this formula is A158869 in OEIS [12], which counts ways of filling a $2 \times 3 \times 2n$ parallelepiped with bricks $1 \times 2 \times 2$.

Corollary 4.2. *For $n \geq 2$, the 3-dimensional 4-bounded Catalan numbers satisfy the recurrence*

$$C_{3,4,n} = 6C_{3,4,n-1} - 3C_{3,4,n-2}.$$

Similarly, we have the following recurrence relation for computing $C^{\vec{b}}_{3,5,n}$.

Proposition 4.4. *For any infinite sequence of integers $\vec{b} = (b_0, b_1, \ldots)$, the 3-dimensional 5-bounded weighted Catalan numbers $C^{\vec{b}}_{3,5,n}$ satisfy the recurrence*

$$\begin{pmatrix} A_n \\ B_n \\ C'_n \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_2 + b_0b_1 & 0 \\ 2 & 2(b_1+b_2+b_3) & b_2+b_3 \\ 1 & b_1+b_2+b_3 & 2(b_1+b_2+b_3) \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C'_{n-1} \end{pmatrix}.$$

Proof. We use the same strategy as the one used to prove Proposition 4.3. There are three possible forms of intermediate points of Balanced ballot paths whose weights we sum when calculating $C^{\vec{b}}_{3,5,n}$: points that are of form (a,a,a) for some integer a, points of form $(a, a-1, a-2)$, and points of form $(a,a,a-3)$.

We first prove $A_n = b_0 A_{n-1} + (b_0b_1 + b_0b_2)B_{n-1}$ Due to the restriction on the height of Balanced ballot paths that we sum over in the formula for $C^{\vec{b}}_{3,5,n}$ (see Definition 2.8), there is only one possible sub-ballot path from (a,a,a) to $(a+1,a+1,a+1)$ that does not exceed height 5; the weight of the sub-ballot path is b_0. Thus, we obtain the summand $b_0 A_{n-1}$. There are two sub-ballot paths from (a,a,a) to $(a+2,a+1,a)$ that have height at most 5: one has weight b_0b_2, the other has weight b_0b_1. Thus, we obtain the summand $(b_0b_1 + b_0b_2)B_{n-1}$. There are zero possible 3-step sub-ballot paths from $(a,a,a-3)$ to $(a+3,a+3,a)$. Therefore, we have summand $0C'_{n-1}$.

Now, we prove $B_n = 2A_{n-1} + 2(b_1+b_2+b_3)B_{n-1} + (b_3+b_2)C'_{n-1}$. First, there are exactly two paths from $(a,a-1,a-2)$ to (a,a,a) that have height at most 5, and each has weight 1. This justifies the summand $2A_{n-1}$. Then, notice that there are six sub-ballot paths from $(a,a-1,a-2)$ to $(a+1,a,a-1)$ with height at most 5—two have weight b_1, two have weight b_2, and two have weight b_3. This yields the summand $2(b_1+b_2+b_3)B_{n-1}$. Lastly, there are two sub-ballot paths from $(a,a-1,a-2)$ to $(a+1,a+1,a-2)$ of height at most 5; one has weight b_3 and the other has weight b_2. This contributes to the summand $(b_3+b_2)C'_{n-1}$.

Finally, we prove $C'_n = A_{n-1} + (b_1+b_2+b_3)B_{n-1} + 2(b_1+b_2+b_3)C'_{n-1}$. First, observe that there is only one sub-ballot path from $(a,a,a-3)$ to (a,a,a) that has height at most 5; this path has weight 1. This contributes to the summand A_{n-1}. Then, notice that there are 3 sub-ballot paths from $(a,a,a-3)$ to $(a+1,a,a-1)$; the weights of these paths are b_1, b_2 and b_3. This contributes the summand $(b_1+b_2+b_3)B_{n-1}$. Lastly, there are 6 sub-ballot paths from $(a,a,a-3)$ to $(a+1,a+1,a-2)$ each of height at most 5; two have weight b_1, two have weight b_2, and two have weight b_3. This justifies the summand $2(b_1+b_2+b_3)C'_{n-1}$. □

For 3-dimensional, 6-bounded Catalan numbers, we have many more possible forms of the intermediate points to consider.

Proposition 4.5. *Let $\vec{b} = (1,1,1,1,1,0,0,0,\ldots)$. The 3-dimensional 6-bounded Catalan numbers $C_{3,6,n} = C^{\vec{b}}_{3,6,n}$ satisfy the following recurrence:*

$$\begin{pmatrix} A_n \\ B_n \\ C'_n \\ D_n \\ E_n \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 6 & 2 & 1 & 1 & 0 \\ 1 & 3 & 3 & 0 & 2 & 2 \\ 0 & 2 & 1 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C'_{n-1} \\ D_{n-1} \\ E_{n-1} \\ F_{n-1} \end{pmatrix}.$$

Proof. The result is obtained in the same manner as Proposition 4.3, but with 6 states instead of 3. □

5. Multidimensional Balanced-Ballot-Path-Height triangles

In this section, we define the k-dimensional Balanced-Ballot-Path-Height triangle, our analog to the triangle of the number of Dyck paths of length $2n$ and height u (A080936 in the OEIS [12]). We then compute values of this triangle in Table 1 for $k = 3$ and $k = 4$ for Table 2 and derive several properties, including periodicity of the entries of the triangles modulo an integer of the entries of the triangle.

Denote by $D_{k,u,n}$ the number of k-dimensional Balanced ballot paths of kn steps such that their heights are exactly u, i.e., for at least one intermediate point $h_k(x) = u$, but for no points $h_k(x) > u$.

Definition 5.1. *For integer $k \geq 2$, the **k-dimensional Balanced-Ballot-Path-Height triangle** is the table of values of $D_{k,u,n}$ over all positive values of u, n.*

Recalling Definition 2.5 for the k-dimensional u-bounded Catalan numbers $C_{k,u,n}$, we have

$$D_{k,u,n} = C_{k,u,n} - C_{k,u-1,n}.$$

In particular, one can find that the 2-dimensional Balanced-Ballot-Path-Height triangle is the same as the triangle of Dyck paths of length $2n$ and height u.

In a tabular representation of this triangle, such as Table 1 for $k = 3$ and Table 2 for $k = 4$, the numbers in each row correspond to the number of Balanced ballot paths of kn steps and height from $k - 1$ to $(k - 1)n$. We compute the values in our tables using Python code that implements a recursive algorithm to generate Balanced ballot paths that meet the specified height restriction.

Table 1 gives the first 36 nonzero terms of the triangular array sequence $D_{3,u,n}$; it can also be found as the new sequence A387912 in OEIS [12]. We observe that, since each entry $D_{3,u,n}$ is the number of 3-dimensional Balanced ballot paths of $3n$ steps and height exactly u, the sum of the numbers in the nth row is equal to the nth 3-dimensional Catalan number (A005789 in the OEIS [12]). Moreover, for any n we have $D_{3,2n,n} = C_n$, where C_n is the classical nth Catalan number. Indeed, for each path counted by $D_{3,2n,n}$, the first n steps are equal to $\vec{e}_1$, which is a necessary and sufficient condition for the path to reach height $2n$. Also, the number of possible sequences of steps after the first n steps is the number of sequences of steps $(\vec{s}_1, \vec{s}_2, \ldots \vec{s}_{2n})$ consisting of n steps of form $\vec{e_2}$ and n steps of form $\vec{e_3}$ such that for each intermediate point $\vec{v}_i = \sum_{j=1}^{i} \vec{s}_j$ the second coordinate is at least the third coordinate. This number is the nth 2-dimensional Catalan number.

$n \backslash u$	2	3	4	5	6	7	8	9	10	11	12
1	1										
2	1	2	2								
3	1	8	18	10	5						
4	1	26	120	142	117	42	14				
5	1	80	720	1480	1789	1130	596	168	42		
6	1	242	4122	13680	23205	20940	14817	6936	2781	660	132

Table 1: Some entries of the 3-dimensional Balanced-Ballot-Path-Height triangle.

$n \backslash u$	3	4	5	6	7	8	9	10	11	12
1	1									
2	1	3	5	5						
3	1	15	68	147	105	84	42			
4	1	63	722	3098	4720	5940	5112	2520	1386	462

Table 2: Some entries of the 4-dimensional Balanced-Ballot-Path-Height triangle.

Table 2 gives the first 22 elements of the sequence $D_{4,u,n}$. This array can also be found as the new sequence A387987 in OEIS [12]. For each n, the sum of the entries of the nth row is the nth 4-dimensional Catalan number (A005790 in the OEIS, [12]). Moreover, for all n we have $D_{4,3n,n} = C_{3,n}$, by arguments analogous to the 3-dimensional case. This suggests that generalizations are possible.

For each row of the k-dimensional Balanced-Ballot-Path-Height triangle we have:

$$\sum_{u=k-1}^{n(k-1)} D_{k,u,n} = C_{k,n} \quad \text{and} \quad D_{k,u,n} = C_{k,u,n} - C_{k,u-1,n}.$$

Proposition 5.1. *For the k-dimensional Balanced-Ballot-Path-Height triangle we have*

$$D_{k,(k-1)n,n} = C_{k-1,n}.$$

Proof. The sequence $D_{k,(k-1)n,n}$ counts the k-dimensional Balanced ballot paths of kn steps, with height $(k-1)n$. The only way to reach this height is when the first n steps are all up-steps. This is since there must exist an i such that the ith intermediate point $\vec{v}_i$ has height $(k-1)n$ and that can only be achieved by having only n up-steps and no other kinds of steps (steps that are not up-steps decrease the height of the intermediate point). Therefore, the first n steps are all $\vec{e_1}$ and the number of k-dimensional sub-ballot paths starting from $(n, 0, 0, \dots, 0) \in \mathbb{Z}^k$ and ending at $(n, n, \dots, n) \in \mathbb{Z}^k$ equals the number of $(k-1)$-dimensional sub-ballot paths from $(0, \dots, 0) \in \mathbb{Z}^{k-1}$ to $(n, \dots, n) \in \mathbb{Z}^{k-1}$, which is $C_{k-1,n}$. □

Proposition 5.2. *For the k-dimensional Balanced-Ballot-Path-Height triangle, we have*

$$D_{k,(k-1)n-1,n} = (n-1)D_{k,(k-1)n,n}.$$

Proof. A k-dimensional Balanced ballot path has height $(k-1)n-1$ only if the first $(k-1)n+1$ elements consist of $(k-1)n$ steps in the $\vec{e}_1$ direction and one step in the direction of $\vec{e}_2$. There are exactly $(n-1)$ k-dimensional Balanced ballot paths that meet this condition. Let S_1 be the collection of pairs whose first coordinate consists of a k-dimensional Balanced ballot path of kn steps and maximum height $(k-1)n$, and whose second coordinate consists of an integer in $\{2, 3, \dots, n\}$. Let S_2 be the collection of k-dimensional Balanced ballot paths of kn steps and height $(k-1)n-1$. It suffices to show $|S_1| = |S_2|$. We illustrate a bijection between S_1 and S_2: given a pair (P, i), we construct a k-dimensional Balanced ballot path P' of kn as follows. The ith step of P' is in the direction $\vec{e}_2$, while all steps from the first to the $(n+1)$st one, except for the ith one, are in the direction $\vec{e}_1$, and all the other steps in P' are the same as those of P. □

We end the section with a result analogous to Theorems 3.1 and 3.2:

Theorem 5.1. *For any fixed positive integers $k \geq 2$ and u, the sequence $D_{k,u,0} \pmod{m}, D_{k,u,1} \pmod{m}, D_{k,u,2} \pmod{m}, \dots$ is eventually periodic.*

Proof. First, recall from earlier in this section that $D_{k,u,n} = C_{k,u,n} - C_{k,u-1,n}$. Then, observe that $C_{k,u,n} = C^{(1,1,\dots)}_{k,u,n}$ and $C_{k,u-1,n} = C^{(1,1,\dots)}_{k,u-1,n}$. From Theorem 3.1 it follows that both $C_{k,u,n}$ and $C_{k,u-1,n}$ are eventually periodic modulo m. It remains to note that if two sequences are eventually periodic modulo m, then their difference is also eventually periodic. □

6. A multidimensional generalization of the Narayana triangle

We now extend the notion of Narayana numbers to the setting of k-dimensional Balanced ballot paths for any integer $k \geq 2$. In the 2-dimensional setting of Dyck paths, the sequence representing the number of Dyck paths of $2n$ steps with a fixed number of peaks is called the Narayana numbers (A001263 in OEIS [12]). Previously, Sulanke [15] proposed a higher-dimensional analog of Narayana numbers $N(k, n, a)$, the number of k-dimensional Balanced ballot paths that have exactly a "ascents", which are instances of a step $\vec{e}_i$ being immediately followed by $\vec{e}_j$ for some $i > j$. Using our peaks rather than Sulanke's ascents, we define a distinct higher-dimensional analogue of the Narayana numbers.

To that end, we first consider the number of peaks, a statistic of the k-dimensional Balanced ballot paths. We formally define them as follows.

Definition 6.1. *Given a Balanced ballot path P, a **peak** in P is an intermediate point, to which we have arrived with an up-step and left with a down-step.*

Example 6.1. *The 3-dimensional Balanced ballot path in Figure 3 has three peaks: the first and second intermediate points with height 2, along with the first point with height 3.*

In the context of k-dimensional Balanced ballot paths, we use this definition of peak; increases in the first coordinate of intermediate points represent a positive change in height, and increases in any other coordinate represent a negative change. Therefore, the first coordinate of intermediate points is the primary coordinate. This explains why we consider $\vec{e}_1$ to be an up-step for any $k \geq 2$ and any k-dimensional Balanced ballot path P.

Definition 6.2. *Denote by $N'_{k,\alpha,n}$ the number of k-dimensional Balanced ballot paths with α peaks. For fixed k, the array of values of $N'_{k,\alpha,n}$ over all possible values of n and numbers of peaks α is the **k-dimensional Narayana triangle**.*

In particular the 2-dimensional Narayana triangle is the Narayana triangle (A001263 in OEIS [12]). Also, observe that for each row of the k-dimensional Narayana triangle, we have:

$$\sum_{\alpha=1}^{n} N'_{k,\alpha,n} = C_{k,n} = \sum_{u=k-1}^{n(k-1)} D_{k,u,n}$$

Furthermore, we find that the k-dimensional Narayana triangle is related to the k-dimensional Balanced-Ballot-Path-Height triangle. More specifically, the numbers in the first column form the sequence of $(k-1)$-dimensional Catalan numbers. Together with Proposition 5.1, this yields a relation between the three types of sequences.

Proposition 6.1. *For the k-dimensional Narayana triangle we have*

$$N'_{k,1,n} = C_{k-1,n} = D_{k,(k-1)n,n}.$$

Proof. A k-dimensional balanced ballot path P has only one peak if and only if there is only one instance in P of a step of the form $\vec{e_1}$ immediately followed by a down-step. The only way for the condition to be satisfied is if the first n steps are up-steps. These paths are the same as in Proposition 5.1. The number of k-dimensional sub-ballot paths starting from $(n, 0, 0, \ldots, 0)$ and ending at $(n, n, \ldots, n)$ equals the number of $(k-1)$-dimensional Balanced ballot paths of $(k-1)n$ steps, $C_{k-1,n}$. □

We compute the first few terms of $N'_{3,\alpha,n}$ and $N'_{4,\alpha,n}$ by using recursive Python code to generate valid Balanced ballot paths. Table 3 and Table 4 show the 3-dimensional and 4-dimensional Narayana triangles, respectively. From the entries of Table 3, observe that the 3-dimensional Narayana triangle differs from the sequence A087647 in the OEIS [12], Sulanke's [15] 3-dimensional analogs of the Narayana numbers; this shows our analog of Narayana numbers is different from Sulanke's.

$n \backslash \alpha$	1	2	3	4	5	6
1	1					
2	2	3				
3	5	23	14			
4	14	131	233	84		
5	42	664	2339	2367	594	
6	132	3166	18520	36265	24714	4719

Table 3: Table of values of $N'_{3,\alpha,n}$, the 3-dimensional Narayana triangle.

$n \backslash \alpha$	1	2	3	4	5	6
1	1					
2	5	9				
3	42	236	184			
4	462	5354	12268	5940		
5	6006	118914	543119	737129	257636	
6	87516	2653224	20245479	53243052	50245691	13754842

Table 4: Table of values of $N'_{4,\alpha,n}$, the 4-dimensional Narayana triangle.

The 4-dimensional Narayana triangle from Table 4 is the new sequence A387936 in the OEIS [12] and has no known combinatorial interpretation other than the one we describe.

In contrast, as shown in Table 3, the first 21 values of the 3-dimensional Narayana triangle equal those of sequence A338403 in OEIS [12]. The OEIS sequence counts the (n, α)-*Duck words*, which are defined in [2] as a special class of 3-dimensional Dyck words of length $3n$ (equivalent to 3-dimensional balanced ballot paths of $3n$ steps) studied in the context of valid hook configurations. There is no known formula for A338403 (Problem 6.3 in [2]). This leads us to *conjecture* that the number of (n, α)-Duck words is equal to the entry $N'_{3,\alpha,n}$ of the 3-dimensional Narayana triangle. This motivates the following directions for future research:

Open Problem 6.1. *Establish a bijection between the set of (n, α)-Duck words and the set of all 3-dimensional Balanced ballot paths with $3n$ steps and α peaks. We believe that these two sets are equinumerous.*

Open Problem 6.2. *Find a closed-form expression for $N'_{3,\alpha,n}$.*

Solving these two open problems would yield a formula for the number of (n, α)-Duck words and could have significant implications in the study of valid hook configurations.

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